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ON UNIFORM RATIONALIZATION OF ULTRAMETRICS

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Abstract

The paper gives a complete solution of the rationalization problem for ultrametric spaces and spaces close to them. It is proved that for any ultrametric space (X, d) for arbitrary prescribed $\varepsilon > 0$ and $K > 1$ there exists a binary-rational ultrametric $r(x, y)$ on X such that the identity map $i : (X, d) \to (X, r)$ is (i) homeomorphism, (ii) uniform equivalence, (iii) non-expanding map, i.e., $d(x, y) > r(x, y)$, (iv) ε-translation, i.e., $|d(x, y) - r(x, y)| < \varepsilon$, (v) inverse K-Lipschitz map, i.e., $d(x, y) \leq Kr(x, y)$. Criterion of uniform rationalability for general metric spaces is stated. It is Smirnov proximate zero-dimensionality, δ dim X = 0. For topological rationalability a necessary condition (ind X = 0) and sufficient condition (Ind X = dim X = 0) are proved. An unsolved problem of rationalization for Roy's type spaces (with ind $X = 0$, dim $X > 0$) is discussed.

A metric space (X, d) is called ultrametric [1] (or non-Archimedean [2], or isosceles [3]) if its metric d satisfies the

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Key words: Metric space, ultrametric space (= non-Archimedean metric space), binary-rational metric, rationalization problem, dimension, zero-dimensionality (in the sense of Menger-Uryson, Brower, Lebesgue), uniform dimension, proximate dimension in the sense of Smirnov.

strong triangle axiom $d(x,z) \leq \max[d(x,y), d(y,z)]$. Examples of these spaces are well known for a long time in many areas of mathematics: in number theory (rings $\mathbf{Z}_p$ of Hensel integers and fields $\mathbf{Q}_p$ of Henselian numbers), in algebra (non-Archimedean normed fields), in topology (Baire space and generalized Baire spaces), in p-adic analysis (field Ω and rings of Ω-valued functions), and so on. Topological [2] and uniform [3] characterizations of these spaces as well as description of their metric [4, 5], geometric [6, 7] and categorical [8-11] properties can be found in the literature. The following theorem gives a lower bound of weight of the universal ultrametric space constructed in [5].

Theorem 1. *For any ultrametric space (X,d) the set of values of its metric $V = \{d(x,y)|x,y \in X\}$ has cardinality no greater than its weight, $|V| \leq w(X)$.*

It leads us immediately to the following corollary.

Corollary. *For any separable ultrametric space the set of values of its metric is at most countable.*

The following question is of importance for applications of the theory of ultrametric spaces to computer science. Given an ultrametric space (X,d), is it possible to approximate its metric by a rational-valued (binary rational-valued) metric $r(x,y)$ close to the initial metric $d(x,y)$ in a certain sense?

The purpose of the paper is to answer this question in affirmative for ultrametric spaces and spaces close to them.

Theorem 2. *If (X,d) is an ultrametric space and $|V| = |\{d(x,y)|x,y \in X\}| \leq \aleph_0$ then for every $\varepsilon > 0$, for every $K > 1$ there exists an ultrametric $r(x,y)$ over X such that*

a) (X,d) and (X,r) are homeomorphic, *b) (X, d) and (X, r) are uniformly equivalent,*

c) the identity map $i : (X, d) \longrightarrow (X, r)$ is non-expanding,

d) the identity map $i : (X, r) \longrightarrow (X, d)$ is K-Lipschitz, i.e. $r(x, y) \leq d(x, y) \leq Kr(x, y)$

e) the difference between d and r is at most ε, $d(x, y) \geq r(x, y) \geq d(x, y) - \varepsilon$,

f) all values of new ultrametric $r(x, y)$ are binary rational,

g) the identity map $i : (X, d) \longrightarrow (X, r)$ induces the map $i*$: $\{d(x, y)|x, y \in X\} \longrightarrow \{r(x, y)|x, y \in X\}$ and the last is one-to-one.

Proof. Let B_+ be the set of binary rational positive numbers. To prove the theorem we define a monotonically increasing function $f : V \longrightarrow B_+$ such that

c) $f(t) < t$,
d) $t < Kf(t)$,
e) $0 < t - f(t) < \varepsilon$,
f) $f(t)$ is binary rational $\forall t \in V$,
g) $f(t)$ is injective,

and then put $r(x, y) = f(d(x, y))$. Since $d(x, y)$ is an ultrametric, the strong triangle inequality for $d(x, y)$ implies the same inequality for $r(x, y)$. So $r(x, y)$ is an ultrametric. The properties c) – g) of the function f imply the statements c) – g) of the theorem. The statements a) and b) are corollaries of c) and d). For a sake of simplicity we first define $f(t)$ for $\varepsilon = 2$ and $K = 4$ and then show how one should modify the definition for arbitrary $\varepsilon > 0$ and $K > 1$.

Let us divide the positive half-line $\mathbf{R}_+$ by integer points n ($n \in \mathbf{N}$). Suppose the set $I_n = V \cap (n, n+1]$ is not empty. We shall define $f : V \longrightarrow B_+$ separately over each I_n ($n > 1$). Since I_n is a countable ordered set, there exists an order-preserving (monotonically increasing) injective function $f_n : I_n \mapsto B \cap (n-$

$1,n]$. For any $n>1$ we put $f|_{I_n}=f_n$. To define $f(t)$ on $(0,2]$ we divide this interval by the points $1, \frac{1}{2}, \frac{1}{4}, \dots 1/2^m, \dots$ and define $f(t)$ as a monotone injection $V\cap(1,2]\to B\cap(\frac{1}{2},1]$, $V\cap(\frac{1}{2},1]\to B\cap(\frac{1}{4},\frac{1}{2}], \dots$ $V\cap(1/2^m,1/2^{m-1}]\to B\cap(1/2^{m+1},1/2^m]$, and so on. By putting $f(0)=0$ we make $f(t)$ continuous at 0. By the definition of f_n we have $0<t-f(t)<2$ and $f(t)>1/2^{m+1}=\frac{1}{4}(1/2^{m-1})>\frac{1}{4}t, \forall t\in V\cap(1/2^m,1/2^{m-1}]$. This implies $t<4f(t)=Kf(t)\forall t\in V$. Therefore $r(x,y)=f(d(x,y))$ satisfies the conditions a) – g) of the theorem for $\varepsilon=2$ and $K=4$.

For arbitrary $K\in(1,4)$ let us find the minimal integer m such that the distance $|(\sqrt{K})^{m+1}-(\sqrt{K})^m|$ between the points of a geometric progression is not less than 1. $|(\sqrt{K})^{m+1}-(\sqrt{K})^m|=(\sqrt{K})^m(\sqrt{K}-1)\geq 1$, $m\geq -2\log_k(\sqrt{K}-1)$, i.e., $m=-2[\ln(\sqrt{K}-1)/\ln K]$ (the function $g(t)=-2\ln(\sqrt{t}-1)/\ln t$ is monotonically decreasing on $(1,+\infty)$, $g(t)\to+\infty$ as $t\to 1$, $g(4)=0$, $g(t)\to -1$ as $t\to\infty$, hence such integer m does exist).

For $t>(\sqrt{K})^m$ we divide the half-line $((\sqrt{K})^m,\infty)$ into the equal half-open intervals $((\sqrt{K})^m+n,(\sqrt{K})^m+n+1]$, where $n\in\mathbf{N}$, and define $f(t)$ on $((\sqrt{K})^m,\infty)$ as above. By the definition of $f(t)$, $0<t-f(t)<2$ for every $t>(\sqrt{K})^m$. It proves e) for $\varepsilon=2$. To prove the property d) let us consider $t\in((\sqrt{K})^m+1,(\sqrt{K})^m+2]$. We have $f(t)\in((\sqrt{K})^m,(\sqrt{K})^m+1]$ or $f(t)>(\sqrt{K})^m$. By the definition of m, $(\sqrt{K})^{m+1}\geq(\sqrt{K})^m+1$. Therefore $(\sqrt{K})^{m+2}\geq\sqrt{K}((\sqrt{K})^m+1)=(\sqrt{K})^{m+1}+\sqrt{K}>(\sqrt{K})^{m+1}+1\geq(\sqrt{K})^m+2$ since $K>1$. Hence $f(t)>(\sqrt{K})^m=(\sqrt{K})^{m+2}/K>((\sqrt{K})^m+2)/K\geq t/K$. Thus $t<Kf(t)$. For $t\in((\sqrt{K})^m+n+1,(\sqrt{K})^m+n+2]$ we have $f(t)>(\sqrt{K})^m+n=(\sqrt{K})^{m+2}/K+n\geq((\sqrt{K})^m+2)/K+n=n-n/K+((\sqrt{K})^m+n+2)/K>((\sqrt{K})^m+n+2)/K\geq t/K$ since $K>1,n>0$. So f satisfies e) for every $t>(\sqrt{K})^m$.

For $t\leq(\sqrt{K})^m$ we divide the half-open interval $(0,(\sqrt{K})^m]$ by the points $\lambda^{-m}, \lambda^{-m+1}, \dots$ λ^n ,... , where $\lambda=1/\sqrt{K}$,

$n \geq -m$, and define $f : V \cap (0, \lambda^{-m}] \to B \cap (0, \lambda^{-m+1}]$ as follows. For any non-empty $J_n = V \cap (\lambda^{n+1}, \lambda^n]$ let f_n be a monotone injection from J_n to $B \cap (\lambda^{n+2}, \lambda^{n+1}]$ and $f|_{J_n} = f_n$. By the definition of m, $0 < t - f(t) < 2$ for every $t \leq (\sqrt{K})^m$. Hence $f(t)$ satisfies the condition e) for $\varepsilon = 2$ for any t. As for the statement d) we have $\lambda^{n+2} < f(t) < t \leq \lambda^n = \lambda^{n+2}/\lambda^2 < f(t)/\lambda^2$, i.e., $t < f(t)/\lambda^2 = Kf(t)$ for any $t \in J_n$ and therefore for any $t \leq \lambda^{-m}$. For $t > \lambda^{-m}$ the inequality is proved above.

If ε is an arbitrary small positive number we should compare the distance $|(\sqrt{K})^{m+1} - (\sqrt{K})^m|$ with $\varepsilon/2$, find the smallest integer m such that $|(\sqrt{K})^{m+1} - (\sqrt{K})^m| \geq \varepsilon/2$, and then divide the half-line $((\sqrt{K})^m, \infty)$ by the points $\{(\sqrt{K})^m + n\varepsilon/2\}$. The rest of the proof is just the same as above. □

Note. If the set V of values of the initial metric d is uncountable, then it is clear that no one-to-one rationalization can exist. However, omitting the requirement g) we can prove the following theorem.

Theorem 3. *For any ultrametric space (X, d) there exists a uniformly equivalent binary rational ultrametric $r(x, y)$ satisfying the statements a) – f) of theorem 2.*

Proof. In order to prove the theorem we define a non-decreasing function $f : \mathbf{R}_+ \to B_+$ that satisfies the conditions c) – f) above and put $r(x, y) = f(d(x, y))$.

For $\varepsilon = 1$ and $K = 2$ it can be done as follows: $f(t) = [t]$ for $t \geq 1$, where $[x]$ is the integral part of x, and $f([1/2^{m+1}, 1/2^m)) = \{1/2^{m+1}\}$ for any integer $m > 0$. It is obvious that

c) $f(t) \leq t$ on R_+

d) $t < 2f(t)$,

e) $t - f(t) < 1$,

f) $f(t)$ is binary rational.

These imply the statements c) – f) of the theorem. The modification of the definition of f for arbitrary $\varepsilon > 0$ and $K > 1$ is similar to that in the proof of theorem 2. □

So we see that an ultrametric $d(x, y)$ can be rationalized arbitrary close in any reasonable sense (up to topological equivalence, uniform equivalence, an arbitrary small ε-translation, etc.).

As for general metric spaces (not ultrametric ones), it is impossible, in general, to rationalize their metric even up to homeomorphism. Actually, if a metric $d(x, y)$ is rational-valued then any closed ball $O(x, q)$ of irrational radius q is open. Hence (X, d) is small-inductive zero-dimensional, $\operatorname{ind} X = 0$. If the stronger equality holds $\operatorname{Ind} X = \dim X = 0$ then, in view of de Groot's theorem [2], (X, d) admits a topologically equivalent ultrametric $\Delta(x, y)$. Theorem 3 then enables us to rationalize it. Thus small inductive zero-dimensionality $\operatorname{ind} X = 0$ is necessary and (large inductive) zero-dimensionality $\operatorname{Ind} X = 0$ is sufficient for a topological rationalability of a space. However, a topological re-metrization is too rough, it preserves neither completeness, nor totally boundedness, nor Cantor connectedness, nor other uniform properties of a space. For a more accurate re-metrization we need the strengthening of de Groot's theorem, proved in [3].

Theorem 4. *On any zero-dimensional metric space (X, d) an ultrametric $\Delta(x, y)$ can be introduced in such a way that the identity map $i : (X, d) \to (X, \Delta)$ is continuous and the inverse map is uniform (and even non-expanding).*

Corollary. [3]. *Any zero-dimensional metric space can be re-metrized into an ultrametric space without worsening its completeness.*

This means that any Cauchy sequence x_n in (X, Δ) without a limit point is a Cauchy sequence in (X, d) without a limit point (and surely any Cauchy sequence x_n in (X, d) with a limit point x is a Cauchy sequence in (X, Δ) with the same limit point). In other words, the set of Cauchy sequences having no limits in (X, Δ) is not greater than that in (X, d). In particular, if (X, d) is complete then so is (X, Δ). Combining this with theorem 3 we get the next corollary.

Corollary. *Any zero-dimensional metric space (X, d) can be supplied with a binary rational ultrametric $r(x, y)$ without worsening its completeness.*

This re-metrization is better than topologically equivalent one. However, in general it is not a uniformly equivalent re-metrization yet. For a uniform re-metrization we need a description of metric spaces that can be mapped uniformly (non-expandingly) onto ultrametric spaces. Their complete characterization can be found in the following theorem.

Theorem 5. [3]. *1) Totally unlinked metric spaces and only they are inverse images of ultrametric spaces under uniformly continuous (and even non-expanding) bijections.*
2) Small-proximate zero-dimensional metric spaces ($in\delta X = 0$) and only they are inverse images of ultrametric spaces under uniformly continuous (and non-expanding) bijections with continuous inverse maps.
3) Large-proximate zero-dimensional metric spaces ($In\delta X = 0$) and only they are inverse images of ultrametric spaces under mutually uniform homeomorphisms (with a non-expanding direct map).

See [3] for proofs and definitions. Recall that a large proximate zero-dimensionality $In\delta X = 0$ is equivalent to a proximate zero-dimensionality $\delta \dim X = 0$ in the sense of Yu. M. Smirnov [14]. It means that Smirnov's compactification $s(X)$ of a space (X, d) is zero-dimensional, $\dim s(X) = 0$ [13]. Combining theorem 3 with the second and the third parts of the last theorem we obtain the following.

Corollary. *1) Small-proximate zero-dimensional metric spaces (X, d) and only they admit a binary rational ultrametrization $r(x, y)$ such that the statements a), c), and f) of theorem 2 hold.*
2) Large-proximate zero-dimensional metric spaces (X, d) and only they admit a binary rational ultrametrization $r(x, y)$ such that the statements a), b), c), and f) of theorem 2 hold.

The last statement implies the following criterion.

Criterion. *Large-proximate zero-dimensionality $In\delta X = 0$ of a metric spaces (X, d) is necessary and sufficient for existence of a uniform ultrametric rationalization of its metric.*

On the other hand, theorem 3 and de Groot's theorem show that zero-dimensionality $\mathrm{Ind}\ X = \dim X = 0$ is necessary and sufficient for existence of a topological ultrametric rationalization. As for a rationalization by general metrics (non-ultrametric ones) we saw that $\mathrm{Ind}\ X = 0$ is sufficient and $\mathrm{ind}\ X = 0$ is necessary for it. The problem rising now is to find a criterion. It seems to be related to difficult problems in the dimension theory for metric spaces and complicated examples of spaces with non-equal dimensions. In particular, we pose the following problem.

Problem. There exist metric spaces (X, d) such that $\mathrm{ind}\ X = 0$ and $\mathrm{Ind}\ X > 0$, e.g., Roy's space [12]. Is it possible to introduce there a rational-valued metric $r(x, y)$ up to a topological equivalence at least?

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