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A BITOPOLOGICAL GELFAND THEOREM FOR C^*-ALGEBRAS

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Abstract

I. M. Gelfand, in his delightfully elegant representation theorem, showed that any commutative C^*-algebra is isometrically isomorphic with an algebra of complex-valued continuous functions. It is the purpose of this paper to obtain a "Gelfand type" representation of an arbitrary (not necessarily commutative) C^*-algebra, as an algebra of continuous mappings. The key idea, here, is to assign two topologies to the base space and then require continuity with respect to both topologies. The first of these topologies is the familiar hull-kernel topology while the second is the co-compact dual of the first.

1. Introduction

In this paper, we give a representation for not necessarily commutative C^*-algebras that is an analogue of Gelfand's beautiful theorem.

Theorem 1.1. *(Gelfand) Let A be a commutative C^*-algebra. Then $Max(A)$, the maximal ideal space of A with the hull-kernel topology, is a locally compact Hausdorff space. Further,*

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A is isometrically isomorphic with $C_\infty(Max(A))$, the algebra of all complex-valued continuous functions that vanish at infinity.

Here we bring together ideas from C^*-algebra representation theory as described by Fell, Dauns and Hofmann in [Fe] and [DH] with the topological and bitopological concepts from [Ko], [HK] and [HKMS] and the lattice theory notions from [G&].

The necessary bitopological concepts are outlined in section 2, while section 3 provides information about the lattice $Id(A)$ of closed ideals of a C^*-algebra A. The basic structure, called a field of C^*-algebras, which is used for the representation theorem, is generated in section 4. The represention is described in two separate theorems 5.5 and 6.4. Theorem 5.5 gives the representation for algebras with identity while 6.4 takes care of C^*-algebras lacking a unit. Part (a) of 6.4 asserts:

Any C^*-algebra is isometrically isomorphic to an algebra of pairwise continuous sections which vanish at infinity.

2. Bitopological Spaces

Definition 2.1. A bitopological space (X, τ, τ^*) consists of a set X with two topologies τ and τ^*.

The generic example for the bitopological spaces that we deal with here is (R, ω, σ), i.e. the reals R with the left ray, ω, and the right ray, σ, topologies. More generally for a complete lattice, L, the *lower topology* ω is the topology that has the collection $\{L\setminus \uparrow a : a \in L\}$ as a subbasis. A subset U of L is *Scott-open* provided (i) U is an upper set (i.e., $U = \uparrow U$) and (ii) $\sup D \in U$ implies that $U \cap D$ is non-empty for each directed set D in L. The *Scott topology* σ on L consists of all Scott-open sets. See 1.3, p. 99 in [G&]. For continuous lattices, such as we consider in this paper, the Scott topology is equal to the *co-compact* topology of ω, that is, the coarsest topology for which each ω-quasicompact, saturated set is closed. See [G&, 5.1, p. 258].

Definition 2.2. For bitopological spaces (X, τ, τ^*) and $(Y, \upsilon, \upsilon^*)$ and a map $f : X \to Y$, the function f is *pairwise continuous* if it is continuous both as a map from (X, τ) to (Y, υ) and as a map from (X, τ^*) to (Y, υ^*). For a property P, the bitopological space (X, τ, τ^*) is *pairwise* P if both (X, τ, τ^*) and (X, τ^*, τ) have the property P.

Remark. A pairwise continuous map is continuous with respect to the join $\tau \vee \tau^*$ of the two topologies, but the converse fails, in general. A real valued function f defined on R is pairwise continuous from (R, ω, σ) to (R, ω, σ) if and only if f is order preserving and continuous with respect to the join topology (= the usual topology on R).

The separation properties for bitopological spaces are analogous to those for topological spaces. We list some of the definitions here for easy reference. For more details, see [Ko].

Separation and Compactness Definitions for Bitopological Spaces 2.3. A bitopological space, (X, τ, τ^*), is defined to be:

(a) *completely regular* if whenever $x \in U \in \tau$, there is a pairwise continuous $f : (X, \tau, \tau^*) \to ([0, 1], \omega, \sigma)$ such that $f(x) = 0$ and $f(y) = 1$ whenever $y \notin U$;

(b) *regular* if whenever $x \in T \in \tau$, then there are a τ-open U and a τ^*-closed D such that $x \in U \subset D \subset T$;

(c) *completely Hausdorff* if whenever $x \notin \tau\text{-}cl\{y\}$, there is a pairwise continuous $f : (X, \tau, \tau^*) \to ([0, 1], \omega, \sigma)$ such that $f(x) = 0$ and $f(y) = 1$;

(d) *pseudoHausdorff* if $x \notin \tau\text{-}cl\{y\}$ implies there are disjoint $T \in \tau$ and $T^* \in \tau^*$ such that $x \in T$ and $y \in T^*$;

(e) *joincompact* if it is pairwise pseudoHausdorff and its join topology $\tau \vee \tau^*$ is compact Hausdorff.

The implications that hold for topological spaces also hold for the bitopological separation properties defined above and each bitopological property implies the corresponding separation property for the join topology, as stated in 2.4 below.

Theorem 2.4. *For any bitopological space (X, τ, τ^*):*
(a) Joincompact $\Rightarrow$ completely regular $\Rightarrow$ regular $\Rightarrow$ pseudo-Hausdorff. Furthermore, this string of implications remains valid when "regular" is replaced by "completely Hausdorff".
(b) For any of the properties P = regular, completely Hausdorff, or completely regular, if (X, τ, τ^*) is pairwise P then $\tau \vee \tau^*$ has P. Additionally, subspaces and products of spaces satisfying P, also satisfy P.
(c) Products and $\tau \vee \tau^*$-closed subspaces of joincompact spaces are joincompact.

3. Closed Ideals of A

For a C^*-algebra A, let $Id(A)$ denote the set of all norm closed, two sided ideals of A. When ordered by set inclusion, $Id(A)$ becomes a complete lattice. The zero ideal is the smallest element of $Id(A)$ while the entire algebra A is the largest. With the lower, ω, and the Scott, σ, topologies as defined in section 2, $(Id(A), \omega, \sigma)$ becomes a joincompact bitopological space. The common refinement $\omega \vee \sigma$ of the lower and Scott topologies is the *Fell topology*, as described in [F]. Each primitive ideal is closed and each proper closed ideal I is the intersection of all primitive ideals which contain I. The set of all primitive ideals is denoted by $Prim(A)$. Unless stated otherwise, the topology on $Prim(A)$ is taken to be the hull-kernel topology (= the restriction of ω to $Prim(A)$).

Notation. We will use the notation of [G&] for lower and upper sets in the lattice $Id(A)$. Specifically, for an ideal I in $Id(A)$, we have $\uparrow I = \{J : J \in Id(A) \;\; I \subset J\}$ and for a subset S of $Id(A)$, we have $\uparrow S = \bigcup\{\uparrow I : I \in S\}$. The set S is an *upper set* if $S = \uparrow S$. The symbols $\downarrow I$, $\downarrow S$ and the term *lower set* are defined in a similar way.

Definition 3.1. (Glimm). For a pair P, Q of primitive ideals, $P \sim Q$ means that $f(P) = f(Q)$ for all complex-valued continuous functions on $Prim(A)$. This relation partions $Prim(A)$ into equivalence classes. An ideal is a *Glimm ideal* if it is the intersection of an $\sim$ equivalence class. A closed ideal is *Glimmal* if it contains a Glimm ideal. $Glimm(A)$ denotes the set of Glimm ideals; $Glimmal(A)$, the set of Glimmal ideals and $Glimmal'(A)$, the set of all proper Glimmal ideals.

Definition 3.2. For a C^*-algebra A (with or without an identity), the *centroid of* A, denoted by $R(A)$ or simply R, is the set of all self maps $r : A \to A$ for which $(r(a))b = a(r(b))$ for all a and b in A. When the coordinate-wise operations of addition, multiplication and involution are assigned to R, then R becomes a commutative C^*-algebra with unit $\mathbf{1}$, where $\mathbf{1}(a) = a$ for all $a \in A$.

Remark. (a) For $r \in R$ and $P \in Prim(A)$, there is a complex number $\tilde{r}(P)$ so that $r(a) - \tilde{r}(P)a \in P$ for all $a \in A$.
(b) If $P \in Prim(A)$, then $M_P = \{r : r[A] \subset P\}$ is a maximal ideal of R. A maximal ideal M of R is *fixed* if $M = M_P$ for some $P \in Prim(A)$. Otherwise M is a *free ideal.*

Theorem 3.3. *(Dauns-Hofmann) The map $r \to \tilde{r}$ is an isometric isomorphism of R onto $C(Prim(A))$, the algebra of all complex-valued continuous functions on $Prim(A)$ with the hull-kernel ($= \omega$) topology. (See page 121 in [DH].)*

Theorem 3.4. *Let M be a maximal ideal of R. Then*
(a) $MA = \{r(a) : r \in M \quad a \in A\}$ is closed ideal of A.
(b) M is a fixed ideal in R if and only if MA is a proper ideal in A. Furthermore if M is fixed, then $M = M_P$ for each $P \in Prim(A)$ for which $MA \subset P$.
(c) $Glimm(A) = \{MA : M \in Max(R),\ M$ is fixed*$\}$.*

Theorem 3.5. *Let I be a proper ideal in $Glimmal(A)$.*
(a) Then there exist $M \in Max(R)$, $G \in Glimm(A)$, each uniquely determined by I, and $P \in Prim(A)$ so that $G = MA \subset I \subset P$ and $M = M_P$. Also $r(a) - \tilde{r}(P)a \in G$ for all $a \in A$.
(b) If $\tilde{r}$ is extended to $Glimmal'(A)$ by the formula $\tilde{r}(I) = \tilde{r}(P)$ where $P \in (\uparrow I) \cap Prim(A)$, then the extended function is continuous with respect to the ω topology. If A has a unit, then this extended map is also continuous with respect to the σ topology. (See section 7 of [HKMS].)

4. Basic Construction

In this section, we develop the basic structure needed for the representation theorems 5.5 and 6.5, which are proved in sections 5 and 6. The construction used here is an adaptation to bitopological spaces of the methods developed in [DH] and [M].

Definition 4.1. (a) A *field of sets* (E, X, Y, ϕ) consists of sets E, X and Y and a surjective map $\phi : X \times Y \to E$, for which there exists a map $p : E \to X$ such that $p \circ \phi$ is the projection of $X \times Y$ onto X. For $x \in X$, the *stalk over* x is the set $\phi(\{x\} \times Y)$ in E. A *section* s is a map from X to E for which $p \circ s$ is the identity map on X.
(b) The field (E, X, Y, ϕ) is a *field of topological spaces* if in addition to the field properties, E, X and Y are topological spaces for which the map ϕ is continuous and open.
(c) The field (E, X, Y, ϕ) is a *field of C^*-algebras* if Y and each

of the stalks are C^*-algebras so that the restriction of ϕ to $\{x\} \times Y$, for each x, is a homomorphism of C^*-algebras.

Let A be a C^*-algebra; denote by $Id(A)$ the lattice of closed ideals of A. Define $E = \cup\{I\} \times A/I$ where the union is taken over $Id(A)$. Let ϕ: $Id(A) \times A \to E$, be the map given by $\phi(I,a) = (I, a + I)$. Define the map $p : E \to Id(A)$ by $p(I, a + I) = I$. Then $(E, Id(A), A, \phi)$ is a field of sets.

Definition 4.2.
(a) Define the map Φ from $Id(A) \times A \times A$ into $E \times E$ by $\Phi(I,a,b) = (\phi(I,a), \phi(I,b))$.
(b) Denote by $E \vee E$ the range in $E \times E$ of the map Φ. That is $E \vee E = \Phi(Id(A) \times A \times A)$.

Remark. Observe that $E \vee E = \bigcup\{p^{\leftarrow}(I) \times p^{\leftarrow}(I) \mid I \in Id(A)\}$.

We now transfer the the operations and the norm on A to E. Note that the domain of definition for addition and multiplication is $E \vee E$.

$$(I, a + I) + (I, b + I) = (I, (a + b) + I)$$

$$(I, a + I) \cdot (I, b + I) = (I, (ab) + I)$$

$$\lambda \cdot (I, a + I) = (I, (\lambda a) + I) \quad \text{for scalars } \lambda$$

$$(I, a + I)^* = (I, a^* + I)$$

$$\|(I, a + I)\| = \|a + I\|$$

Remark. Note that for addition we have $(+) \circ \Phi = \phi \circ (id \times (+))$ where id denotes the identity map on $Id(A)$. A similar equality holds for multiplication. Thus the map Φ is very useful in establishing the continuity of addition and multiplication in E.

Notation. Let η be the norm topology on A. Recall that ω and σ denote the lower and Scott topologies on $Id(A)$. Define $\mathcal{L}$ (respectively, $\mathcal{U}$) to be the ϕ-quotient topology on E generated by the $\omega \times \eta$ ($\sigma \times \eta$, respectively) topology on $Id(A) \times A$. For any two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, the symbol $\mathcal{T}_1 \vee \mathcal{T}_2$ will denote the common refinement.

Theorem 4.3. *If the Fell topology, $\omega \vee \sigma$, is assigned to $Id(A)$ and $\mathcal{L} \vee \mathcal{U}$ to E, then $(E, Id(A), A, \phi)$ becomes a field of C^*-algebras.*

Proof. This theorem follows from 4.4(b) below.

Theorem 4.4. *(a) The map ϕ is open with respect to the $\sigma \times \eta$ and $\mathcal{U}$ topologies while Φ is open with respect to $\sigma \times \eta \times \eta$ and $\mathcal{U} \times \mathcal{U}$.*
(b) The map ϕ is open with respect to the $(\omega \vee \sigma) \times \eta$ and $\mathcal{L} \vee \mathcal{U}$ topologies.

Proof. (a) Let G be a $\sigma \times \eta$-open set and choose $(I_1, a_1) \in \phi^{\leftarrow}\phi(G)$. Then there exist $b_1 \in A$ so that $(a_1 - b_1) \in I_1$ and $(I_1, b_1) \in G$. Thus there are $V \in \sigma$ and a positive real number ε so that $(I_1, b_1) \in V \times \{b : \|b - b_1\| < 2\varepsilon\} \subset G$. Define the set Q by $Q = \{J : \|(a_1 - b_1) + J\| \geq \varepsilon\}$. Then Q is an ω-saturated, ω-quasicompact set. (See [D: 3.3.7, p. 75]). Thus by [G&, 5.1, p. 258] or [HKMS, 4.4], Q is σ-closed. Thus $(V \setminus Q) \times \{a : \|a - a_1\| < \varepsilon\}$ is a $\sigma \times \eta$-neighborhood of (I_1, a_1) that is contained in $\phi^{\leftarrow}\phi(G)$. This proves that $\phi(G)$ is $\mathcal{U}$-open; whence ϕ is an open map. That Φ is open follows in a similar way.
(b) The proof of this part is similar to that of (a). This time, let G be $(\omega \vee \sigma) \times \eta$-open and let I_1, a_1 and b_1 have the same

meaning as in (a). Then there exist $U \in \omega$, $V \in \sigma$ and $\varepsilon > 0$ so that $(I_1, b_1) \in (U \cap V) \times \{b : \|b - b_1\| < 2\varepsilon\} \subset G$. Define Q as in (a). Then $(U \cap (V \setminus Q)) \times \{a : \|a - a_1\| < \varepsilon\}$ is a $(\omega \vee \sigma) \times \eta$-neighborhood of (I_1, a_1) that is contained in $\phi^{\leftarrow}\phi(G)$. □

Lemma 4.5. *Let $a \in A$ and a real number $r \geq 0$ be given. Then*
(a) $\phi(Id(A) \times \{b : \|b - a\| \leq r\})$ is $\mathcal{L}$-closed and
(b) $\phi(Id(A) \times \{b : \|b - a\| < r\})$ is $\mathcal{U}$-open.
(c) If D is ω-closed, then $\phi(D \times \{b : \|b - a\| \leq r\})$ is $\mathcal{L}$-closed.

Proof. Part (b) follows from the fact that ϕ is a $(\sigma \times \eta)$-$\mathcal{U}$-open map.
(a) For fixed a and r, set $F = Id(A) \times \{b : \|b - a\| \leq r\}$. Next, let (I_1, a_1) be in the complement of $\phi^{\leftarrow}\phi(F)$. Then $\|(a_1 - a) + I_1\| > r$. Define $\varepsilon = (1/2)(\|(a_1 - a) + I_1\| - r)$. Then $U = \{J : \|(a_1 - a) + J\| > r + \varepsilon\}$ is ω-open. Let $(I, c) \in U \times \{b : \|b - a_1\| < \varepsilon\}$. If $x \in I \in U$ then $\|(c - a) + x\| \geq \|(a_1 - a) + x\| - \|c - a_1\|$. Thus $\|(c - a) + I\| \geq r + \varepsilon - \|c - a_1\| > r$. Therefore $U \times \{b : \|b - a_1\| < \varepsilon\}$ is disjoint from $\phi^{\leftarrow}\phi(F)$.
(c) For any $D \subset Id(A)$, $\phi(D \times \{b : \|b - a\| \leq r\}) = \phi(D \times A) \bigcap \phi(Id(A) \times \{b : \|b - a\| \leq r\}$ is $\mathcal{L}$-closed by (a). □

Definition 4.6. For $c \in A$ the *translation* T_c in E by c is the map $T_c : E \to E$ so that $T_c \circ \phi(I, a) = \phi(I, a + c)$.

Proposition 4.7. *Each translation T_c is a bitopological homeomorphism from $(E, \mathcal{L}, \mathcal{U})$ to itself.*

Proof. It is easy to show that T_c is a well-defined map and that $T_{-c} \circ T_c$ is the identity on E. Thus it suffices to prove that each T_c is pairwise continuous. Since ϕ is a bitopological quotient map and $T_c \circ \phi$ is pairwise continuous for each c, it follows that each translation is pairwise continuous from $(E, \mathcal{L}, \mathcal{U})$ to itself. □

We next define two special functions and prove a technical lemma that we will use later. For $c \in A$, define the real valued functions f_c and g_c on E by

$$f_c(I, a+I) = \|(a-c)+I\|$$
$$g_c(I, a+I) = \|a-c\| - f_c(I, a+I).$$

Lemma 4.8. *For each $c \in A$,*
(a) f_c is lower (upper, respectively) semicontinuous with respect to the $\mathcal{L}$ ($\mathcal{U}$, respectively) topology on E.
(b) g_c is upper (lower, respectively) semicontinuous with respect to the $\mathcal{L}$ ($\mathcal{U}$, respectively) topology on E.
(c) f_c is pairwise continuous as a map from $(E, \mathcal{U}, \mathcal{L})$ to $([0,1], \omega, \sigma)$.
(d) g_c is pairwise continuous as a map from $(E, \mathcal{L}, \mathcal{U})$ to $([0,1], \omega, \sigma)$.

Proof. (a) Since ϕ is a quotient map it suffices for the first part of (a) to show that $f_c \circ \phi$ is lower semicontinous on $(Id(A) \times A,\ \omega \times \eta)$. To this end, let $r \geq 0$ and (I_1, a_1) be so that $f_c \circ \phi(I_1, a_1) > r$. Define $\varepsilon = (1/2)(f_c \circ \phi(I_1, a_1) - r)$. Then $U = \{J : \|(a_1 - c) + J\| > r + \varepsilon\}$ is ω-open. If $(I, a) \in U \times \{b : \|b - a_1\| < \varepsilon\}$, then $f_c \circ \phi(I, a) = \|(a-c)+I\| \geq \|(a_1 - c) + I\| - \|a - a_1\| > r$. Thus f_c is lower semicontinuous. The second half of (a) is proved in a similar way.
(b) This follows from (a) and the fact that the norm on A is η continuous.
(c) and (d) are restatements of (a) and (b) in bitopological notation. □

Theorem 4.9. (Structure Theorem) *With the topologies $\mathcal{L}$, $\mathcal{U}$ on E defined above, then $(E, \mathcal{U}, \mathcal{L})$ becomes a completely regular bitopological space so that ϕ is pairwise continuous. Furthermore, $(E, \mathcal{L}, \mathcal{U})$ is completely Hausdorff while $(E, Id(A), A, \phi)$*

is a field of C^*-algebras with respect to the $(\omega \vee \sigma) \times \eta$ topology on $X \times A$ and the $\mathcal{L} \vee \mathcal{U}$ topology on E.

Proof. First we show that $(E, \mathcal{U}, \mathcal{L})$ is completely regular. To prove this, let $I_1 \in Id(A), a_1 \in A$ and $G \subset E$ be chosen so that $(I_1, a_1 + I_1) \in G \in \mathcal{U}$. Then by the definition of $\mathcal{U}$, there exist $V \in \sigma$ and an $\varepsilon > 0$ so that $I_1 \in V$ and $V \times \{b : \|b - a_1\| < \varepsilon\} \subset \phi^{\leftarrow}(G)$. By the definition of the Scott topology, we know that $Q = Id(A) \setminus V$ is ω-quasicompact and saturated; whence $Q = \downarrow Q$. Thus for each ideal q in Q, the ideal I_1 is not a subset of q. So for each $q \in Q$, there exists a positive $b_q \in I_1$ so that $\|b_q + q\| > 1$. Define $U_q = \{J : \|b_q + J\| > 1\}$; then $q \in U_q \in \omega$. Since Q is quasicompact there is a finite $F \subset Q$, so that $\{U_q : q \in F\}$ covers Q. Set $b = \Sigma\{b_q : q \in F\}$ and define h on $Id(A)$ by $h(I) = min\{1, \|b+I\|\}$. Then h is pairwise continous (see 7.2 in [HKMS]) from $(Id(A), \sigma, \omega)$ to $([0,1], \omega, \sigma)$ while $h(I_1) = 0$ and $h(J) = 1$ for all $J \in Q$. Next define $\overline{h}$ on E by $\overline{h}(I, a + I) = h(I)$. Since $\overline{h} \circ \phi = h \times id_A$, the map $\overline{h}$ is pairwise continuous as a map from $(E, \mathcal{U}, \mathcal{L})$ to $([0,1], \omega, \sigma)$. Finally set $g = (\overline{h} \vee (1/\varepsilon) f_{a_1}) \wedge \mathbf{1}$. Then g is pairwise continuous and $g(I_1, a_1 + I_1) = 0$ while g is identically 1 on the complement of G. This completes the proof that $(E, \mathcal{U}, \mathcal{L})$ is a completely regular bitopological space.

To show that $(E, \mathcal{L}, \mathcal{U})$ is completely Hausdorff, let $p, q \in E$ be so that $p \notin \mathcal{L}\text{-}cl\{q\}$. It is clear from properties of the topologies ω and σ that $p \notin \mathcal{L}\text{-}cl\{q\}$ if and only if $q \notin \mathcal{U}\text{-}cl\{p\}$. By the previous paragraph, there is a pairwise continuous g from $(E, \mathcal{U}, \mathcal{L})$ to $([0,1], \omega, \sigma)$ so that $g(q) = 0$ and $g(p) = 1$. Then $f = 1 - g$ is a pairwise continuous function from $(E, \mathcal{L}, \mathcal{U})$ to $([0,1], \omega, \sigma)$ that separates p and q.

It is immediate from 4.4(b) that $(E, Id(A), A, \phi)$ is a field of C^*-algebras. □

5. Continuous Sections on E

In this part of the paper we will consider pairwise continuous maps $s : (Id(A), \omega, \sigma) \to (E, \mathcal{L}, \mathcal{U})$ which are sections for $(E, Id(A), p)$, i.e. maps s so that $p \circ s$ is the identity map on $Id(A)$. Here $p(I, a+I) = I$. These pairwise continuous sections are the maps that we use to represent C^*-algebras with unity. (See Theorem 5.5)

Notation. For a subspace X of $Id(A)$, the set of all pairwise continuous sections from (X, ω, σ) to $(E, \mathcal{L}, \mathcal{U})$ will be denoted by $\Sigma(X, E)$. Here ω and σ denote the lower and Scott topologies as well as their relative topologies on the subset X. The symbol for the set $\Sigma(Id(A), E)$ will often be shortened to $\Sigma(E)$ or simply Σ. For $a \in A$, the section $\breve{a}$ is defined by $\breve{a}(I) = \phi(I, a)$. Clearly $\breve{a}$ is pairwise continuous and $\breve{A} = \{\breve{a} : a \in A\}$ is a subset of $\Sigma(X, E)$ for any X containing $Prim(A)$.

Definition 5.1. For $s, t \in \Sigma(X, E)$ and $\lambda \in$ C, define addition, multiplication, scalar multiplication, involution and norm by

(i) $(s + t)(I) = s(I) + t(I)$;

(ii) $s \cdot t(I) = s(I) \cdot t(I)$;

(iii) $(\lambda s)(I) = \lambda \cdot s(I)$;

(iv) $s^*(I) = (s(I))^*$;
(v) $\|s\|_X = \sup\{\|s(I)\| \mid I \in X\}$.

Recall that these operations were defined for E in 4.2 and following.

Theorem 5.2. *With the operations defined above $\Sigma(E)$ becomes a C^*-algebra.*

Proof. This fact is implicit in the proof of 5.5 below.

Lemma 5.3. *Let s be a section for $(E, Id(A), A, \phi)$.*
(a) If s is $(\omega, \mathcal{L})$ continuous, then $I \to \|s(I)\|$ is lower semi-continuous.
(b) If s is $(\sigma, \mathcal{U})$ continuous, then $I \to \|s(I)\|$ is upper semi-continuous.

Proof. (a) Let s be $(\omega, \mathcal{L})$ continuous and r be a real number ≥ 0. Set $F = \{I : \|s(I)\| \leq r\}$. We must show that F is ω closed for each r. Note that $F = s^{\leftarrow}(\phi(Id(A) \times \{a : \|a\| \leq r\}))$. By 4.5(a) and the $(\omega, \mathcal{L})$ continuity of s, the set F is ω closed. This completes the proof of (a).
(b) Since ϕ is an $(\sigma \times \eta, \mathcal{U})$-open map, this part follows in a manner similar to (a). □

Convention. For the remaining part of this section, we will assume that the C^*-algebra A has a multiplicative identity, denoted by $\mathbf{1}$. Also, the center of A will be denoted by $Z(A)$ or simply Z.

Proposition 5.4. *If $z \in Z$, then $\check{z}(I) = \phi(I, \check{z}(I)\mathbf{1})$ for $I \in Glimmal'(A)$.*

Proof. This assertion follows immediately from 3.5 and the definition of $\check{a}$.

Theorem 5.5. (Representation Theorem) *Let A be a C^*-algebra with identity. Set $X = Glimmal'(A)$; then X is an $\omega \vee \sigma$ closed subspace of $Id(A)$ that contains $Prim(A)$. Also*
(a) The normed algebra of all pairwise continuous sections $s : X \to E$, is isometrically isomorphic to A.
(b) If A is a commutative C^*-algebra, then $X = Max(A)$ and the representation in part (a) reduces to the standard Gelfand transform.

(c) If $Prim(A)$ is Hausdorff for the hull-kernel topology, then $X = Prim(A)$ and the representation in (a) is equivalent to the Fell representation of A as a continuous field of C^*-algebras.

Proof. Since A has a unit, it follows from the Dauns-Hofmann Theorem that $(Glimm(A), \sigma)$ is compact Hausdorff. (See page 120 in [DH] or section 7 of [HKMS].) We interrupt the proof of 5.5 to prove a pair of lemmas.

Lemma 5.6. *If s is a $(\sigma, \mathcal{U})$ continuous section and $\varepsilon > 0$, then there exists an $a \in A$ so that $\|s(I) - \phi(I,a)\| < \varepsilon$ for all Glimm ideals I.*

Proof. For $J \in Id(A)$, there exists $a_J \in A$ so that $s(J) = \phi(J, a_J)$. By 5.3(b) the set $V_J = \{I : \|s(I) - \phi(I, a_J)\| < \varepsilon\}$ is σ-open. Thus the collection $\mathcal{V} = \{V_J\}$ is a σ open cover of $Id(A)$. Since $Glimm(A)$ is compact there is a finite partition of unity that is subordinate to $\mathcal{V}$ on $Glimm(A)$. By the Dauns-Hofmann Theorem, page 120 in [DH], the center Z of A is isometrically isomorphic with the algebra of all complex valued σ continous functions on $Glimm(A)$. Thus the aforementioned partition of unity can be selected from Z. Specifically, there is a family $\{z_J \mid J \in F\}$ in Z^+, indexed by a finite subset F of $Glimm(A)$, whose sum is $\mathbf{1}$ and is so that the cozero set associated with z_J is a subset of $V_J \cap Glimm(A)$. We claim that $a = \sum z_J a_J$, (the sum is over $J \in F$) is the element of A that satisfies the conclusion of this lemma. For if $I \in Glimm(A)$ then $s(I) - \phi(I,a) = s(I) - \sum \tilde{z}_J(I)\phi(I,a_J) = \sum \tilde{z}_J(s(I) - \phi(I,a_J))$. Therefore, by 5.4, $\|s(I) - \phi(I,a)\| \leq \sum \tilde{z}_J(I)\|s(I) - \phi(I,a_J)\| < \varepsilon$. □

Lemma 5.7. *If s is a $(\sigma, \mathcal{U})$ continuous section, then there exists an $a \in A$ so that $s(I) = \phi(I,a)$ for all Glimm ideals I.*

Proof. By induction and the Lemma above, there exists a sequence $\{a_k\}$ in A so that for each positive integer n and each Glimm ideal I, $\|s(I) - \sum_{k=1}^{n} \breve{a}_k(I)\| < 2^{-n}$. By taking the difference between the partial sums for n and $n-1$, it follows that $\|\breve{a}_n(I)\| < 2^{-n} + 2^{-(n-1)} = 3 \cdot 2^{-n}$. Now $\sup\{\|\breve{a}_n(I)\| \mid I \in \mathit{Glimm}(A)\} = \sup\{\|\breve{a}_n(P)\| \mid P \in \mathit{Prim}(A)\} = \|a_n\|$. So $\|a_n\| < 3 \cdot 2^{-n}$; whence the series $\sum_{n=1}^{\infty} a_n$ converges to an element $a \in A$. It now follows that $s(I) = \breve{a}(I) = \phi(I, a)$ for all Glimm ideals I. □

We now return to the proof of 5.5.

Assume that s is a pairwise continuous section. By Lemma 5.7 above, there exists $a \in A$ so that $s(I) = \breve{a}(I)$ for all Glimm ideals I. Since $\{I \mid s(I) = \breve{a}(I)\}$ equals $s^{\leftarrow}(\phi(Id(A) \times \{b : \|b - a\| \leq 0\}))$, it follows from Lemma 4.5(a) and the $(\omega, \mathcal{L})$ continuity of s that s and $\breve{a}$ agree on the ω-closure of $\mathit{Glimm}(A)$. But the latter is, by definition, equal to $\mathit{Glimmal}(A)$. Thus $\Sigma(X, E) = \breve{A} = \{\breve{a} : a \in A\}$; clearly $a \to \breve{a}$ is an isomorphic isometric embedding of A into $\Sigma(X, E)$. This completes the proof of 5.5(a).

When A is a commutative C^*-algebra with identity, then A reduces to Z and E is the product $X \times \mathbf{C}$ where $\mathbf{C}$ denotes the complex numbers. (b) is now clear.

When $\mathit{Prim}(A)$ is Hausdroff, it is clear that $X = \mathit{Prim}(A)$. Also, the topologies ω and σ are each equal to the hull-kernel topology. So the complication of dealing with bitopologies is removed and we are back in well-established territory. □

6. Algebras Lacking a Unit

Recall (3.2 above) that for any C^*-algebra A (with or without an identity), the *centroid of* A, denoted by $R(A)$ or simply R, is the set of all self maps $r : A \to A$ for which $(r(a))b = a(r(b))$ for all a and b in A. For $z \in Z = Z(A)$, the center of A, define r_z by $r_z(a) = za$; then $\{r_z : z \in Z\}$, as a subset of R, can be identified with Z. When the operations and the norm are defined in the canonical way, R becomes a commutative C^*-algebra with $\mathbf{1}$ which contains the center Z of A and equals the center when A has a unit. Also, R is the center of the multiplier algebra of A.

In $R \times A$, define the product by $(r, a) \cdot (s, b) = (rs,\ r(b) + s(a) + ab)$. All other operations are defined co-ordinate wise and the operator semi-norm is given by

$$\|(r, a)\|_A = \sup\{\|r(x) + ax\| : x \in A,\ \|x\| \leq 1\}.$$

When assigned the norm $\|(r, a)\| = \max\{\|r\|, \|(r, a)\|_A\}$, the algebra $R \times A$ becomes a C^*-algebra. The null space of the operator semi-norm is the ideal $\{(r_z, -z) : z \in Z\}$ which we denote by $\Delta(-Z)$. Let B be the quotient space $(R \times A)/\Delta(-Z)$. Then the usual quotient space operations make B a C^*-algebra which contains isometrically, isomorphic copies of both A and R. Furthermore, the center $Z(B)$ of B is that copy of R. Finally, by the Dauns-Hofmann Theorem, both $Z(B)$ and R are isometrically isomorphic with $C^b(Prim(A))$, the algebra of all bounded complex-valued continuous functions on $Prim(A)$ with the hull-kernel topology. Note that if $M \in Max(R)$ is so that $MA = A$, then $M + A$ is a primitive ideal of B. Also, if $Q \in Prim(B)$ is so that $Q \supset A$, then $Q \cap R = M \in Max(R)$ and $Q = M + A$. Thus $Prim(B)$ (See [D, 2.11.5, p. 61]) can be identified with $Prim(A) \cup \{M : M \in Max(R), M$ is free$\}$ and $A = \bigcap\{M + A : M \in Max(R), M$ is free$\}$. (See the Remark below 3.2 for the definition of a free ideal.) Likewise $Glimm(B)$ can be identified with $Glimm(A) \cup \{M : M \in Max(R), M$ is free$\}$ and $Glimmal(B)$ with $Glimmal(A) \cup \{M : M \in Max(R), M$ is free$\}$.

Definition 6.1. When A is a C^*-algebra that lacks a unit, let B be the algebra with unit constructed above and let $(E, Id(B), B, \phi)$ be the corresponding field of C^*-algebras. For $X = Glimmal'(B)$, denote by $\Sigma_0(X, E)$ the set of all pairwise continuous sections from (X, σ, ω) to $(E, \mathcal{L}, \mathcal{U})$ which vanish outside $Glimmal(A)$, i.e. vanish on the set $\{M + A : M \in Max(R), M \text{ is free}\}$. A pairwise continuous section s is said to *vanish at* ∞ if it vanishes at $M + A$ for each free ideal $M \in Max(R)$.

Remark. In the remaining part of this paper, A will be dealt with as a subalgebra of B. The base space will be $Id(B)$ with its lower ω and Scott σ topologies. This will enable us to use 5.5 to establish our main theorem.

Definition 6.2. Within $(E, Id(A), B, \phi)$, define E_0 as $E_0 = \phi[Id(B) \times A]$.

Clearly, all of the sections in $\Sigma(X, E_0)$ vanish at ∞; i.e., $\Sigma(X, E_0) \subset \Sigma_0(X, E)$.

Lemma 6.3. *(a) If $r \in R$ is so that $\check{r} \in \Sigma(Id(B), E_0)$, then $r = z$ for some $z \in Z(A)$.*
(b) If for $X = Glimmal'(B)$, $r \in R$ is so that $\check{r} \in \Sigma_0(X, E)$, then $r = z$ for some $z \in Z(A)$.

Proof. (a) Let $s = \check{r}$. Since $s[Id(B)] \subset E_0$, there exists $a_J \in A$ so that $s(J) = \phi(J, a_J)$ for each $J \in Id(B)$. We may now proceed as in Lemmas 5.6 and 5.7, except that here, the z_J are in $Z(B) = R$. Thus there exists $z \in A$ so that $\check{r}(I) = \check{z}(I)$ for all ideals I in $Glimm(B)$. As in the proof of 5.5, the ω-continuity of these sections, it follows that they agree on $Glimmal'(B)$. In particular $\check{r}(P) = \check{z}(P)$ for all primitive ideals $P \in Prim(B)$. Whence $r = z \in R \cap A = Z(A)$.
(b) The fact that $\check{r}$ vanishes at ∞ implies that $r \in \bigcap\{M : M \in Max(R), M \text{ is free}\}$. Thus $r \in \bigcap\{M + A : M \in Max(R), M \text{ is free}\} = A$. Whence $r \in Z(A)$. □

Theorem 6.4. (Main Theorem) *For a C^*-algebra A and its*

enlarged unitary algebra B, let $X = Glimmal'(B)$; then X is an $\omega \vee \sigma$ closed subspace of $Id(B)$ that contains $Prim(A)$. Also
(a) The C^*-algebra $\Sigma_0(X, E)$ of all pairwise continuous sections $s : X \to E$ that vanish at ∞, is isometrically isomorphic to A.
(b) If A is a commutative C^*-algebra, then $X = Max(R)$ and the representation in part (a) reduces to the standard Gelfand transform.
(c) If $Prim(A)$ is Hausdorff for the hull-kernel topology, then $X = Prim(B)$ and the representation in (a) is equivalent to the Fell representation of A as a continuous field of C^*-algebras.
(d) (Baker [Ba]) If for $Z = Z(A)$, it is true that $ZA = A$, then the C^*-algebra $\Sigma_0(X, E)$ of all pairwise continuous sections $s : X \to E$, which vanish at infinity, is isometrically isomorphic to A. (In this statement, $X = Glimmal'(B)$ can be replaced by the one-point compactification of $Max(Z)$.)

Proof. (a) Since $a \to \check{a}$ is an isometric homomorphic imbedding of A into $\Sigma_0(X, E)$, it suffices to show that the image of A exhausts $\Sigma_0(X, E)$. Let $s \in \Sigma_0(X, E)$. Then by 5.5, applied to B, there exist $r \in R$ and $a \in A$ so that $s = (r + a)\check{}$. Define the section t by $t = s - \check{a}$. Then $t \in \Sigma_0(X, E)$ and $t = \check{r}$. By 6.3(b), there exists $z \in Z(A)$ so that $t = \check{z}$. Thus $s = \check{b}$ for $b = a + z \in A$. This completes the proof of (a).

Under the hypotheses of (b) or (c), it is easy to show that $Prim(A)$ is a dense, locally compact subspace of the compact Hausdorff space $Prim(B)$. Moreover, the two topologies ω and σ agree on $Prim(B)$. In (d), $Glimm(A)$ is a dense, locally compact subspace of $Glimm(B)$. Once these facts are known, (b), (c) and (d) become corollaries of part (a). □

Example. Let A be the C^*-algebra of all compact operators on a separable, infinite dimensional Hilbert space. Then $Prim(A)$ consists of a single point; so $R = \mathbf{C} \cdot \mathbf{1}$ where $\mathbf{C}$ is the algebra of complex numbers. In this case, $B = R \times A$, while $Id(B)$ consists of three ideals: $\{0\}$, A, B and $E = (\{0\} \times B) \cup (\{A\} \times R) \cup (\{B\} \times \{0\})$. In this example, the ideal A is the point at ∞. Thus $\Sigma(X, E_0) = \Sigma_0(X, E)$. Note that X is, in this case, the set of proper ideals in $Id(A)$.

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