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EMBEDDING OF SIMPLICIAL ARCS INTO THE PLANE

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Abstract

Suppose $\varphi : G_1 \to G_0$ is a simplicial map between graphs and h_0 is an embedding of G_0 into the plane. In this paper we study under what conditions G_1 can be embedded into the plane with an embedding sufficiently close to $h_0 \circ \varphi$. We answer this question in the case when G_1 is an arc.

1. Introduction

By a *graph* we understand a one-dimensional finite simplicial complex. If G is a graph then $\mathcal{V}(G)$ will denote the set of vertices of G and $\mathcal{E}(G)$ will denote the set of its edges. By an edge we understand the closed segment between two vertices. Two vertices belonging to an edge are called *adjacent*. A *simplicial map* of a graph G_1 into a graph G_0 is a function from $\mathcal{V}(G_1)$ into $\mathcal{V}(G_0)$ taking every two adjacent vertices either onto a pair of adjacent vertices or onto a single vertex. In this paper we will not distinguish between a graph and its geometric realization. We will assume that every graph is a space which is the union of finitely many arcs (edges) that may intersect only at common endpoints (vertices). We will assume that each edge is parametrized by some homeomorphism of the interval $[0, 1]$. It is important to note, however, that a graph, either abstract

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or geometric, has a fixed collection of vertices and any change in this collection changes the graph. Each simplicial map between graphs can be uniquely extended linearly (according to the parametrization) from vertices edges to form a continuous map. We will not distinguish between the simplicial map and its continuous representation.

A graph with a geometric realization homeomorphic to an arc is simply called an *arc*. Observe that two arcs are isomorphic in the simplicial category if and only if they have the same number of vertices. A connected graph without a simple closed curve is called a *tree*. A tree consisting of three edges having a common vertex is called a *simple triod*. A graph with three vertices and three edges is called a *simple triangle*. If u and v are two adjacent vertices of a graph, by $[u, v]$ we will denote the edge between u and v. Additionally, if a and b are two points (not necessarily vertices) of a tree, by $[a, b]$ we will denote the arc between a and b.

Generally it is difficult to establish whether an atriodic one-dimensional continuum can be embedded into the plane. For instance, it is still unknown whether an atriodic version of the fixed-point-free tree-like continuum constructed by D. P. Bellamy in [1] can be embedded in the plane (see comments to Problem 1 in [3]. In many cases, where one can embed a given continuum in the plane, the following technique is used. Suppose $G_0 \xleftarrow{f_1} G_1 \xleftarrow{f_2} G_2 \xleftarrow{f_3} \ldots$ is an inverse system of graphs with continuous (not necessarily simplicial) bonding maps. It is very well known that if we place each G_i into the plane with an embedding h_i so that h_{i+1} is sufficiently close to $h_i \circ f_i$, then the limit of $\{h_i(G_i)\}$ in the plane is homeomorphic to the inverse limit of the system. For example, this technique is used to embed in the plane continua constructed by W. T. Ingram ([5], [6]) and by J. F. Davis and W. T. Ingram ([4]. An earlier application of the technique in its version for patterns was given by R. H. Bing who proved in [2, theorem 4] that each snake-like continuum can be embedded into the plane. Snake-like continua (also known as chainable continua) are inverse limits of arcs. Bing observed, in the language of patterns, that if $f : I_1 \to I_0$ is a map between two arcs, h_0 is an embedding of I_0 into the plane and ϵ is a positive number, then there is an embedding h_1 of I_1 into the plane such that h_1 is ϵ-close to $h_0 \circ f$. Observe that the restriction of the last theorem to simplicial arcs I_0 and I_1 and a simplicial map f yields an equivalent statement. It is essential, however, that both I_0 and I_1 are arcs. Motivated by the embedding technique and the Bing theorem, we will study the combinatorics of the following problem.

Question 1.1. *Suppose a graph G_0 is embedded in the plane and φ is a simplicial map of a graph G_1 into G_0. Under what conditions can φ be be approximated by an embedding of G_1 into the plane?*

We will answer this question fully in the case when G_1 is an arc and give a necessary condition for an arbitrary G_1. Before we explain the results of the paper let us first consider a few examples.

Example 1.2. Let G_0 be a graph with vertices $v_0, \dots, v_4$ embedded into the plane as shown in the left side of Figure 1. Suppose that G_1 contains two exclusive arcs $a_0 - a_1 - a_2$ and $b_0 - b_1 - b_2$ with $\varphi(a_0) = v_1$, $\varphi(a_1) = v_0$, $\varphi(a_2) = v_3$, $\varphi(b_0) = v_2$, $\varphi(b_1) = v_0$ and $\varphi(b_2) = v_4$. If G_1 is mapped into the plane in such a way that each edge e is mapped close to the embedding $\varphi(e)$, then the images of the arcs $a_0 - a_1 - a_2$ and $b_0 - b_1 - b_2$ must intersect (see the right side of Figure 1).

Example 1.3. Let G_0 be a simple triod embedded into the plane (see the left side of Figure 2). Suppose G_1 contains two exclusive simple triods G_1' and G_1''. Let $\varphi : G_1 \to G_0$ be a simplicial map sending each of G_1' and G_1'' onto G_0. If G_1

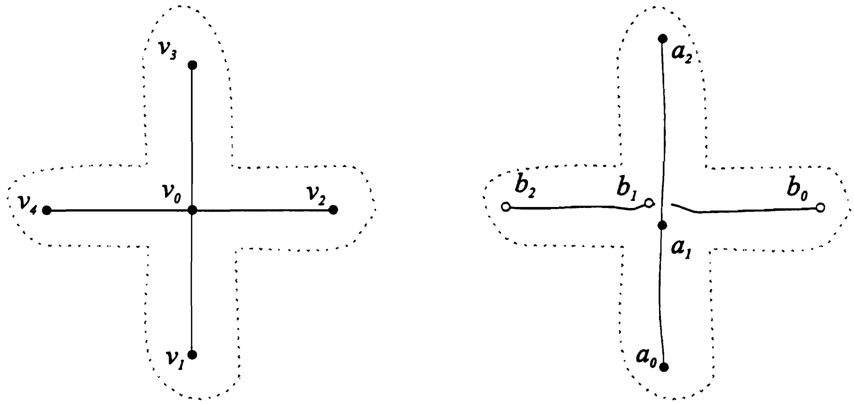


Fig. 1.

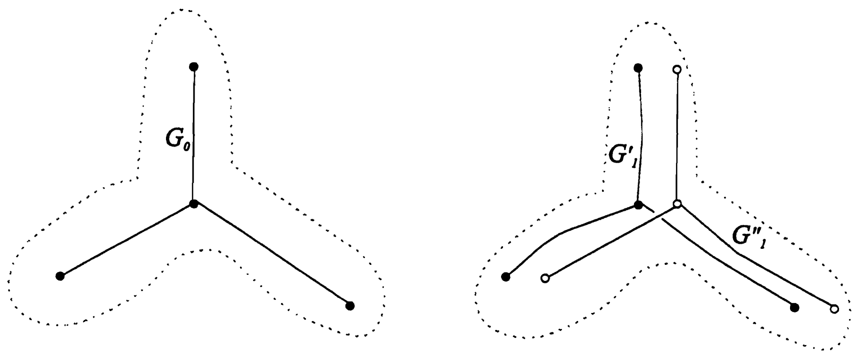


Fig. 2.

is mapped into the plane in such a way that each edge e is mapped close to the embedding $\varphi(e)$, then the images of the triods G_1' and G_1'' must intersect (see the right side of Figure 2).

Examples 1.2 and 1.3 illustrate two basic obstacles not allowing the simplicial map φ from Question 1.1 to be approximated by an embedding. In the first example φ forces two arcs to cross each other. In the second example we cannot embed to disjoint copies of the same triod close to each other.

In this paper we consider the operation d introduced by the author in [8]. If φ is a simplicial map of a graph G_1 to a graph G_0, then $d[\varphi]$ is a simplicial map mapping some graph $D(\varphi, G_1)$ to some graph $D(G_0)$. The n-th iteration of the operation is denoted by $d^n[\varphi]$. It maps a graph $D^n(\varphi, G_1)$ into a graph $D^n(G_0)$. (We will recall the definitions in the next section.) It was proven in [8] that if G_1 is an arc with k vertices, then $D(\varphi, G_1)$ is an arc (possibly degenerate) with less than k vertices. We will notice that any embedding of G_0 into an oriented 2-manifold, induces an embedding of $D(G_0)$ into another oriented 2-manifold. We will prove (Theorem 4.3) that if φ can be approximated by an embedding, then $d[\varphi]$ can also be approximated by an embedding. In the case where G_1 is an arc and φ has no self crossing as in Example 1.2, we will prove (Theorem 4.10) that if $d[\varphi]$ can be approximated by an embedding, then φ can also be approximated by an embedding. It follows that a simplicial map φ of an arc G_1 with k vertices into a plane graph G_0 can be approximated by an embedding if and only if $d^n[\varphi]$ has no self crossing for each $n = 0, 1, \ldots, k$ (Theorem 4.11). The last statement yields a computer algorithm to check whether a simplicial map from an arc into a plane graph can be approximated by an embedding (Remark 4.13). Theorem 4.3 also lets to generalize Example 1.3 to the following result (see Theorem 4.5). Suppose G is a graph embedded in the plane. Suppose φ is a simplicial map of a tree T into G such that φ cannot be factored through an arc. Then there are no two exclusive embedding of T approximating φ.

2. Embedding Graphs in Oriented 2-Manifolds

In this section we observe that every graph can be embedded into an oriented 2-manifold with an embedding that is determined globally by its local behavior at every vertex. We start with the following definitions.

Definition 2.1. If $v \in \mathcal{V}(G)$ then $E(v)$ will denote the collection of all edges of G that have v as a vertex. Let $\mu(v)$ denote the number of elements of $E(v)$. For each $v \in \mathcal{V}(G)$ arrange elements of $E(v)$ into a sequence $A(v) = e_1(v), e_2(v), \ldots, e_{\mu(v)}(v)$. The collection $\mathcal{A} = \{A(v)\}_{v \in \mathcal{V}(G)}$ will be called a *local ordering of edges* of G.

Definition 2.2. Suppose G is a graph embedded in the interior of an oriented 2-manifold M (possibly with boundary). A closed disk $B \subset M$ will be called a *regular ball* around a vertex $v \in \mathcal{V}(G)$ if v is contained in the interior of B, v is the only vertex of G contained in B and each edge in $E(v)$ intersects the boundary of B at exactly one point. If B is a regular ball around a vertex v and $e \in E(v)$, then we can arrange elements of $E(v)$ into a sequence starting with e and going counterclockwise along the boundary of B. Since the order clearly does not depend on the choice of B, we may define the counterclockwise order on $E(v)$ starting with e. Let $\mathcal{A} = \{A(v)\}_{v \in \mathcal{V}(G)}$ be a local ordering of edges of G. We will say that $\mathcal{A}$ *agrees with the embedding of* G *into* M, if the counterclockwise order on $E(v)$ starting with $e_1(v)$ coincides with $A(v)$ for each $v \in \mathcal{V}(G)$.

The proof of the following proposition is easy and will be omitted.

Proposition 2.3. *Suppose G is a graph with a local ordering $\mathcal{A}$. Then there is an oriented 2-manifold M and an embedding of G into the interior of M that agrees with $\mathcal{A}$.*

Definition 2.4. Suppose G is a graph embedded in the interior an oriented 2-manifold M. We will say that a compact manifold $N \subset M$ containing G in its interior is a *normal neighborhood of* G if there are two collections of closed discs $\{B_v\}_{v \in \mathcal{V}(G)}$ and $\{C_e\}_{e \in \mathcal{E}(G)}$ such that (see Figure 3)

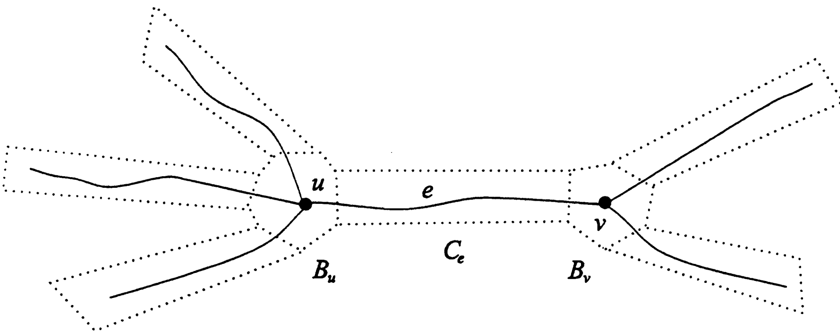


Fig. 3.

(1) $N = \bigcup_{v\in\mathcal{V}(G)} B_v \cup \bigcup_{e\in\mathcal{E}(G)} C_e$,

(2) for each $v \in \mathcal{V}(G)$, B_v is a regular ball around v,

(3) elements of $\{B_v\}_{v\in\mathcal{V}(G)}$ are mutually disjoint,

(4) elements of $\{C_e\}_{e\in\mathcal{E}(G)}$ are mutually disjoint,

(5) $B_v \cap C_e \neq \emptyset$ if and only if v is a vertex of e, and

(6) if v is a vertex of e then $B_v \cap C_e$ is an arc containing the point $e \cap \mathrm{Bd}(B_v)$ in its interior.

It follows from the above conditions that if e is an edge with vertices a and b, then $e \setminus (B_a \cup B_b)$ is an open arc contained in the interior of the disk C_e. The collections $\{B_v\}_{v\in\mathcal{V}(G)}$ and $\{C_e\}_{e\in\mathcal{E}(G)}$ will be called a *normal structure on* N.

The proof of the next two propositions is left to the reader.

Proposition 2.5. *Suppose G is a graph embedded in the interior an oriented 2-manifold M. Then there is a normal neighborhood N of G in M.*

Proposition 2.6. *Let $\mathcal{A}$ be a local ordering of edges of a graph G. Let, for $i = 1, 2$, h_i be an embedding of G into an oriented*

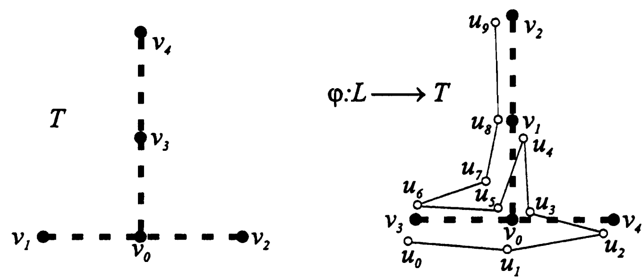


Fig. 4.

2-manifold M_i that agrees with $\mathcal{A}$ and let N_i be a normal neighborhood of $h_i(G)$ in M_i. Then N_1 and N_2 are homeomorphic.

Definition 2.7. Suppose G is a graph contained in a normal neighborhood N with a normal structure $\mathcal{S}$ given by collections $\{B_v\}_{v\in\mathcal{V}(G)}$ and $\{C_e\}_{e\in\mathcal{E}(G)}$. Let φ be a simplicial map of a graph G' into G and let f be a map of G' into N. We will say that f is an $\mathcal{S}$-approximation of φ provided that for each $e' \in \mathcal{E}(G')$ with vertices a' and b'

(1) if $\varphi(a') = \varphi(b') = v$, then $f(e') \subset B_v$ and

(2) if $\varphi(a') = a$ and $\varphi(b') = b$ where a and b are two distinct vertices of an edge $e \in \mathcal{E}(G)$, then there are points $a'', b'' \in e'$ such that $f([a', a'']) \subset B_a$, $f([b', b'']) \subset B_b$ and $f((a'', b'')) \subset C_e \setminus (B_a \cup B_b)$.

Example 2.8. Let T be the graph obtained from a simple triod by attaching an additional edge to one of its endpoints. Let $v_0, \ldots, v_4$ denote the vertices of T as it shown on the left side of Figure 4. Suppose L is a simplicial arc with vertices $u_0, \ldots, u_9$. Let $\varphi : L \to T$ be the simplicial map given by $\varphi(u_0) = \varphi(u_6) = v_1$, $\varphi(u_1) = \varphi(u_3) = \varphi(u_5) = \varphi(u_7) = v_0$, $\varphi(u_2) = v_2$, $\varphi(u_4) = \varphi(u_8) = v_3$ and $\varphi(u_9) = v_4$. The map

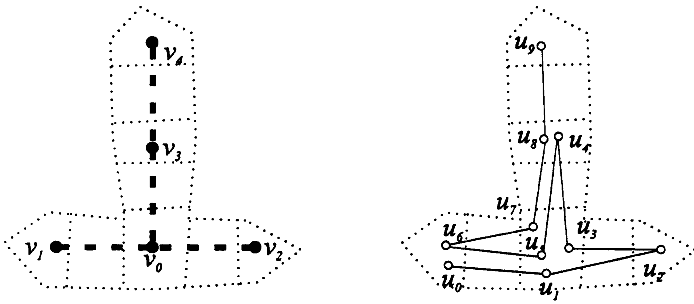


Fig. 5.

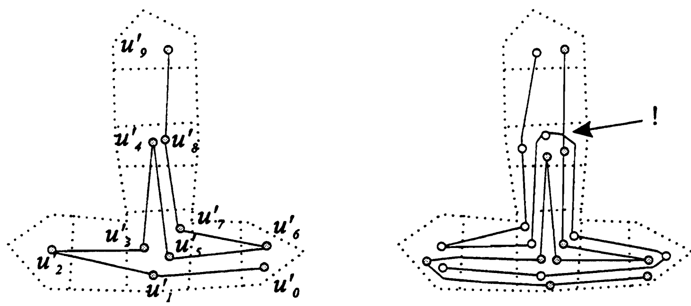


Fig. 6.

φ is illustrated on the right side of Figure 4. Each vertex $u \in \mathcal{V}(L)$ is close to its image $\varphi(u) \in \mathcal{V}(T)$. The left side of Figure 5 shows a normal structure $\mathcal{S}$ on some neighborhood of T in the plane. The right side of the same figure shows an embedding of L into the plane which is an $\mathcal{S}$-approximation of φ.

Let $\sigma : T \to T$ be the symmetry on T that swaps v_1 with v_2 and keeps the remaining vertices of T fixed. Let L' be a simplicial arc with vertices $u_0', \ldots, u_9'$. Define $\varphi' : L' \to T$ by setting $\varphi'(u_i') = \sigma(\varphi(u_i))$ for $i = 0, \ldots, 9$. The left side of Figure 6 illustrates an embedding of L' into the plane which is an $\mathcal{S}$-approximation of φ'. The right side of Figure 6 indicates that the images of $\mathcal{S}$-approximations of φ and φ' must intersect.

The next proposition will allow us to use $\mathcal{S}$-approximations instead of ϵ-approximations. (A map $f_\epsilon : X \to Y$ is an *ϵ-approximation* of a map $f : X \to Y$ if the distance between f and f_ϵ is less than ϵ.)

Proposition 2.9. *Suppose G is a graph contained in a normal neighborhood N with a normal structure $\mathcal{S}$. Let φ be a simplicial map of a graph G' into G. Then the following conditions are equivalent:*

(a) There is an embedding $h : G' \to N$ which is an $\mathcal{S}$-approximation of φ.

(b) For each positive number ϵ there is an embedding $h_\epsilon : G' \to N$ which is an ϵ-approximation of φ.

Proof of 2.9. Let $\mathcal{S}$ be given by the collections $\{B_v\}_{v\in\mathcal{V}(G)}$ and $\{C_e\}_{e\in\mathcal{E}(G)}$.

We will outline the idea of the proof before we proceed with its formal implementation. If (a) is true, we will first shrink all B_v's to make their diameters less than ϵ. We will then define h_ϵ satisfying the condition (b) by using the image of h intersected with $\bigcup_{v\in\mathcal{V}(G)} B_v$ and then, for each edge $w \in \mathcal{E}(G')$ with vertices a and b such $\varphi(w) \in \mathcal{E}(G)$, connecting $h(w) \cap B_{\varphi(a)}$ with $h(w) \cap B_{\varphi(b)}$ with an arc almost parallel to $\varphi(w) \cap C_{\varphi(w)}$. If (b) is true, we will choose sufficiently small ϵ, and then replace each disk B_v by a larger disk B'_v in such a way that the intersection $B'_v \cap h_\epsilon(w)$ is connected for each $w \in \mathcal{E}(G')$. The disks B'_v will be mutually exclusive. Then we will define a homeomorphism g of N onto itself such that $g(B'_v) = B_v$ for $v \in \mathcal{V}(G)$. Finally, we will define a map h satisfying (a) as $g \circ h_\epsilon$. Since implementing this very simple idea is somewhat cumbersome, the detailed proof is supplied below.

Suppose (a) is true. Let ϵ be an arbitrary positive number. There is a homeomorphism g_ϵ of N into itself such that $g_\epsilon(G) = G$ and for each $v \in \mathcal{V}(G)$ we have $g_\epsilon(v) = v$ and $\operatorname{diam}(g_\epsilon(B_v)) < \epsilon$. Clearly, $g_\epsilon(\mathcal{S})$, defined by the collections $\{g_\epsilon(B_v)\}_{v\in\mathcal{V}(G)}$ and $\{g_\epsilon(C_e)\}_{e\in\mathcal{E}(G)}$, is a normal structure on $g_\epsilon(N)$ satisfying (a) with N replaced by $g_\epsilon(N)$. Since $g_\epsilon(N) \subset N$, the version of the condition (b) with N replaced by $g_\epsilon(N)$ implies (b) stated in the proposition. Without loss of generality we may therefore replace $\mathcal{S}$ by $g_\epsilon(\mathcal{S})$ and assume that the original normal structure $\mathcal{S}$ has the property that $\operatorname{diam}(B_v) < \epsilon$ for each $v \in \mathcal{V}(G)$.

For each edge $e \in \mathcal{E}(G)$ let $p_e : [0,1] \to e$ be the parametrization of e. Let s_e be the first point in $[0,1]$ such that $p_e(s_e) \in C_e$ and let t_e be the last point in $[0,1]$ such that $p_e(t_e) \in C_e$. Let r_e be a homeomorphism of C_e onto $[s_e, t_e] \times [-1,1]$ such that $r_e\left(C_e \cap B_{p_e(0)}\right) = \{s_e\} \times [-1,1]$, $r_e\left(C_e \cap B_{p_e(1)}\right) = \{t_e\} \times [-1,1]$ and $r_e(p_e(x)) = (x, 0)$ for each $x \in [s_e, t_e]$. Since diameters of $B_{p_e(0)}$ and $B_{p_e(1)}$ are less than ϵ, there are numbers s'_e and t'_e such that $s_e < s'_e < t'_e < t_e$, $\operatorname{diam}(r_e^{-1}([s_e, s'_e] \times [-1,1])) < \epsilon$ and $\operatorname{diam}(r_e^{-1}([t'_e, t_e] \times [-1,1])) < \epsilon$. There is a positive number η_e such that $\operatorname{diam}(r_e^{-1}(\{x\} \times [-\eta_e, \eta_e])) < \epsilon$ for each $x \in [s_e, t_e]$. Let $q_e : [s_e, t_e] \to [\eta_e, 1]$ be a map such that $q_e(s_e) = 1$, $q_e(t_e) = 1$ and $q_e(x) = \eta_e$ for each $x \in [s'_e, t'_e]$. Finally, let τ_e denote the map of $[s_e, t_e] \times [-1,1]$ into itself given by $\tau_e(x, y) = (x, y q_e(x))$.

Let W_e denote the set of edges $w \in \mathcal{E}(G')$ such that $\varphi(w) = e$. For each $w \in W_e$ let $p_w : [0,1] \to w$ be the parametrization of w such that $\varphi(p_w(x)) = p_e(x)$ for each $x \in [0,1]$. Let a_w be the point $h(w) \cap B_{p_e(0)} \cap C_e$ and let b_w be the point $h(w) \cap B_{p_e(1)} \cap C_e$. By ℓ_w we denote the linear function from $[s_e, t_e]$ into $[s_e, t_e] \times [-1,1]$ such that $\ell_w(s_e) = r_e(a_w)$ and $\ell_w(t_e) = r_e(b_w)$. Observe that since h is an embedding, the images of ℓ_w's are mutually exclusive for $w \in W_e$.

For each $w \in W_e$, we will define a map $\sigma_w : [0,1] \to N$ in the following way:

(i) σ_w restricted to $[0, s_e]$ is an embedding of $[0, s_e]$ into $h(w)$ such that $\sigma_w(0) = h(p_w(0))$ and $\sigma_w(s_e) = a_w$,

(ii) σ_w restricted to $[s_e, t_e]$ is equal to $r_e{}^{-1} \circ \tau_e \circ \ell_w$ and

(iii) σ_w restricted to $[t_e, 1]$ is an embedding of $[t_e, 1]$ into $h(w)$ such that $\sigma_w(t_e) = b_w$ and $\sigma_w(1) = h(p_w(1))$.

Observe that σ_w is an embedding of $[0,1]$ into N such that the distance between $\sigma_w(x)$ and $p_e(x)$ is less than ϵ for each $x \in [0,1]$.

Finally, we are now able to define h_ϵ. For each edge $u \in \mathcal{E}(G')$ such that $\varphi(u)$ is a vertex of G, let h_ϵ restricted to u be the same as h restricted to u. For each edge $w \in \mathcal{E}(G')$ such that $\varphi(w)$ is an edge of G, let h_ϵ restricted to w be equal to $\sigma_w \circ p_w{}^{-1}$. It may be verified that so defined h_σ satisfies the condition (b).

Now, we will suppose (b) is true and prove (a).

Let ϵ_1 be a positive number less than the distance between v and the complement of B_v for each vertex $v \in \mathcal{V}(G)$. Let ϵ_2 be a positive number less than the distance between e and the complement of $B_v \cup C_e \cup B_u$ for each edge $e \in \mathcal{E}(G)$ with vertices v and u.

For each edge $e \in \mathcal{E}(G)$ let m_e be a point in e that belongs to the interior of C_e. Recall that $E(v)$ we denote the set of edges of G having v as a vertex for each vertex $v \in \mathcal{V}(G)$. If $e \in E(v)$, let $I[v,e]$ denote the subarc of e between v and m_e. Let u denote the other vertex of e and let $J[v,e]$ be the subarc of e between m_e and u. Notice that $I[v,e] = J[u,e]$ and $J[v,e] = I[u,e]$. Clearly, $J[v,e] \cap B_v = \emptyset$. Let δ be a positive number such that the distance between B_v and $J[v,e]$ is greater than δ for each vertex $v \in \mathcal{V}(G)$ and each edge $e \in E(v)$.

Choose $\epsilon > 0$ less than the minimum of ϵ_1, ϵ_2 and δ. Let $h_\epsilon : G' \to N$ be an embedding ϵ-approximating φ.

As before, for each edge $e \in \mathcal{E}(G)$, let W_e denote the set of edges $w \in \mathcal{E}(G')$ such that $\varphi(w) = e$.

Suppose e is an edge of G and $w \in W_e$. Let v and u denote the vertices of e. Let w_v and w_u denote the vertices of w in such an order that $\varphi(w_v) = v$ and $\varphi(w_u) = u$. Let w'_v be the last point on the edge w directed from w_v to w_u such that $h_\epsilon(w'_v) \in \mathrm{Bd}(B_v)$. Similarly, let w'_u be the last point on the edge w directed from w_u to w_v such that $h_\epsilon(w'_u) \in \mathrm{Bd}(B_u)$. Let $A_{w,v}$ be the subarc of w between w_v and w'_v. Similarly, let $A_{w,u}$ be the subarc of w between w_u and w'_u. Since $\epsilon < \delta$, we have that $\varphi(w'_v) \in I[v,e]$. Since $\varphi(w_v) = v \in I[v,e]$ and φ is linear on each edge of G', it follows that $\varphi(A_{w,v}) \subset I[v,e]$. By the choice of ϵ, we get the result that $h_\epsilon(A_{w,v}) \cap B_u = \emptyset$. By the symmetry between v and u, it follows that that $h_\epsilon(A_{w,u}) \cap B_v = \emptyset$.

There is a collection of disks $\{P_{w,z}\}_{w\in W_e, z=v,u}$ with the following properties:

> $h_\epsilon(A_{w,z})$ is contained in the interior of $P_{w,z}$ for each $w \in W_e$ and $z = v, u$,
>
> $P_{w,z}$ is contained in the interior of $C_e \cup B_z$ for each $w \in W_e$ and $z = v, u$,
>
> $P_{w',v} \cap P_{w'',u} = \emptyset$ for any $w', w'' \in W_e$,
>
> $P_{w',z} \cap P_{w'',z} \subset B_z$ for any $w' \neq w'' \in W_e$, and $z = u, v$,
>
> $\mathrm{Bd}(P_{w,z}) \cap h_\epsilon(w)$ consists of a single point for each $w \in W_e$ and $z = v, u$.

Let $Q_{e,v}$ be the union of $B_v \cup \bigcup_{w\in W_e} P_{w,v}$ and all components of $C_e \setminus \bigcup_{w\in W_e} P_{w,v}$ that do not contain $C_e \cap B_u$. Similarly. let $Q_{e,u}$ be the union of $B_u \cup \bigcup_{w\in W_e} P_{w,u}$ and all components of $C_e \setminus \bigcup_{w\in W_e} P_{w,u}$ that do not contain $C_e \cap B_v$. Observe that

$Q_{e,v}$ and $Q_{e,u}$ are disjoint disks. Let C'_e denote the closure of $C_e \setminus (Q_{e,v} \cup Q_{e,u})$. Notice that C'_e is a disk whose intersection with each of the disks $Q_{e,v}$ and $Q_{e,u}$ is an arc. It follows from the construction that the set $h_\epsilon(w) \cap C'_e \cap Q_{e,z}$ is a single point for $z = v, u$.

For each $v \in \mathcal{V}(G)$, let $B'_v = \bigcup_{e \in E(v)} Q_{e,v}$. Clearly,

(1) elements of $\{B'_v\}_{v \in \mathcal{V}(G)}$ are mutually disjoint,

(2) elements of $\{C'_e\}_{e \in \mathcal{E}(G)}$ are mutually disjoint,

(3) $B'_v \cap C'_e \neq \emptyset$ if and only if v is a vertex of e, and

(4) if v is a vertex of e then $B'_v \cap C'_e$ is an arc.

It follows that there is a homeomorphism g of N onto itself such that $g(B'_v) = B_v$ for $v \in \mathcal{V}(G)$ and $g(C'_e) = C_e$ for $e \in \mathcal{E}(G)$. It may be verified that $h = g \circ h_\epsilon : G' \to N$ satisfies the condition (a). □

Definition 2.10. (See Figure 1 and Example 1.2.) Let G be a graph with a local ordering of edges $\mathcal{A} = \{A(v)\}_{v \in \mathcal{V}(G)}$, where $A(v) = e_1(v), e_2(v), \ldots, e_{\mu(v)}(v)$. Suppose that, for a certain vertex v, i, j, i' and j' are four integers such that $1 \leq i < j \leq \mu(v)$ and $1 \leq i' < j' \leq \mu(v)$. We will say that the pair $\{e_i(v), e_j(v)\}$ *crosses* the pair $\{e_{i'}(v), e_{j'}(v)\}$ if either $i < j' < j < j'$ or $i' < i < j' < j$. Suppose that φ and φ' are simplicial maps mapping graphs G_1 and G'_1, respectively, into G. We will say that φ *crosses* φ' if there are arcs $L \subset G_1$ and $L' \subset G'_1$, there is a vertex $v \in \mathcal{V}(G)$ and there are four edges $e_i(v)$, $e_j(v)$, $e_{i'}(v)$ and $e_{j'}(v)$ such that $\varphi(L) = e_i(v) \cup e_j(v)$, $\varphi'(L') = e_{i'}(v) \cup e_{j'}(v)$ and the pair $\{e_i(v), e_j(v)\}$ crosses the pair $\{e_{i'}(v), e_{j'}(v)\}$. If $\varphi = \varphi'$, we will say that φ *crosses itself* (or has a *self crossing*). Observe that a crossing (or self crossing) depends on the local ordering $\mathcal{A}$. We will always understand that a crossing (or self crossing) occurs in some local ordering. If the range graph is embedded in an oriented

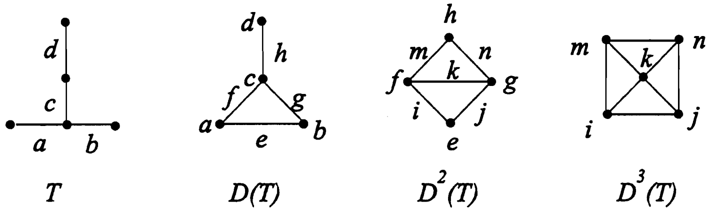


Fig. 7.

2-manifold we will implicitly assume that the local ordering agrees with the embedding.

3. The Operation d

We start this section by recalling some definitions introduced in [8].

Definition 3.1. (See [8, 2.1.].) For a graph G, let $D(G)$ be the graph such that

(1) the set of vertices of $D(G)$ consists of edges of G and

(2) two vertices of $D(G)$ are adjacent if and only if they intersect (as edges of G).

In particular, in the trivial case, when G contains no edges, $D(G)$ is empty. Since $\mathcal{E}(G) = \mathcal{V}(D(G))$, we will use the same notation for vertices of $D(G)$ and edges of G. Sometimes, however, it will be more convenient for us to denote by v^* the edge of G corresponding to a vertex v of $D(G)$. We will use the notation $D^2(G) = D(D(G))$ and, in general, $D^n(G) = D(D^{n-1}(G))$. Figure 7 illustrates the first tree iterations of the operation D on the graph T defined in Example 2.8. Edges of T are denoted by a, b, c and d. The same letters are used to denote the corresponding vertices of $D(T)$. Edges of the graph $D(T)$ and the corresponding vertices of $D^2(T)$

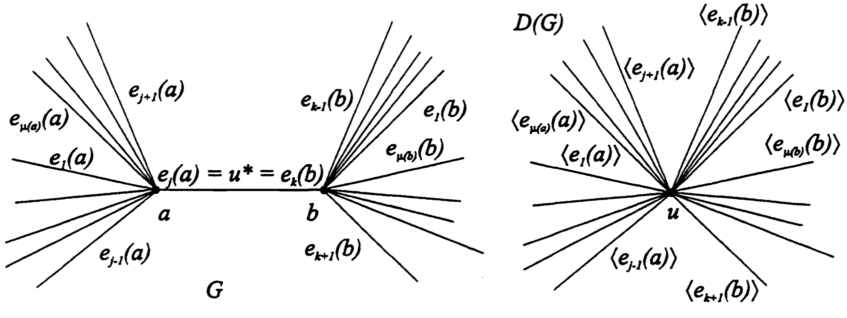


Fig. 8.

are denoted by e, f, g and h. Finally, edges of $D^2(T)$ and the corresponding vertices of $D^3(T)$ are denoted by letters i, j, k, m and n.

Definition 3.2. (See Figure 8.) Suppose G is a graph and $\mathcal{A} = A(v)_{v\in\mathcal{V}\ of G}$ is a local ordering of edges of G, where $A(v) = e_1(v), e_2(v), \ldots, e_{\mu(v)}(v)$. We will define $D(\mathcal{A}) = \{A(v)\}_{v\in\mathcal{V}(D(G))}$. Let u be a vertex of $D(G)$. Then u^* is an edge of G. Let a and b be vertices of u^*. Let $E(u)$ be the set of edges in $D(G)$ that have u as a vertex. Recall that $E(a)$ and $E(b)$ denote the sets of edges of G with one vertex at a and b, respectively. For each $e \in E(a) \cup E(b)$ that is different than u^*, let $\langle e\rangle$ denote the edge in $E(u)$ between u and the vertex of $D(G)$ representing e. Observe that every edge of $E(u)$ is equal to $\langle e\rangle$ for some edge $e \neq u^*$ of either $E(a)$ or $E(b)$. The edge u^* is present in both $A(a) = e_1(a), e_2(a), \ldots, e_{\mu(a)}(a)$ and $A(b) = e_1(b), e_2(b), \ldots, e_{\mu(b)}(b)$. There are integers $j = 1, \ldots, \mu(a)$ and $k = 1, \ldots, \mu(b)$ such that $e_j(a) = e_k(b) = u^*$. We will define $A(u)$ as the following sequence: $\langle e_1(a)\rangle$, $\langle e_2(a)\rangle$, $\ldots$, $\langle e_{j-1}(a)\rangle$, $\langle e_{k+1}(b)\rangle$, $\langle e_{k+2}(b)\rangle$, $\ldots$, $\left\langle e_{\mu(b)}(b)\right\rangle$, $\langle e_1(b)\rangle$, $\langle e_2(b)\rangle$, $\ldots$, $\langle e_{k-1}(b)\rangle$, $\langle e_{j+1}(a)\rangle$, $\langle e_{j+2}(a)\rangle$, $\ldots$,

$\left\langle e_{\mu(a)}(a)\right\rangle$.

We will denote $D(D(\mathcal{A}))$ by $D^2(\mathcal{A})$ and, in general, we will use the notation $D^n(\mathcal{A}) = D(D^{n-1}(\mathcal{A}))$.

Observe that if $\mathcal{A}$ is a local ordering of edges that agrees with the embedding of the graph T shown in Figure 7, then $D(\mathcal{A})$, $D^2(\mathcal{A})$ and $D^3(\mathcal{A})$ agree with the shown embedding of $D(T)$, $D^2(T)$ and $D^3(T)$, respectively.

All graphs in Figure 7 are planar. It should be noted, however, that $D(G)$ is not necessary planar even if G is. For example, if G consists of 5 edges meeting at the common vertex, then $D(G)$ is the Kuratowski graph K_5. Definition 3.2 allows us to circumvent this difficulty. If a graph G is embedded in the plane, or more general, in an oriented 2-manifold, there is a local ordering $\mathcal{A}$ that agrees with the embedding. Proposition 2.3 lets us embed $D(G)$ in an oriented 2-manifold so that the embedding agrees with $D(\mathcal{A})$. This natural embedding will be essential in our further considerations.

Definition 3.3. (See [8, 2.4.]) Suppose $\varphi : G' \to G$ is a simplicial map between graphs. For every (closed) edge $e \in \mathcal{E}(G)$, let $\mathcal{K}(e)$ denote the set of components of $\varphi^{-1}(e)$ which are mapped by φ onto e. Denote by $\mathcal{K}(\varphi)$ the union of all $\mathcal{K}(e)$. Let $D(\varphi, G')$ be the graph such that

(i) the vertices of $D(\varphi, G')$ are elements of $\mathcal{K}(\varphi)$, and

(i) two vertices of $D(\varphi, G')$ are adjacent if and only if they intersect (as subgraphs of G').

Let $d[\varphi] : D(\varphi, G') \to D(G)$ be the map defined by $d[\varphi](v) = \varphi(v)$ for every vertex $v \in D(\varphi, G')$.

Every vertex $v \in D(\varphi, G')$ is also a subgraph of G'. We will denote this subgraph by v^* when we need to distinguish between the two roles of the same object.

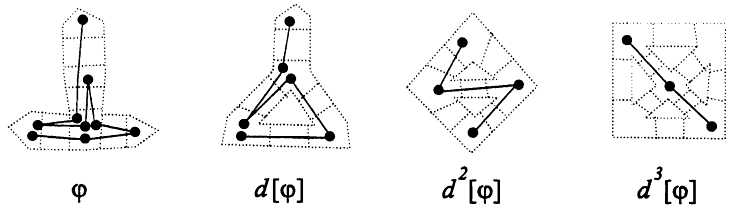


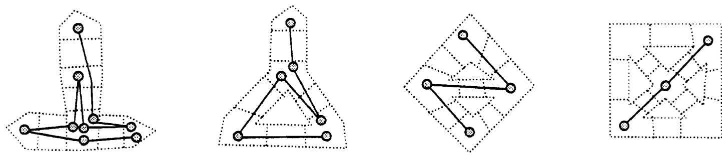
Fig. 9.

Observe that $d[\varphi]$ is empty if no edge of G is in the image of φ.

We will denote the map $d[d[\varphi]]$ by $d^2[\varphi]$ and its domain $D(d[\varphi], D(\varphi, G'))$ by $D^2(\varphi, G')$. In general, we will use the notation $d^n[\varphi] = d[d^{n-1}[\varphi]]$ for the n-th iteration of the operation d on φ. Also, $D^n(\varphi, G') = D(d^{n-1}[\varphi], D^{n-1}(\varphi, G'))$ will denote the domain of $d^n[\varphi]$. Thus, we have $d^n[\varphi] : D^n(\varphi, G') \to D^n(G)$.

Example 3.4. Let $\varphi : L \to T$ and $\varphi' : L' \to T$ be as in Example 2.7. The first graph in Figure 9 indicates an embedding of L into a normal neighborhood of T approximating the map φ (see also Figure 5). The next three graphs in Figure 9 indicate embedding of $D(\varphi, L)$, $D^2(\varphi, L)$ and $D^3(\varphi, L)$ into normal neighborhoods of $D(T)$, $D^2(T)$ and $D^3(T)$ approximating the maps $d[\varphi]$, $d^2[\varphi]$ and $d^3[\varphi]$, respectively (see also Figure 7). Observe that $D(\varphi, L)$, $D^2(\varphi, L)$ and $D^3(\varphi, L)$ are arcs. In fact, if the domain of a simplicial map ψ is an arc with n vertices, then the domain of $d[\psi]$ is an arc (possibly degenerate) with at most $n-1$ vertices ([8, Proposition 2.7.]).

Figure 10 indicates the similar embeddings for φ'. Notice that $d^3[\varphi]$ crosses $d^3[\varphi']$ in the sense of Definition 2.10. We will show that only such crossings may prevent existence of an embedding approximating a simplicial map defined on a graph whose components are arcs.



φ' $\quad d[\varphi']$ $\quad d^2[\varphi']$ $\quad d^3[\varphi']$

Fig. 10.

4. Approximations of φ and $d[\varphi]$ By Embedding

We will need the following proposition in the proof of Lemma 4.2.

Proposition 4.1. *Suppose R is a disk and H is a connected graph contained in R in such a way that the set $Q = H \cap \mathrm{Bd}(R)$ consists of finitely many points. Let U be a neighborhood of H in R. Then there is a point p and a set of arcs $\{P(q)\}_{q \in Q}$ contained in U such that for each $q \in Q$ we have*

(1) $P(q)$ is an arc with endpoints at p and q,

(2) $P(q) \cap \mathrm{Bd}(R) = \{q\}$ and

(3) $P(q) \cap \bigcup_{r \in Q \setminus \{q\}} P(r) = \{p\}$.

The proof of the above proposition is easy and will be omitted.

Lemma 4.2. *Let G be a graph embedded in a normal neighborhood N with a normal structure S. Let $\mathcal{A}$ be a local ordering of edges that agrees with the embedding of G into N. Suppose the graph $D(G)$ is embedded in a normal neighborhood $\widetilde{N}$ with a normal structure $\widetilde{S}$ in such a way that the embedding agrees with $D(\mathcal{A})$. Suppose φ is a simplicial map of a graph G_1 into G. Finally, suppose that there exists an embedding f of G_1*

into N that is an $\mathcal{S}$-approximations of φ. Then there exists an embedding $\tilde{f}$ of $D(\varphi, G_1)$ into $\widetilde{N}$ which is an $\tilde{\mathcal{S}}$-approximations of $d[\varphi]$.

Proof of 4.2. Let the normal structure $\mathcal{S}$ be defined by the collections $\{B_v\}_{v\in\mathcal{V}(G)}$ and $\{C_e\}_{e\in\mathcal{E}(G)}$. Similarly, suppose the normal structure $\tilde{\mathcal{S}}$ is defined by the collections $\left\{\tilde{B}_v\right\}_{v\in\mathcal{V}(D(G))}$ and $\left\{\tilde{C}_e\right\}_{e\in\mathcal{E}(D(G))}$.

Let s be an arbitrary edge of G and let a and b denote the vertices of s. Recall that $E(a)$ (or $E(b)$) denotes the set of edges of G that have a (or b) as a vertex. Since s is an edge of G, it is also a vertex of $D(G)$. In this context, $E(s)$ denotes the set of edges of $D(G)$ that have s as a vertex. As in Definition 3.2., for each $t \in E(a) \cup E(b)$ that is different than s, let $\langle t\rangle$ denote the edge of $E(s)$ between s and the vertex of $D(G)$ representing t. Let R_s denote the union $B_a \cup C_s \cup B_b$. Since the embedding of $D(G)$ into $\widetilde{N}$ agrees with $D(\mathcal{A})$, there is an orientation preserving homeomorphism h_s of R_s onto $\tilde{B}_s$ such that $h_s(R_s \cap C_t) = \tilde{B}_s \cap \tilde{C}_{\langle t\rangle}$ for each $t \in E(a) \cup E(b)$ that is different than s.

For an arbitrary edge $e \in \mathcal{E}(D(G))$, let $s(e)$ and $t(e)$ denote the vertices of e. Each vertex of $D(G)$ is an edge of G. Since $s(e)$ and $t(e)$, understood as vertices of $D(G)$, belong to the same edge, they must have a common vertex as edges of G. We will denote this vertex as $\alpha(e)$. Let g_e be a homeomorphism of $B_{\alpha(e)}$ onto $\tilde{C}_e$ such that $g_e(x) = h_{s(e)}(x)$ for each $x \in B_{\alpha(e)} \cap C_{t(e)}$ and $g_e(y) = h_{t(e)}(y)$ for each $y \in B_{\alpha(e)} \cap C_{s(e)}$.

Let w be an arbitrary vertex of $D(\varphi, G_1)$. Let s denote $d[\varphi](w)$ and let H_w be the component of $f(w^*)$ in $R_s \cap f(G_1)$. We will prove the following claim.

Claim. *If $u \neq w$ is another vertex of $D(\varphi, G_1)$ such that $d[\varphi](u) = s$, then $H_w \cap H_u = \emptyset$.*

Proof of Claim. Let S_v denote the union of v^* and all the open edges of G_1 that have a vertex in v^* where v is either u or w. Since $H_v = f(S_v) \cap R_s$, to prove the claim, it is enough to show that $S_u \cap S_w = \emptyset$. Suppose to the contrary that S_u and S_w intersect. Since $u^* \cap w^* = \emptyset$, there is an edge $z \in \mathcal{E}(G_1)$ with one vertex in u^* and another in w^*. It follows that $\varphi(z)$ is either s or a vertex of s. Since both u^* and w^* are components of $\varphi^{-1}(s)$, the edge z must be contained in both u^* and w^*. Consequently, we have that $u^* = w^*$ which contradicts $u \neq w$. So the claim is true. □

Let U_w be a neighborhood of H_w in R_s chosen so that U_w's are mutually exclusive. We will use Proposition 4.1 with $R = h_s(R_s) = \widetilde{B}_s$, $H = h_s(H_w)$ and $U = h_s(U_w)$. Let Q_w, p_w and $P_w(q)$ denote, respectively, Q, p and $P(q)$ from the conclusion of the proposition.

For an arbitrary vertex $w \in \mathcal{V}(D(\varphi, G_1))$, we define $\tilde{f}(w) = p_w$. We will extend this definition to an arbitrary edge $z \in \mathcal{E}(D(\varphi, G_1))$. By [8, Proposition 2.6], $d[\varphi](z)$ is an edge of $D(G)$. We will denote this edge by e. Let u and v denote the vertices of z in such an order that $d[\varphi](u) = s(e)$ and $d[\varphi](v) = t(e)$. Observe that $u^* \cap v^* \neq \emptyset$ and $\varphi(u^* \cap v^*) = \alpha(e)$. Let c be a component of $u^* \cap v^*$. Since u^* is connected and $\varphi(u^*) = s(e)$, there is an edge u' of u^* intersecting c but not contained in c. Observe that $\varphi(u') = s(e)$, because otherwise u' would be contained in c. Similarly, there is an edge v' of v^* intersecting c but not contained in c such that $\varphi(v') = t(e)$. The intersection $f(u') \cap B_{\alpha(e)} \cap C_{s(e)}$ is a one point set will be denoted by q_u. Similarly, let $q_v = f(v') \cap B_{\alpha(e)} \cap C_{t(e)}$. Since $u' \cup c \cup v'$ is connected, there is an arc $L \subset f(u' \cup c \cup v')$ with endpoints at q_u and q_v. Notice that $L \subset B_{\alpha(e)}$ and the intersection of L with the boundary of $B_{\alpha(e)}$ consists of q_u and q_v. Observe that $g_e(q_u) = h_{s(e)}(q_u) \in Q_u$ and $g_e(q_v) = h_{t(e)}(q_v) \in Q_v$. The set $Z = P_u(g_e(q_u)) \cup g_e(L) \cup P_v(g_e(q_v))$ is an arc with its endpoints at p_u and p_v. Let $\tilde{f}$ restricted to the edge

z be a homeomorphism of z onto Z such that $\tilde{f}(u) = p_u$ and $\tilde{f}(v) = p_v$. It may be verified that so defined $\tilde{f}$ satisfies the conclusion of the lemma. □

The following theorem is a corollary of the lemma and Proposition 2.9.

Theorem 4.3. *Let G be a graph embedded in an oriented 2-manifold M and let $\mathcal{A}$ be a local ordering of edges of G that agrees with the embedding. Suppose the graph $D(G)$ is embedded in an oriented 2-manifold $\widetilde{M}$ in such a way that the embedding agrees with $D(\mathcal{A})$. Let φ be a simplicial map of a graph G_1 into G with the property that for each positive number ϵ there is an embedding $h_\epsilon : G_1 \to M$ which is an ϵ-approximation of φ. Then, the map $d[\varphi]$ has the same property. That is, for each positive number ϵ, there is an embedding $\tilde{h}_\epsilon : D(\varphi, G_1) \to \widetilde{M}$ which is an ϵ-approximation of $d[\varphi]$.*

Example 4.4. (Ingram map [5]) We will consider here an inverse system $T^{(0)} \xleftarrow{f_0} T^{(1)} \xleftarrow{f_1} T^{(2)} \xleftarrow{f_2} \dots$ whose inverse limit defines the Ingram (non chainable and atriodic) continuum. Let T be the extended simple triod from Example 2.8. For each $n = 0, 1, \dots$, let h_n be a homeomorphism of T onto a certain graph $T^{(n)}$. Let $v_i^{(n)}$ denote $h_n(v_i)$ for $i = 0, \dots, 4$. Let $f_n : T^{(n+1)} \to T^{(n)}$ be a continuous map defined in the following way (see Figure 11):

(1) $f_n\left(v_0^{(n+1)}\right) = v_1^{(n)}$,

(2) the arc $\left[v_0^{(n+1)}, v_1^{(n+1)}\right]$ is mapped by a homeomorphism onto $\left[v_1^{(n)}, v_2^{(n)}\right]$,

(3) a point $p^{(n+1)}$ is selected in the interior of $\left[v_0^{(n+1)}, v_2^{(n+1)}\right]$,

(4) the arc $\left[v_0^{(n+1)}, p^{(n+1)}\right]$ is mapped by a homeomorphism

onto $\left[v_1^{(n)}, v_3^{(n)}\right]$,

(5) the arc $\left[p^{(n+1)}, v_2^{(n+1)}\right]$ is mapped by a homeomorphism onto $\left[v_3^{(n)}, v_2^{(n)}\right]$,

(6) the arc $\left[v_0^{(n+1)}, v_3^{(n+1)}\right]$ is mapped by a homeomorphism onto $\left[v_1^{(n)}, v_4^{(n)}\right]$, and

(7) the arc $\left[v_3^{(n+1)}, v_4^{(n+1)}\right]$ is mapped by a homeomorphism onto $\left[v_4^{(n)}, v_2^{(n)}\right]$.

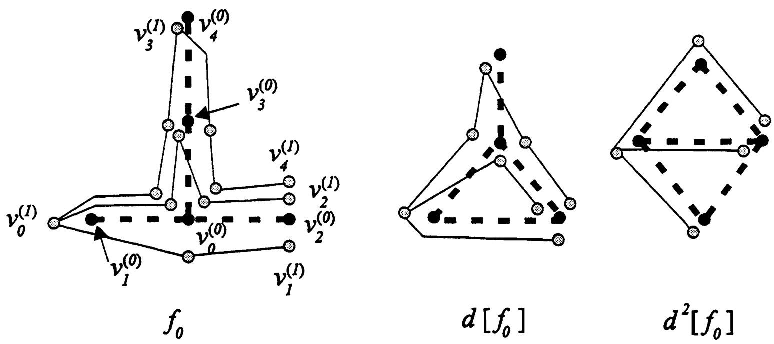


Fig. 11.

Suppose the graph $T^{(0)}$ has only five vertices $v_0^{(0)}, \ldots, v_4^{(0)}$. By using piece-wise linear homeomorphisms in the definition of f_n and selecting vertices of $T^{(n+1)}$ we can make f_n to be a simplicial map. Figure 11 illustrates the maps $f_0 : T^{(1)} \to T^{(0)}$, $d[f_0] : D\left(f_0, T^{(1)}\right) \to D\left(T^{(0)}\right)$ and $d^2[f_0] : D^2\left(f_0, T^{(1)}\right) \to D^2\left(T^{(0)}\right)$ (each vertex of the domain is close to its image). Notice that $d^2[f_0]$ maps a simple triod contained in $D^2\left(f_0, T^{(1)}\right)$ onto a simple triod contained in $D^2\left(T^{(0)}\right)$. This makes it impossible to find two embeddings closely approximating $d^2[f_0]$

with disjoint images. It follows from Theorem 4.3 that any two embeddings sufficiently closely approximating f_0 must intersect. It should be noted here that it would be easy to prove the last statement elementarily by drawing one embedding in the plane and eliminating all possible positions for the other. The proof by elimination of all possible cases does not work, however, for more complicated functions. Consider for example $\varphi_n = f_0 \circ f_1 \circ \dots f_{n-1}$ mapping $T^{(n)}$ onto $T^{(0)}$. By using the argument from [8, 5.12], one could prove that $d^{2n}[\varphi_n]$ maps a simple triod contained in $D^{2n}\left(\varphi_n, T^{(n)}\right)$ onto a simple triod contained in $D^{2n}\left(T^{(0)}\right)$. It would follow from Theorem 4.3 that any two embeddings sufficiently closely approximating φ_n must intersect. An alternate proof will follow from Theorem 4.5 generalizing the statement to any simplicial map that cannot be factored through an arc. Since the Ingram continuum is not chainable, φ_n cannot be factored through an arc (see also [8, 5.12]).

Theorem 4.5. *Suppose φ is a simplicial map of a tree Y into a graph G embedded into an oriented 2-manifold M_0. If φ cannot be factored through an arc, then there is a positive number ϵ such that any two embeddings of Y into M_0 ϵ-approximating φ must intersect.*

Before we prove Theorem 4.5 we need to state the following proposition. We leave its proof to the reader.

Proposition 4.6. *Suppose ψ is a simplicial map of a tree G' into a graph G_0 embedded into an oriented 2-manifold M_0. Suppose G' contains a connected (possibly degenerate) subgraph Y' and tree edges A, B and C each intersecting Y' such that $\psi(Y')$ is a single vertex and $\psi(Y' \cup A \cup B \cup C)$ is a simple triod. Then there is a positive number ϵ such that any two embeddings of G' into M_0 ϵ-approximating φ must intersect.*

Proof of 4.5. Let $\mathcal{A}$ be a local ordering of edges of G that agrees with the embedding of G into M_0. By Proposition 2.3, we may assume that $D^n(G)$ is embedded into an oriented 2-manifold M_n with an embedding that agrees with $D^n(\mathcal{A})$ for each positive integer n.

By [8, Theorem 2.13.], $d^n[\varphi]$ is not empty for each positive integer n. It follows from [8, Proposition 2.16] that there is a positive integer n such that $D^n(\varphi, Y)$ contains a simple triangle. Let k be the least integer such that $D^{k+1}(\varphi, Y)$ contains a simple triangle. By using [8, Proposition 2.16] again, we establish that $D^k(\varphi, Y)$ is a tree. It follows from [7, Proposition 3] that $D^k(\varphi, Y)$ contains a connected subgraph Y' and tree edges A, B and C each intersecting Y' such that $d^k[\varphi](Y')$ is a single vertex and $\psi(Y' \cup A \cup B \cup C)$ is a simple triod. By Proposition 4.6, there is a positive number ϵ such that any two embeddings of $D^k(\varphi, Y)$ into M_k ϵ-approximating $d^k[\varphi]$ must intersect. Now, Theorem 4.5 follows from Theorem 4.3 used k times. □

The next lemma shows that the implication from Lemma 4.2 may be reversed if the domain of φ is an arc without self crossings (see Definition 2.10).

Lemma 4.7. *Let G be a graph with a local ordering of edges $\mathcal{A}$. Suppose the graphs G and $D(G)$ are embedded into their respective normal neighborhoods N (with a normal structure $\mathcal{S}$) and $\widetilde{N}$ (with a normal structure $\widetilde{\mathcal{S}}$) such that the embeddings agree with $\mathcal{A}$ and $D(\mathcal{A})$, respectively. Let L be a graph whose components are simplicial arcs (possibly degenerate) and let $\varphi : L \to G$ be a simplicial map that does not cross itself. If there is an embedding $\tilde{f}$ of $D(\varphi, L)$ into $\widetilde{N}$ which is an $\widetilde{\mathcal{S}}$-approximations of $d[\varphi]$, then there is an embedding f of L into N that is an $\mathcal{S}$-approximations of φ.*

In the proof of the lemma we need the following two propositions.

Proposition 4.8. *Suppose R is a disk and $I_1, I_2, \ldots, I_k$ are mutually exclusive arcs contained in the boundary of R in the counterclockwise order. Suppose that, for each $i = 1, \ldots, k$, there is a sequence of mutually exclusive arcs $I_i^{i-1}, I_i^{i-2}, \ldots, I_i^1$, $I_i^k, I_i^{k-1}, \ldots, I_i^{i+1}$ contained in I_i in the counterclockwise order inherited from the boundary of R. Let $\mathcal{K}$ be a collection of pairs of integers such that*

(1) if $(i,j) \in \mathcal{K}$ then $1 \le i < j \le k$ and

(2) if i, j, i' and j' are integers such that $1 \le i < i' < j < j' \le k$ and $(i,j) \in \mathcal{K}$ then $(i',j') \notin \mathcal{K}$.

Then there is a collection of mutually exclusive disks $\{Z_{i,j}\}_{(i,j)\in\mathcal{K}}$ contained in R such that $Z_{i,j} \cap \mathrm{Bd}\,(R) = I_i^j \cup I_j^i$ for each $(i,j) \in \mathcal{K}$.

Proof of 4.8. We will prove the proposition by induction on the number of elements of $\mathcal{K}$. Suppose that the proposition has been proven for any disk R' and any collection $\mathcal{K}'$ which has less elements than $\mathcal{K}$. Take $(i,j) \in \mathcal{K}$. Let $\mathcal{K}_1$ be the set of the pairs $(i',j') \in \mathcal{K} \setminus \{(i,j)\}$ such that $i \le i' < j' \le j$. Let $\mathcal{K}_2$ be the set of the pairs $(i',j') \in \mathcal{K} \setminus \{(i,j)\}$ such that either $j' \le i$ or $j \le i'$. Observe that $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K} \setminus \{(i,j)\}$. There a disk $Z_{i,j} \subset R$ such that $Z_{i,j} \cap \mathrm{Bd}\,(R) = I_i^j \cup I_j^i$. The set $I_j \setminus \mathrm{Int}\,(Z_{i,j})$ has two components. We will denote them I_j' and I_j'' in the *counterclockwise* order. Observe that the arcs $I_j^{j-1}, I_j^{j-2}, \ldots, I_j^{i+1}$ are contained in I_j'. The arcs $I_j^{i-1}, I_j^{i-2}, \ldots, I_j^1 I_j^k, I_j^{k-1}, \ldots, I_j^{j+1}$ are contained in I_j''.

The set $I_i \setminus \mathrm{Int}\,(Z_{i,j})$ also has two components. We will denote them I_i' and I_i'' in the *clockwise* order. Observe that the arcs $I_i^{j-1}, I_i^{j-2}, \ldots, I_i^{i+1}$ are contained in I_i'. The arcs I_i^{i-1}, $I_i^{i-2}, \ldots, I_i^1 I_i^k, I_i^{k-1}, \ldots, I_i^{j+1}$ are contained in I_i''.

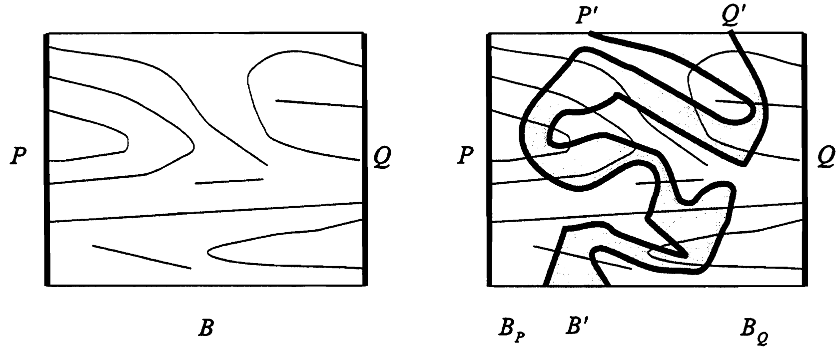


Fig. 12.

The set $R \setminus \mathrm{Int}\,(Z_{i,j})$ consists of two disjoint disks. We will name them R_1 and R_2 in such an order that I'_i and I'_j is contained in R_1 and I''_i and I''_j is contained in R_2. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ have less elements than $\mathcal{K}$ we may now apply the proposition to the pair $R_1, \mathcal{K}_1$ and to the pair $R_2, \mathcal{K}_2$. The resulting collections $\{Z_{i',j'}\}_{(i',j')\in\mathcal{K}_1}$ and $\{Z_{i'',j''}\}_{(i'',j'')\in\mathcal{K}_2}$ together with $Z_{i,j}$ form the required collection $\{Z_{i,j}\}_{(i,j)\in\mathcal{K}}$. □

Figure 12 is an illustration for the next proposition. The left side shows the disk B (the rectangle) with the collection of arcs $\mathcal{F}$ inside. The right side shows the same disk B partitioned by arcs P' and Q' into the disks B_P, B' (gray) and B_Q. The proof of the proposition is left to the reader.

Proposition 4.9. *Suppose B is a closed disk and P and Q are two disjoint arcs in the boundary of B. Let $\mathcal{F}$ be a finite collection of mutually exclusive arcs contained in B so that, for each $F \in \mathcal{F}$, the intersection of F with the boundary of B is equal to $F \cap (P \cup Q)$ and it is either empty or consists of one or both endpoints of F. Then B can be partitioned into three closed discs B_P, B' and B_Q such that $P \subset B_P$, $Q \subset B_Q$, $B_P \cap B_Q = \emptyset$, the sets $P' = B_P \cap B'$ and $Q' = B_Q \cap B'$ are* *arcs and, for each $F \in \mathcal{F}$, we have*

(1) the set $F \cap B'$ is not empty and it has at most two components,

(2) each component of $F \cap B'$ is an arc with one end in P' and the other in Q' and

(3) the $F \cap B'$ has two components only when both endpoints of F belong to one of the the arcs P and Q.

Proof of 4.7. Let the normal structure $\mathcal{S}$ be defined by the collections $\{B_v\}_{v\in\mathcal{V}(G)}$ and $\{C_e\}_{e\in\mathcal{E}(G)}$. Similarly, suppose the normal structure $\widetilde{\mathcal{S}}$ is defined by the collections $\left\{\widetilde{B}_v\right\}_{v\in\mathcal{V}(D(G))}$ and $\left\{\widetilde{C}_e\right\}_{e\in\mathcal{E}(D(G))}$.

Let a be an arbitrary vertex of G and let s be an arbitrary element of $E\left(a\right)$. Let b denote the other vertex of s. Recall that $E\left(a\right)$ (or $E\left(b\right)$) denotes the set of edges of G that have a (or b) as a vertex. Since s is an edge of G, it is also a vertex of $D\left(G\right)$. In this context, $E\left(s\right)$ denotes the set of edges of $D\left(G\right)$ that have s as a vertex. As before, for each $e \in E\left(a\right) \cup E\left(b\right)$ that is different than s, let $\langle e\rangle$ denote the edge of $E\left(s\right)$ between s and the vertex of $D\left(G\right)$ representing e. Let $E_a\left(s\right)$ be the set of edges of $E\left(s\right)$ in the form $\langle e\rangle$ where $e \in E\left(a\right) \setminus \{s\}$. Similarly, let $E_b\left(s\right)$ be the set of edges of $E\left(s\right)$ in the form $\langle e\rangle$ where $e \in E\left(b\right) \setminus \{s\}$. Clearly, $E\left(s\right) = E_a\left(s\right) \cup E_b\left(s\right)$. Since the embedding of $D\left(G\right)$ into $\widetilde{N}$ agrees with $D\left(\mathcal{A}\right)$, there are disjoint arcs $J\left[a,s\right]$ and $J\left[b,s\right]$ in the boundary of $\widetilde{B}_s$ such that $\widetilde{B}_s \cap \widetilde{C}_{\langle e\rangle} \subset J\left[a,s\right]$ for each $e \in E_a\left(s\right)$, and $\widetilde{B}_v \cap \widetilde{C}_{\langle e\rangle} \subset J\left[b,s\right]$ for each $e \in E_b\left(s\right)$.

Let $\mathcal{F}'\left(s\right)$ be the collection of components of the set $\tilde{f}\left(D\left(\varphi,L\right)\right) \cap \widetilde{B}_s$. Observe that all nondegenerate elements of $\mathcal{F}'\left(s\right)$ are arcs intersecting the boundary of $\widetilde{B}_s$ at either one or both endpoints and contained in the interior of $\widetilde{B}_s$ otherwise. All degenerate elements of $\mathcal{F}'\left(s\right)$ are single points contained in the interior of $\widetilde{B}_s$. Let $\mathcal{F}(s)$ be a collection of mutually exclusive arcs obtained from $\mathcal{F}'(s)$ by including all of its arcs and replacing each singleton in $\mathcal{F}'(s)$ by a small arc contained in the interior of $\widetilde{B}_s$.

Now use Proposition 4.9. with $B = \widetilde{B}_s$, $P = Ja, s$, $Q = J[b,s]$ and $\mathcal{F} = \mathcal{F}$ of s. Set $\widetilde{B}'_s = B'$, $J'[a,s] = P'$, $J'[b,s] = Q'$, $\widetilde{B}[a,s] = B_P$ and $\widetilde{B}[b,s] = B_Q$ where B', P', Q', B_P and B_Q are obtained from the proposition. Additionally, for each $F \in \mathcal{F}(s)$, let $U(F)$ be a neighborhood of F in $\widetilde{B}_s$ chosen so that the sets $U(F)$'s are mutually exclusive.

For each edge $s \in \mathcal{E}(G)$ with vertices a and b, we may choose a a neighborhood H_s of C_s in $C_s \cup B_a \cup B_b$ in such a way that the sets H_s's are mutually exclusive. Let h be an orientation preserving homeomorphism of $\bigcup_{s\in\mathcal{E}(G)} \widetilde{B}_s$ into N such that $h\left(\widetilde{B}_s\right) \subset H_s$, $h(J'[a,s]) = C_s \cap B_a$ and $h(J'[b,s]) = C_s \cap B_b$ for each $s = [a,b] \in \mathcal{E}(G)$. Observe that $h(J[a,s]) \subset B_a$ and $h(J[b,s]) \subset B_b$.

For each vertex $a \in \mathcal{V}(G)$, let $R_a = B_a \backslash \bigcup_{e\in E(a)} \operatorname{Int}\left(h\left(\widetilde{B}_e\right)\right)$. Observe that R_a is a disk. Let $e_1(a), e_2(a), \ldots, e_{\mu(a)}(a)$ be the edges of $E(a)$ arranged according to the local ordering $\mathcal{A}$. For each $i = 1, 2, \ldots, \mu(a)$, let $I_i(a) = h(J[a, e_i(a)])$. Observe that $I_1(a), I_2(a), \ldots, I_{\mu(a)}(a)$ are mutually exclusive arcs contained in the boundary of R_a in the counterclockwise order. For each $i = 1, 2, \ldots, \mu(a)$ and for each $j = 1, 2, \ldots, \mu(a)$ such that $j \neq i$, let $I_i^j(a) = h\left(\widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_j(a)\rangle}\right)$. Since the embedding of $D(G)$ into $\widetilde{N}$ agrees with $D(\mathcal{A})$, the arcs $\widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_{i+1}(a)\rangle} \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_{i+2}(a)\rangle}, \ldots, \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\left\langle e_{\mu(a)-1}(a)\right\rangle}$, $\widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\left\langle e_{\mu(a)}(a)\right\rangle}, \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_1(a)\rangle}, \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_2(a)\rangle}, \ldots, \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_{i-2}(a)\rangle}, \widetilde{B}_{e_i(a)} \cap \widetilde{C}_{\langle e_{i-1}(a)\rangle}$, are contained in the boundary of $\widetilde{B}_{e_i(a)}$ in the counterclockwise order. Since h is an orientation preserving homeomorphism, the images of the arcs are contained in the same order in the boundary of $h\left(\widetilde{B}_{e_i(a)}\right)$. Since

R_a is outside of $h\left(\widetilde{B}_{e_i(a)}\right)$, the order of the arcs on the boundary of R_a is reversed and the arcs $I_i^{i-1}(a), I_i^{i-2}(a), \ldots, I_i^1(a)$, $I_i^{\mu(a)}(a), I_i^{\mu(a)-1}(a), \ldots, I_i^{i+1}(a)$ contained in $I_i(a)$ in the counterclockwise order inherited from the boundary of R_a.

Let $\mathcal{K}_a$ be the collection of pairs of integers such that if $(i,j) \in \mathcal{K}_a$ then $1 \le i < j \le \mu(a)$ and there is an arc $L_{i,j} \subset L$ such that $\varphi(L_{i,j}) = e_i(a) \cup e_j(a)$. Since L does not cross itself the collection $\mathcal{K}_a$ satisfies the second condition in the statement of Proposition 4.8. It follows from the proposition that there is a collection of mutually exclusive discs $\{Z_{i,j}(a)\}_{(i,j)\in\mathcal{K}_a}$ contained in R_a such that $Z_{i,j}(a) \cap \operatorname{Bd}(R_a) = I_i^j(a) \cup I_j^i(a)$.

For an arbitrary edge $e \in \mathcal{E}(D(G))$, let s_e and v_e denote the vertices of e. Each vertex of $D(G)$ is an edge of G. Since s_e and v_e, understood as vertices of $D(G)$, belong to the same edge, they must have a common vertex as edges of G. We will denote this vertex as $\alpha(e)$. For any vertex $a \in \mathcal{V}(G)$, let $\widetilde{\mathcal{K}}_a$ be the set of these edges $e \in \mathcal{E}(D(G))$ for which $\alpha(e) = a$ and there is $(i,j) \in \mathcal{K}_a$ such that $e_i(a)$ and $e_j(a)$ coincide with s_e and v_e in some order. We will refer to the pair i,j as $i[e]$ and $j[e]$. Observe that $\widetilde{\mathcal{K}}_a \cap \widetilde{\mathcal{K}}_b = \emptyset$ for $a \neq b$.

For each $a \in \mathcal{V}(G)$ and each $e = [s,t] \in \widetilde{\mathcal{K}}_a$ there is a homeomorphism h_e of $\widetilde{C}_e$ onto $Z_{i[e],j[e]}(a)$ such that h_e coincides with h on each of the sets $\widetilde{B}_s \cap \widetilde{C}_e$ and $\widetilde{B}_v \cap \widetilde{C}_e$.

All nondegenerate components of L are arcs. We choose a direction on each of them. As usual, if b and c are two points of the same component, $[b,c]$ denote the arc between b and c.

For an arbitrary vertex $w \in \mathcal{V}(D(\varphi, L))$, let v denote the vertex $d[\varphi](w)$. Since v is a vertex of $D(G)$ it is also an edge of G. We will denote this edge by v^*. The set $w^* \subset L$ is a maximal arc mapped by φ onto the edge v^*. Let b_w and c_w denote the endpoints of w^* with $b_w < c_w$. Let p_w and q_w denote the vertices $\varphi(b_w)$ and $\varphi(c_w)$, respectively. Notice that p_w and q_w may coincide. Let b'_w be the last vertex on w^* such that $\varphi([b_w, b'_w]) = p_w$. Let c'_w be the first vertex such that $\varphi([c'_w, c_w]) = q_w$. Clearly $b'_w < c'_w$ and φ maps the arc $[b'_w, c'_w]$ onto v^*. Let b''_w a point inside of the first edge of $[b'_w, c'_w]$ and let c''_w a point inside of the last edge of $[b'_w, c'_w]$ such that $b''_w < c''_w$. Observe that the arcs in the form $[b''_w, c''_w]$ are mutually exclusive.

Suppose b_w is not the first vertex in its component in L. In this case, let z_ℓ be the last vertex of L less than b_w. Since $z_\ell \notin w^*$, $\varphi(z_\ell)$ is not a vertex v^*. Let $\tilde{z}_\ell$ denote the edge in G between $\varphi(b_w)$ and $\varphi(z_\ell)$. The component of z_ℓ in $\varphi^{-1}(\tilde{z}_\ell)$ is a vertex of $D(\varphi, L)$. We will denote this vertex by $\ell(w)$. Observe that $c'_{\ell(w)} = b_w$ and $b'_w = c_{\ell(w)}$.

Now, suppose c_w is not the last vertex in its component in L. In this case, let z_r be the first vertex of L greater than b_w. Since $z_r \notin w^*$, $\varphi(z_r)$ is not a vertex v^*. Let $\tilde{z}_r$ denote the edge in G between $\varphi(b_w)$ and $\varphi(z_r)$. The component of z_r in $\varphi^{-1}(\tilde{z}_r)$ is a vertex of $D(\varphi, L)$. We will denote this vertex by $r(w)$. Observe that $b'_{r(w)} = c_w$ and $c'_w = b_{r(w)}$.

Let ℓ denote the edge between w and $\ell(w)$ and let r denote the edge between w and $r(w)$. Observe that each edge of $D(\varphi, L)$, with w as a vertex, is either ℓ or r.

Let F_w denote the only element of $\mathcal{F}(v)$ such that $\tilde{f}(w) \in F_w$. If neither ℓ nor r is defined, F_w is contained in the interior of $\widetilde{B}_v$. If, on the other hand, at least one of the points ℓ and r is defined, then $F_w = \widetilde{B}_v \cap \tilde{f}(\ell \cup r)$. Consequently, the set $F_w \cap \mathrm{Bd}\left(\widetilde{B}_v\right) = \mathrm{Bd}\left(\widetilde{B}_v\right) \cap \tilde{f}(\ell \cup r)$ consists with at most two points $x_w \in \tilde{f}(\ell)$ and $y_w \in \tilde{f}(r)$. Observe that, if defined, the points x_w and y_w belong to $J[p_w, v]$ and $J[q_w, v]$, respectively.

Claim. *There is an embedding g_w of $[b'_w, c'_w]$ into $U(F_w)$ such that*

(1) if $\varphi(u) = t$ then $g_w(u) \in \widetilde{B}[t, v]$ for any vertex $u \in \mathcal{V}([b'_w, c'_w])$ and a vertex t of v^*,

(2) if $\varphi(e) = t$ then $g_w(e) \subset \widetilde{B}[t, v]$ for any edge $e \in \mathcal{E}([b'_w, c'_w])$ and a vertex t of v^*,

(3) the set $g_w(e) \cap J'[t,v]$ *has at most one point for each edge* $e \in \mathcal{E}([b'_w, c'_w])$ *and a vertex* t *of* v^**,*

(4) $g_w(b''_w) = x_w$ *if* ℓ *exists and*

(5) $g_w(c''_w) = y_w$ *if* r *exists.*

Proof of Claim. We will prove the claim in the case where both points x_w and y_w are defined. The proof in the remaining case is very similar with less conditions to satisfy.

Suppose $p_w \neq q_w$. In this case $J[p_w, v] \neq J[p_w, v]$ and, consequently, $F_w \cap \widetilde{B}'_v$ has only one component. We start the construction of g_w by embedding the arc $[b'_w, b''_w]$ into $\widetilde{B}[p_w, v]$ such that $g_w(b''_w) = x_w$. Then we continue embedding the first edge, say e_1, of $[b'_w, c'_w]$ in the following way. If $\varphi(e_1) = p_w$, we embed e_1 in $\widetilde{B}[p_w, v]$. Otherwise, we cross $\widetilde{B}'_v$ parallel to F_w. We continue the process extending g_w to other edges one by one. Suppose that g_w is defined on the first vertex u of an edge e. If $\varphi(e) = \varphi(u)$, we embed e in $\widetilde{B}[\varphi(u), v]$. Otherwise, we cross $\widetilde{B}'_v$ parallel to F_w. While zigzagging through $\widetilde{B}'_v$, we make each new crossing outside of the previous ones so that the point y_w may be reached by $g_w(c''_w)$.

Now, suppose $p_w = q_w$. In this case $J[p_w, v] = J[p_w, v]$ and, consequently, $F_w \cap \widetilde{B}'_v$ has two components. We can denote these components by F' and F'' in such a way that F' is between x_y and F''. Since $\varphi([b'_w, c'_w]) = v^*$, there is vertex $z \in [b'_w, c'_w]$ such that $\varphi(z) \neq p_w$. We construct g_w like in the previous case. The only difference is that while crossing $\widetilde{B}'_v$ with edges from $[b'_w, z]$ we keep close to F' and then we keep close to F'' when crossing $\widetilde{B}'_v$ with edges from $[z, c'_w]$ so that the points x_w and y_w may be reached by $g_w(b''_w)$ and $g_w(c''_w)$, respectively. □

Now, we will construct an embedding $f : L \to N$ by patching different definitions on different portions of L.

1. Define f on every interval of the form $[b''_w, c''_w]$ by setting $f = h \circ g_w$.

2. Suppose u and w are vertices of $D(\varphi, L)$ such that $u = \ell(w)$ and, consequently, $w = r(u)$. Observe that $\varphi([c'_u, b'_w]) = q_u = p_w$. We will denote this vertex by a. Let $s = d[\varphi](u)$ and $v = d[\varphi](w)$. So defined s and v are vertices of $D(G)$ and as such are edges of G. Since a is a common vertex of s and v, understood as edges of G, s and v, understood as vertices of $D(G)$, belong to an edge which we will denote by e. Since $\varphi([b_u, c_w])$ is equal to the union of edges s and v, we have the result that $e \in \widetilde{\mathcal{K}}_a$. The vertices u and w form an edge $[u, w] \in \mathcal{E}(D(\varphi, L))$. The set $T = \tilde{f}([u, w]) \cap \widetilde{C}_e$ is an arc with ends at y_u and x_w. Let $g_{u,w}$ be a homeomorphism of $[c''_u, b''_w]$ onto T such that $g_{u,w}(c''_u) = y_u$ and $g_{u,w}(b''_w) = x_w$. Finally, we may define f on $[c''_u, b''_w]$. Observe that $f(c''_u) = h(y_u)$ and $f(b''_w) = h(x_w)$ have been already defined. Since h_e coincides with h on each of the sets $\widetilde{B}_s \cap \widetilde{C}_e$ and $\widetilde{B}_v \cap \widetilde{C}_e$, we may extend the embedding by defining f on $[c''_u, b''_w]$ as the composition $h_e \circ g_{u,w}$.

3. Suppose, for some $w \in \mathcal{V}(D(\varphi, L))$, b_w is the first point of its component in L. Observe that $f(b''_w) \in B_{p_w}$ have been already defined. We extend the definition of f on $[b_w, b''_w]$ by embedding $[b_w, b''_w]$ into B_{p_w} sufficiently close to $f(b''_w)$ so that $f([b_w, b''_w])$ meets the image of previously defined portion of f only at $f(b''_w)$.

4. Suppose, for some $w \in \mathcal{V}(D(\varphi, L))$, c_w is the last point of its component in L. Observe that $f(c''_w) \in B_{p_w}$ have been already defined. We extend the definition of f on $[c''_w, c_w]$ by embedding $[c''_w, c_w]$ into B_{q_w} sufficiently close to $f(c''_w)$ so that $f([c''_w, c_w])$ meets the image of previously defined portion of f only at $f(c''_w)$.

5. Finally, if L' is a component of L mapped by φ onto a single vertex a, let f on L' be defined as an embedding into B_a in such a way that $f(L')$ does not intersect the image of previously defined portion of f.

One may verify that so defined f is an embedding of L into N satisfying the conclusion of the lemma. □

The following theorem is a corollary of 4.3, 4.7 and 2.9.

Theorem 4.10. *Let G be a graph embedded in an oriented 2-manifold M and let $\mathcal{A}$ be a local ordering of edges that agrees with the embedding. Suppose the graph $D(G)$ is embedded in an oriented 2-manifold $\widetilde{M}$ with an embedding agreeing with $D(\mathcal{A})$. Let L be a graph whose every component is an arc (possibly degenerate) and let $\varphi : L \to G$ be a simplicial map. Then the following two conditions are equivalent:*

(1) For each positive number ϵ, the map φ can be ϵ-approximated by an embedding of L into M.

(2) The map φ does not have a self crossing and for each positive number ϵ, the map $d[\varphi]$ can be ϵ-approximated by an embedding of $D(\varphi, L)$ into $\widetilde{M}$.

The following theorem gives a combinatorial condition equivalent to the existence of an embedding closely approximating a simplicial map from an arc to a plane graph.

Theorem 4.11. *Let G be a graph embedded in an oriented 2-manifold M and let $\mathcal{A}$ be a local ordering of edges that agrees with the embedding. Suppose L is graph whose every component is a simplicial arc (possibly degenerate) and $\varphi : L \to G$ is a simplicial map. Then the following two conditions are equivalent:*

(1) For each positive number ϵ, the map φ can be ϵ-approximated by an embedding of L into M.

(2) The map $d^n[\varphi]$ does not cross itself in $D^n\mathcal{A}$ for each $n = 0, 1, \ldots$.

Proof of 4.11. Applying repeatedly Theorem 4.10. we get the result that the condition 1 implies 2. It follows from [8, ?] that $D^k(\varphi, L) = \emptyset$. Since the empty map $d^k[\varphi]$ can be trivially approximated by an embedding, Theorem 4.10 can be used to establish the implication $2 \Rightarrow 1$. □

Remark 4.12. *Let k be the number of vertices of the longest component of the graph L from the statement of theorem 4.11. It follows from [8, ?] that each component of $D^{k-2}(\varphi, L)$ has no more than 2 vertices. Thus, it is enough to take $n = 0, 1 \ldots, k-3$ in the condition 2 of Theorem 4.11.*

Remark 4.13. *Observe that the condition 2 of Theorem 4.11 can be easily verified on a computer. Hence, Theorem 4.11 and Remark 4.12 yield a computer algorithm to check whether a simplicial map from an arc (or a collection of arcs) into a graph contained in an oriented 2-manifold can be approximated by an embedding.*

Theorem 4.11 shows that self crossing on some level of the operation d is the only obstacle preventing a map φ of an arc into a planar graph to be approximated by an embedding. If the domain of φ is a tree then an ϵ-approximation could be prevented by two triods as in Example 1.3. (See also Theorem 4.5.)

Question 4.14. *Let G be a graph embedded in an oriented 2-manifold M and let $\mathcal{A}$ be a local ordering of edges that agrees with the embedding. Suppose φ is a simplicial map of tree T into G. Are the following two conditions equivalent?*

(1) For each positive number ϵ, the map φ can be ϵ-approximated by an embedding of T into M.

(2) For each $n = 0, 1, \ldots$ we have that the map $d^n[\varphi]$ does not

cross itself in $D^n(\mathcal{A})$ and $d^n[\varphi]$ does not map two disjoint open simple triods onto the same triod.

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