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ON THE LINDELÖF PROPERTY AND TIGHTNESS OF PRODUCTS

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Abstract

We show that for every $n \leq \omega$ there is a family $\mathcal{C} = \{ X_i : i < n \}$ of Tychonoff spaces such that all but one spaces in $\mathcal{C}$ are second-countable, the countable power of the product of every proper subfamily of $\mathcal{C}$ is Lindelöf, and the product of $\mathcal{C}$ has the extent equal to $\mathfrak{c}$. We also prove the existence for every $n \in \omega$ of a family $\mathcal{B} = \{ Y_i : i < n \}$ of σ-compact spaces all of which but one are countable, such that the countable power of the product of every proper subfamily of $\mathcal{B}$ has countable tightness, but the product of $\mathcal{B}$ has the tightness equal to $\mathfrak{c}$.

In this paper we present two series of examples. The first is a series of examples of families of n (for every $n \leq \omega$) Lindelöf spaces such that all spaces but one in each family are second-countable, the product of the whole family is not Lindelöf, but the product of every proper subfamily is Lindelöf (and even with its countable power). The idea is to continue the examples constructed earlier by Przymusiński [Prz], Lawrence [Law], and Okunev and Tamano [OT]; the construction in fact is based on a construction in the last paper.

The second series of examples is of families of n (for every natural n) spaces such that in each family all spaces but one are countable, the product of every proper subfamily has countable

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tightness (again, together with its countable power), but the product of the whole family has uncountable tightness. An example of two spaces one of which is countably tight, the other countable, and whose product has uncountable tightness was given in [Arh1]. The only uncountable space in each of the families is σ-compact; note that by a theorem of Malykhin [Mal], if one of two spaces is compact, then the tightness of their product does not exceed the maximum of the tightnesses of factors. The question whether the product of two countably tight σ-compact spaces can have uncountable tightness was open for some time, until it was solved (along with many other problems) by Todorčević in [Tod]. In [Ok2] the Todorčević's example was slightly improved (in one respect) by showing that we may assume one of the factors countable. The example here uses an idea similar to that used in [Ok2], the duality between the Lindelöf number and tightness that arises in the theory of continuous functions with the topology of pointwise convergence. Of course, the sum of all spaces in the nth family gives an example of a σ-compact, countably tight space whose $(n-1)$th power has countable tightness, but the nth is not.

All spaces below are assumed Tychonoff (= Hausdorff completely regular). The symbol $\mathfrak{c}$ denotes the cardinality of a continuum. For a space X and a subset A of X, we denote by $X_{(A)}$ the space obtained from X by retaining the original topology at the points of the set A and making the points of $X \setminus A$ isolated. It is well-known that the space X is Tychonoff (see e.g. [Eng]). Obviously, $X = X_{(X)}$, and the space $X_{(\emptyset)}$ is discrete. For every two sets $A, B \subset X$ we have the natural bijection $i_{AB} : X_A \to X_B$ that coincides with the identity mapping of the space X.

The following observation is obvious.

Proposition 1. *If $A \subset B \subset X$, then the mapping $i_{AB} : X_{(A)} \to X_{(B)}$ is continuous.*

A subset A of a space X is called *holding* [OT] if the countable power of the space $X_{(A)}$ is Lindelöf.

From Proposition 1 immediately follows that every superset of a holding set is holding. The following statement is proved in [OT].

Theorem 2. *If X is an uncountable Polish space, then there is a disjoint family of cardinality $\mathfrak{c}$ of holding sets in X.*

The first main theorem of this paper is

Theorem 3. *For every $n \leq \omega$ there is a family $\mathcal{C} = \{X_i : i < n\}$ of spaces such that*

(1) For every $i > 0$, $i < n$ the space X_i is second-countable,

(2) For every proper subfamily $\mathcal{C}'$ of $\mathcal{C}$, the space $(\prod \mathcal{C}')^\omega$ is Lindelöf,

(3) The extent of the product $\prod \mathcal{C}$ is equal to $\mathfrak{c}$.

Proof. Using Theorem 2, we may fix a disjoint family $\{ A_i : i < n \}$ of holding sets in the Cantor discontinuum $\mathbf{C} = \mathbf{2}^\omega$. For every $i < n$, put $B_i = \bigcup \{ A_j : j < n,\ j \neq i \}$. Obviously, $\bigcap \{ B_i : i < n \} = \emptyset$, and for every $i < n$, $\bigcap \{ B_j : j < n,\ j \neq i \} = A_i$.

Put $X_0 = \mathbf{C}_{(\mathbf{B_0})}$, and for every $i > 0$, $i < n$, $X_i = B_i$. Obviously, the condition (1) holds for the family $\mathcal{C} = \{ X_i : i < n \}$.

Let us verify (2). Let $\mathcal{C}'$ be a proper subfamily of $\mathcal{C}$. Then for some $i_0 < n$, the space X_{i_0} is not in $\mathcal{C}'$. Each of the sets B_i, $i < n$, $i \neq i_0$ contains the set A_{i_0}. From Proposition 1 now follows that each of the spaces $\mathbf{C}_{(\mathbf{B_i})}$, $i < n$, $i \neq i_0$ is a continuous image of the space $\mathbf{C}_{(\mathbf{A_{i_0}})}$. Since the set A_{i_0} is holding, the countable power of the space $\mathbf{C}_{(\mathbf{A_{i_0}})}$ is Lindelöf. Hence, the product $\left(\prod \{ \mathbf{C}_{(B_i)} : i < n,\ i \neq i_0 \}\right)^\omega$ is Lindelöf, because it is a continuous image of the space $(\mathbf{C}_{(A_{i_0})})^\omega$. To end verification of (2), note that since each X_i, $i < n$ is a closed subspace of $\mathbf{C}_{(\mathbf{B_i})}$, the product $(\prod \mathcal{C}')^\omega$ is homeomorphic to a closed subspace of $(\prod\{\mathbf{C}_{(B_i)} : i < n,\ i \neq i_0\})^\omega$.

Finally, let D be the diagonal in the product $\prod \mathcal{C}$, that is, the family of all points of $\prod \mathcal{C}$ whose all coordinates are equal; from the definition of the spaces B_i follows that D is actually the set of all points in $\prod \mathcal{C}$ whose all coordinates are equal and belong to A_0.

Obviously, the set D is closed in $\prod \mathcal{C}$ (it is closed even in the weaker topology of the product $\mathbf{C} \times \prod\{B_i : 1 \leq i < n\}$). The projection $\pi_0 : \prod \mathcal{C} \to X_0$ maps D onto $i_{XB_0}(A_0)$; since every holding set in an uncountable Polish space has cardinality $\mathfrak{c}$ ([OT]; or we obviously may choose A_0 of cardinality $\mathfrak{c}$), we have $|D| = \mathfrak{c}$. Since $A_0 \cap B_0 = \emptyset$, the set $i_{XB_0}(A_0) \subset X_0$ is discrete in $X_0 = \mathbf{C}_{(\mathbf{B_0})}$, and the restriction of π_0 to D is continuous and one-to-one. Hence, D is also discrete.

The proof is complete. □

Now we are ready to prove the statement about the tightness. Recall that the *tightness* of a space X is the minimal cardinal τ with the property that the closure of every set A in X is equal to the union of the closures of subsets of A of cardinality $\leq \tau$.

The construction here is dual to Theorem 3; the duality is provided by the topology of pointwise convergence on function spaces. Recall that for every space X, $C_p(X, 2)$ is the space of all continuous functions on X with the range $2 = \{0, 1\}$ endowed with the topology of pointwise convergence, which coincides with the topology on $C(X, 2)$ induced by the Tychonoff product topology on the set 2^X of *all* $\{0,1\}$-valued functions on X (see, e.g. [Arh2]).

Theorem 4. *For every $n \in \omega$ there is a family $\mathcal{B} = \{Y_i : i < n\}$ of spaces such that*

(1) Y_0 is σ-compact, and for every $i > 0$, $i < n$ the space Y_i is countable,

(2) For every proper subfamily $\mathcal{B}'$ of $\mathcal{B}$, the space $(\prod \mathcal{B}')^\omega$ has countable tightness,

(3) The tightness of the product $\prod \mathcal{B}$ is equal to $\mathfrak{c}$.

Proof. Let $\mathcal{C} = \{ X_i : i < n \}$ be the family constructed in the proof of Theorem 3 (we keep further the same notation as in the proof of Theorem 3). We will construct the spaces Y_i as subspaces of the spaces $C_p(X_i, \{0,1\})$.

Let $\mathcal{O}$ be the family of all nonempty clopen sets in the Cantor discontinuum $\mathbf{C}$. Obviously $|\mathcal{O}| = \omega$. Denote $p_0 = i_{B_0 X} : X_0 \to \mathbf{C}$, and for every $i < n$, $i > 0$ let $p_i : X_i = B_i \to \mathbf{C}$ be the embedding.

Let $X = \bigoplus \{ X_i : i < n \}$; assign to every n-tuple $(F, U_1, \ldots, U_{n-1})$ consisting of a finite set F and a sequence of clopen sets $U_1, \ldots, U_{n-1} \in \mathcal{O}$ such that $F \subset U_1 \cap \cdots \cap U_{n-1} \cap A_0$ the function $f_{F,U_1,\ldots,U_{n-1}} : X \to \{0,1\}$ defined by the rule:

$$f_{F,U_1,\ldots,U_{n-1}}(x) = \begin{cases} 1 & \text{if } x \in X_0 \text{ and} \\ & p_0(x) \in \bigcap \{ U_i : 1 \le i < n \} \setminus F \\ & \text{or } x \in X_i,\ 1 \le i < n \text{ and } p_i(x) \notin U_i. \\ 0 & \text{otherwise.} \end{cases}$$

and put $S = \{ f_{F,U_1,\ldots,U_{n-1}} : U_1, \ldots, U_{n-1} \in \mathcal{O},\ F \subset U_1 \cap \ldots U_{n-1} \cap A_0 \text{ and } |F| < \omega \}$. Obviously, $S \subset C_p(X, \{0,1\})$ (note that the points of F are isolated in X_0).

CLAIM 1. *The zero function* 0 *is a limit point (in $C_p(X,2)$) for the set S, but is not a limit point for any subset of S whose cardinality is less than $\mathfrak{c}$.*

A generic neighborhood of 0 in $C_p(X, \{0,1\})$ is of the form $O(K) = \{ f \in C_p(X,2) : f|K = 0 \}$ where K is a finite set in

X. Let $K_i = K \cap X_i$ and $F_i = p_i(K_i)$, $i = 0, \dots, n-1$; put $F = \bigcup\{ F_i : i < n \} \cap A_0$. Fix clopen sets $U_1, \dots, U_{n-1} \in \mathcal{O}$ so that $F_i \subset U_i$, $\bigcap\{ U_i : 1 \le i < n \} \ne \emptyset$ and $\bigcup\{ U_i : 1 \le i < n \} \cap \bigcup\{ F_i : 0 \le i < n \} = F$. It is easy to see that the function $f_{F,U_1,\dots,U_{n-1}}$ is equal to 0 on the set K_i, so it is in $O(K)$. From $\bigcap\{ U_i : 1 \le i < n \} \ne \emptyset$ follows that $f_{F,U_1,\dots,U_{n-1}} \ne 0$, so we found a nonzero function from S in every neighborhood of 0, and the first part of the claim is proved.

To prove the second part, let S_0 be a subset of S and $|S_0| < \mathfrak{c}$. Let $M = \bigcup\{ F \subset \mathbf{C} : f_{F,U_1,\dots,U_{n-1}} \in S_0$ for some $U_1, \dots, U_{n-1} \in \mathcal{O} \}$. Then $|M| < \mathfrak{c}$, and since $|A_0| = \mathfrak{c}$, there is a point $c_0 \in A_0 \setminus M$. Let $x_i \in X_i$ be the points in X_i such that $p_i(x_i) = c_0$, $i = 0, \dots, n-1$. Then for every sequence $U_1, \dots, U_{n-1} \in \mathcal{O}$ and every finite $F \subset \bigcap\{ U_i : 1 \le i < n \}$ such that $c_0 \notin F$ we have either $c_0 \in \bigcap\{ U_i : 1 \le i < n \}$, and then $f_{F,U_1,\dots,U_{n-1}}(x_0) = 1$, or for some $i \in \{1, \dots, n-1\}$, $c_0 \notin U_i$, and then $f_{F,U_1,\dots,U_{n-1}}(x_i) = 1$. Thus, the set $\{ f \in C_p(X,2) : f(x_0) = f(x_1) = \dots = f(x_{n-1}) = 0 \}$ is a neighborhood of 0 in $C_p(X,2)$ disjoint with S_0.

Claim 2. *The set S is σ-compact.*

We have

$$S = \bigcup_{U_1,\dots,U_{n-1} \in \mathcal{O}} \bigcup_{k \in \omega} S_{U_1,\dots,U_{n-1},k}$$

where $S_{U_1,\dots,U_{n-1},k} = \{ f_{F,U_1,\dots,U_{n-1}} : F \subset A_0 \cap \bigcap\{ U_i : 1 \le i < n \}, |F| \le k \}$. The set $S_{U_1,\dots,U_{n-1},k}$ is the family of all functions in 2^X that differ from the function $f_{\emptyset,U_1,\dots,U_{n-1}}$ at finitely many points belonging to a fixed subset of X; a standard argument shows that $S_{U_1,\dots,U_{n-1},k}$ is closed in 2^X, hence compact. Since $\mathcal{O}$ is countable, the set S is σ-compact.

Thus, $S \cup \{0\}$ is a σ-compact subspace of $C_p(X,2)$ of uncountable tightness.

We may identify $C_p(X,2)$ with $\prod\{ C_p(X_i,2) : i < n \}$; let $r_i \; C_p(X,2) \to C_p(X_i,2)$, $i < n$, be the projections (the restriction mappings). Put $Y_i = \pi_i(S)$. Obviously, the product $\prod\{ Y_i : i < n \}$ contains S, hence has uncountable tightness. This demonstrates (3) for the family $\mathcal{B} = \{ Y_i : i < n \}$.

Since S is σ-compact, Y_0 is σ-compact; by the construction, the restrictions of the functions in S to X_i with $i \geq 1$ are characteristic functions of intersections of elements of $\mathcal{O}$ with B_i; it follows that the spaces $Y_1, \dots, Y_{n-1}$ are countable.

For every $k < n$ we have $l((\prod\{ X_j : j < n,\ j \neq k \})^\omega) = \omega$; obviously, all finite powers of the space $(\bigoplus\{ X_j : j < n,\ j \neq k \}) \times \omega$ are Lindelöf. The product $(\prod\{ Y_j : j < n,\ j \neq k \})^\omega$ lies in the product $(\prod\{ C_p(X_j, 2) : j < n,\ j \neq k \})^\omega = C_p((\bigoplus\{ X_j : j < n,\ j \neq k \}) \times \omega)$, and by the Arhangel'skiĭ-Pytkeev theorem (see [Arh2, II.1.1]) the latter space has countable tightness. This proves (2). □

Remark. Malykhin proved in [Mal] that for every countable family of spaces, the tightness of product of the whole family does not exceed the supremum of the tightnesses of the products of finite subfamilies. Therefore, an example as in Theorem 4 cannot be constructed for $n = \omega$.

Remark. If desired, it is easy to make the spaces Y_i in Theorem 4 topological groups, see [Ok2]. Similarly, using the techniques in [Ok1] and [OT], we may prove that the countable power of $C_p(Y_0)$ in Theorem 4 is Lindelöf and that X_i is homeomorphic to the closed subspace of $C_p(Y_i)$, $i = 0, \dots, n-1$, so a family as in Theorem 3 the spaces X_i may be constructed as $C_p(Y_i)$.

The techniques used above appear to be specifically "countable"; there is little doubt that the answers to the following questions are positive, but there seem to be no ready examples.

Problems. *Is it true that for every cardinal τ there is a family $\mathcal{C}$ of cardinality τ such that the Lindelöf number of the τ-power* *of the product of every proper subfamily of $\mathcal{C}$ does not exceed τ, yet the Lindelöf number of the product of $\mathcal{C}$ is greater than τ? Is there such family with the weight of all spaces but one not exceeding τ?*

Is it true that for every cardinal τ there is a family $\mathcal{B}$ such that the tightness of the τ-power of the product of every proper subfamily of $\mathcal{B}$ does not exceed τ, yet the tightness of the product of $\mathcal{B}$ is greater than τ? Is there such family with the cardinality of all spaces but one not exceeding τ?

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