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ON A CLASS OF SPECIAL NAMIOKA SPACES

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Abstract

We show that any Baire strict p-space is special Namioka. For this purpose we introduce and study a class of spaces defined by means of a topological game. We also give some applications to semitopological groups

1. Introduction

The problem of determining the points of joint continuity of a separately continuous functions dates back to the second half of the 19th century (see Piotrowski's survey paper [P] for a historic background). A real breakthrough in this area has been done by Isaak Namioka who proved the following remarkable theorem in 1974 (see [N, Theorem 1.2]):

If $f : X \times Y \to M$ is a separately continuous function on the product of a compact space Y and a strongly countably complete space X into a metric space M, then there exists a dense G_δ subset A of X such that f is jointly continuous at each point of $A \times Y$.

This result, beyond its many applications in functional analysis, the theory of topological groups and others, spurred extensive further research (see [P] for more details). It has been also generalized in many ways. Our paper contains a game-theoretic description of a class of topological spaces that is substantially wider than, for example, the class of strongly countably complete spaces or the class of metric Baire spaces, yet the conclusion of the Namioka theorem holds for spaces

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from this class (see Theorem 6). We apply our results to the theory of semitopological groups.

We thank the referee for his important suggestions.

All spaces considered here are topological regular spaces. If X and Y are spaces, then $X \times Y$ denotes their *Cartesian product*, and π_X, π_Y denote the *projections* onto X, Y, respectively.

Let F be a subset of $X \times Y$, let $x \in X$, and let $y \in Y$. Then $F_x = \{y \in Y : (x, y) \in F\}$, $F^y = \{x \in X : (x, y) \in F\}$.

Let $f : X \times Y \to Z$ be a function from the product $X \times Y$ into a space Z. If $x \in X$ and $y \in Y$, then

$f_x : Y \to Z$ is given by $f_x(y) = f(x, y)$, and

$f^y : X \to Z$ is given by $f^y(x) = f(x, y)$.

We say that f is *separately continuous* if the functions f_x, f^y are continuous for each $x \in X$ and $y \in Y$; we say that f is *jointly continuous* (at a point) if f is continuous (at that point).

Let $f : X \to M$ be a function into the metric space M with a metric d, let $p \in X$, and let A be a subset of M. Then $diam(A) = \sup\{d(a, b) : a, b \in A\}$, $\omega(f; p) = \inf\{diam(f(U)) : U$ is an open neighborhood of $p\}$. It is well known that f is continuous at a point p if and only if $\omega(f; p) = 0$.

If $\mathcal{S}$ is a family of subsets of a set X and p is a point in X, then $st(p, \mathcal{S}) = \bigcup\{A : A \in \mathcal{S}$ and $p \in A\}$, which is called the *star of $\mathcal{S}$ at the point p.*

2. A Class of Spaces Determined by a Topological Game

Let us consider the following game played by two players, say α and β, on a topological space X.

Player β starts by choosing only one non-empty open subset U_0 of X. Suppose that one of the sets chosen by player β in his $n-th$ move was U_n. Then player α chooses for each point x of U_n an open neighborhood $V_n(x)$ of x contained in U_n. Player β responds by selecting a non-empty open subset $U_{n+1}(x)$ in each of the set $V_n(x)$; note that, in general, the set $U_{n+1}(x)$

need not be a neighborhood of x. The above rule of play is repeated in each of the sets player β has chosen on its $n-th$ move.

Player α wins if there exists a sequence $\{x_n\}$ such that:

(i) $x_1 \in U_0$ and $x_{n+1} \in U_n(x_n)$ for each $n = 1, 2, ...,$

(ii) there exists an accumulation point of the sequence $\{x_n\}$.

According to the rules of the game, we say that player α has a *winning strategy* if there exists a function K defined on the set $\bigcup\{\{U\} \times U : U$ is a non-empty open subsets of the space $X\}$ such that:

(a) for each $x \in U$, $K(U, x)$ is an open neighborhood of x contained in U;

(b) player α wins the game if his responses are given by the function K.

The game itself and the rules under which the player α wins the game may be compared with and related to a game considered by J. Christensen [C]. In his game, there are again two players α and β and β starts by choosing a non-empty open set U_1 of X. Then player α chooses an open subset V_1 of U_1 and a point x_1 belonging to V_1. Next β chooses a non-empty open subset U_2 of V_1, and so on. Player α wins in the Christensen game if any subsequence of the sequence $\{x_n : n = 1, 2, ..\}$ accumulates to at least one point of the set $\bigcap\{V_n : n = 1, 2, ...\}$. Otherwise, player β wins.

This game resembles a singular case of our game however it is more restrictive on the winning conditions for the player α. Spaces in which player α has a winning strategy in the Christensen game are called $\sigma - well\ \alpha - favorable$ and they constitute a generalization of Čech-complete spaces. Spaces in which player α has a winning strategy in our game will be called $fuzzy\ \ \alpha - favorable$. From the previous discussion it follows that $\sigma - well\ \alpha - favorable$ spaces are $fuzzy\ \ \alpha - favorable$.

We will exhibit a class of $fuzzy\ \ \alpha - favorable$ spaces (cf. Theorem 4, below).

Definition 1. A space X is said to be a *strict p-space* if there exists a sequence $\{\mathcal{G}_n : n = 1, 2, ...\}$ of open covers of X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for each $n = 1, 2, ..$, and for $x \in X$, the set $P_x = \bigcap\{st(x, \mathcal{G}_n) : n = 1, 2, ...\}$ is compact and $\{st(x, \mathcal{G}_n) : n = 1, 2, ...\}$ is a base of neighborhoods of P_x, i.e., if U is an open neighborhood of the set P_x, then $st(x, \mathcal{G}_n) \subseteq U$ for some n.

Lemma 1. *Let X be a strict $p-$space. If U is an open subset of X, then there exists an open base of U of cardinality not greater than that of U.*

Proof. Let x be a non-isolated point of U. There exists a compact subset F of X such that $x \in F$ and F has a countable base of neighborhoods $\{G_1, G_2, ...\}$.. If $\mathcal{B}$ is a family of open subsets of X such that $\{V \cap F : V \in \mathcal{B}\}$ is a base of x in F, then the family $\{V \cap G_i : V \in \mathcal{B}$ and $i = 1, 2, ...\}$ is a base of x in X. By compactness of F, $\chi(x, F) \leq |U|$. Hence $\chi(x, X) = \chi(x, U) \leq |U| \cdot \omega = |U|$. □

Lemma 2. *If X is a fuzzy $\alpha - favorable$ space, then X is Baire.*

Proof. Let $E_1, E_2, ...$ be nowhere dense subsets of X. Suppose U is a non-empty open subset of X. Let player β start by choosing the set U. Player α plays using his winning strategy, of course, and suppose that $V_n(x)$ is one of the determined choices of player α on its $n-th$ stage. Then we want player β to select such a non-empty open subset $U_{n+1}(x)$ of $V_n(x)$ that $clU_{n+1}(x) \subseteq V_n(x)$ and $U_{n+1}(x) \cap (E_1 \cup E_2 \cup .... \cup E_n) = \emptyset$. If p is an accumulation point of any sequence $\{x_n\}$ guaranteeing player α wins, then $p \in U - (E_1 \cup E_2 \cup ....)$. □

Theorem 3. *If X a Baire strict $p-$space, then X is $fuzzy$ $\alpha - favorable$.*

Proof. Let $\{\mathcal{G}_n : n = 1, 2, ...\}$ be a sequence witnessing that X is strictly $p - space$.

Suppose player β has chosen a non-empty open set U_0. There exists a non-empty open set V_0 of U_0 such that each non-empty open subset of V_0 has the same cardinality as the entire set V_0. We require that α chooses, as one of its responses just the set V_0 , say it has been chosen for a point $x_0 \in V_0$; the other choices by the player α are unessential. Also, we may assume that the set V_0 is infinite. If β chooses again a non-empty open subset $U_1(x_0)$ of $V_0(x_0)$, then this set will have the same cardinal property as V_0 does, so, as the matter of fact, our game could start assuming that cardinal property for U_0, what we shall do for the sequel.

By Lemma 2, there exists an open base $\mathcal{B}$ in U_0 such that $|\mathcal{B}| \leq |U_0|$. If $|U_0| = \kappa$, then let $U_0 = \{x_\alpha : \alpha < \kappa\}$ and let $\{W_\alpha : \alpha < \kappa\}$ be a transfinite sequence consisting only of the members of base $\mathcal{B}$ such that each member of $\mathcal{B}$ is listed κ times in that sequence (this is possible because $\kappa \cdot |\mathcal{B}| = \kappa$). Now we are ready to describe a strategy for player α.

Suppose one of the sets chosen by player β on its $n - th$ stage was U_n. Let $U_n = \{x_{\alpha_\xi} : \xi < \kappa\}$. If $\xi < \kappa$ and $V_n(x_{\alpha_\eta})$ has been defined for each $\eta < \xi$, then $V_n(x_{\alpha_\xi}) = W_\gamma$, where γ is the least index among the indices of sets W_β satisfying the following conditions:

(1) $x_{\alpha_\xi} \in W_\beta \subseteq U_n$,

(2) W_β is a subset of some member of $\mathcal{G}_n$,

(3) W_β is not among those sets chosen previously for points x_{α_η}, where $\eta < \xi$.

Such a γ exists; if W is a member of $\mathcal{B}$ that is contained in a member of the family $\mathcal{G}_n$ and $x_{\alpha_\xi} \in W \subseteq U_n$, then this set appears κ times in the sequence $\{W_\alpha : \alpha < \kappa\}$ and therefore, the previous choices have not exhausted completely this element yet. Consequently, there exists a $\eta < \kappa$ such that conditions (1), (2) and (3) are satisfied simultaneously by W_η.

We shall show that this is a winning strategy for player α.

It follows from conditions (1), (2), (3) that the sets $\{V_n(x_{\alpha_\xi}) : \xi < \kappa\}$, chosen by player α in response to a move U_n by player β, form a base in U_n. Indeed, if G is a non-empty open subset of U_n and $x \in G$, then there exists a member W of $\mathcal{B}$ that is contained in a member of $\mathcal{G}_n$ and such that $x \in W \subseteq G$. Because there are κ points in W and the set W appears κ times in the sequence $\{W_\alpha : \alpha < \kappa\}$, by virtue of the property (3), this set must be chosen by α as one of the sets $V_n(x_{\alpha_\xi})$. As a consequence, if player β chose sets $U_{n+1}(x_{\alpha_\xi})$, $\xi < \kappa$, in response to the choices $V_n(x_{\alpha_\xi})$, $\xi < \kappa$, by player α, then those sets form a $\pi - base$ in U_n.

Let $\mathcal{R}_1$ be a maximal family consisting of pairwise disjoint sets of the form $U_1(x)$. Then $\bigcup \mathcal{R}_1$ is a dense subset of U_0. Let us suppose that a family $\mathcal{R}_n$ has been already defined. If $R \in \mathcal{R}_n$ and $R = U_n(x)$, then we set $\mathcal{R}_{n+1}(R)$ to be a maximal family consisting of pairwise disjoint sets chosen among sets of the form $U_{n+1}(z)$ which are responses of player β on selections $V_n(z)$, $z \in U_n(x)$, by player α following its strategy. Hence $\mathcal{R}_{n+1}(R)$ is a dense subset of the set R. We set $\mathcal{R}_{n+1} = \bigcup\{\mathcal{R}_{n+1}(R) : R \in \mathcal{R}_n \text{ and } R = U_n(x) \text{ for some } x\}$.

Note the following properties of families $\mathcal{R}_n$ (that are immediate consequences of their inductive definition):

(a) the family $\mathcal{R}_n$ is a refinement of the family $\mathcal{G}_n$,

(b) $\mathcal{R}_n$ is a disjoint family,

(c) $\mathcal{R}_{n+1}$ is a refinement of $\mathcal{R}_n$,

(d) $\bigcup \mathcal{R}_n$ is a dense subset of U_0.

Since X is Baire, by (d), there exists a point p from U_0 belonging to each of the sets $\bigcup \mathcal{R}_n$, $n = 1, 2, \ldots$ By (b), there exists exactly one member R_n in each $\mathcal{R}_n$ containing the point p. Thus each set R_n is of the form $U_n(x_n)$. By (c), $x_{n+1} \in U_{n+1}(x_{n+1}) \subseteq U_n(x_n)$. Since the set $P_p = \bigcap\{st(p, \mathcal{G}_n) : n = 1, 2, \ldots\}$ is compact and $\{st(x, \mathcal{G}_n) : n = 1, 2, \ldots\}$ is a base of neighborhoods of P_p, the sequence $\{x_n\}$ must have an accumulation point in the set P_p. In fact, the same conclusion holds for any subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$. This means that the player α won the game while playing by its strategy. □

Following J.S. Raymond [SR], we say that a space X is $\sigma - well\ \beta - unfavorable$ if player β cannot have a winning strategy in the Christensen game.

Proposition 4. *Any fuzzy $\alpha-favorable$ space is $\sigma-well\ \beta-unfavorable$.*

Proof. Suppose to the contrary that it is possible to have a Baire space X that is both *strict* $p-space$ and $\sigma - well\ \beta - favorable$. Let L be a winning strategy for player β in the Christensen's game and let K be a winning strategy for player α playing our game. Now the two players α and β are going to play our game in such a way that the moves of player α are done according to the strategy K, and the moves of player β are done according to the strategy L provided that a set $U(x)$ chosen by player α is treated as the pair (U, x) in Christensen's game. A sequence $\{x_n\}$, witnessing that α won in our game, contradicts the fact that L is a winning strategy for player β in the Christensen game. □

Following R. Hansell, J. Jayne, and M. Talagrand [HJT], a space X is said to be a *Namioka space* if for every compact space Y, for every metric space M, and for every separately continuous function $f : X \times Y \to M$ there exists a dense G_δ subset A of X such that f is continuous at each point of $A \times Y$. X is said to be a *special Namioka space* if each perfect irreducible preimage of the space X is a Namioka space (let us recall that a function is *perfect* if it is continuous, closed and the preimage of each singleton is compact; it is *irreducible* if there does not exist a proper closed subset of the domain whose image is the entire range). Metric Baire spaces are special Namioka (see [SR]).

Theorem 5. *Any Baire strict $p-space$ is special Namioka.*

Proof. A perfect preimage of a *strict* $p-$ *space* is a *strict* $p-$ *space* (proof of this fact is straightforward and therefore omitted here). An irreducible preimage of a Baire space is a Baire space (proof of this fact is also straightforward and therefore omitted here). In consequence, a perfect and irreducible preimage of a Baire *strict* $p-$ *space* is a Baire *strict* $p-$ *space*. Since $\sigma-$ *well* $\beta-$ *unfavorable* spaces are Namioka (cf. [SR]) and Baire *strict* $p-$ *spaces* are $\sigma-$ *well* $\beta-$ *unfavorable* (cf. Theorem 6), Baire *strict* $p-$ *spaces* are special Namioka. □

Definition 2. A space X is said to be a *Moore space* if there exists a sequence $\{\mathcal{Q}_n : n = 1, 2, ...\}$ of open covers of X such that $\mathcal{Q}_{n+1}$ refines $\mathcal{Q}_n$ for each $n = 1, 2, ..$, and for $x \in X$, the family $\{st(x, \mathcal{Q}_n) : n = 1, 2, ...\}$ is a base of neighborhoods of the singleton x.

It is obvious that any Moore space is a *strict* $p-$ *space* but not vice versa. A good account on Moore spaces and *strict* $p-$ *spaces* can be found in [G].

It should be pointed out that *strict* $p-$ *spaces* are referred to as $p-$ *spaces* in Bouziad's papers (cf. [B1], [B3]).

Theorem 6. *Let X be a Baire strict $p-$ space, let Y be a Namioka space, and let Z be a Baire Moore space. If $f : X \times Y \to Z$ is separately continuous, then for each $x \in X$ there exists a dense G_δ subset D_x of Y such that the function f is jointly continuous at each point (x, y), where $y \in D_x$.*

Proof. Let $\{\mathcal{G}_n\}$ be a sequence witnessing that X is a *strict* $p-$ *space* and let $\{\mathcal{Q}_n\}$ be a sequence witnessing that Z is a Moore space. We set

$\omega(f; \mathcal{Q}_n) = \{(x, y)$: there exists a W in $\mathcal{Q}_n$ and there exists an open neighborhood N of (x, y) such that $f(N) \subseteq W\}$.

Each of the sets $\omega(f; \mathcal{Q}_n)$ is open in $X \times Y$. We shall show that each of these sets is dense in the subspace $\{x\} \times Y$, for every $x \in X$. For this purpose let us take a non-empty open set

V in the space Y. We will construct an open neighborhood U of x and a non-empty open subset V' of V such that $U \times V' \subseteq \omega(f; \mathcal{Q}_n)$.

Let y be an arbitrary point of V and let W be a member of the cover $\mathcal{Q}_n$ containing the point $f(x,y)$. Let W' be an open set such that $f(x,y) \in W' \subseteq clW' \subseteq W$. By continuity of the partial function f_x , there exists an open subset V_1 of V containing y and such that $f(\{x\} \times V_1) \subseteq W'$. The set $E_x = \bigcap\{st(x, \mathcal{G}_n) : n = 1, 2, ...\}$ is compact and contains the point x. Since the space Y is Namioka, there exists a dense G_δ subset D of Y such that the restriction of the function f to the subspace $E_x \times Y$, $f \mid E_x \times Y$, is continuous at each point of $E_x \times D$. Let $y_1 \in D \cap V_1$. Then $f \mid E_x \times Y$ is continuous at the point (x, y_1). Since $f(x, y_1) \in W'$, there exist an open neighborhood U_1 of x and an open neighborhood V_2 of y_1 such that $V_2 \subseteq V_1$ and $f((U_1 \cap E_x) \times V_2) \subseteq W'$. The family $\{st(x, \mathcal{G}_n) : n = 1, 2, ...\}$ forms a countable base of neighborhoods of the set E_x in the space X. If we take arbitrary closed G_δ subset of E_x, it will also have a countable base of neighborhoods in the space X. Let us select such a set, say H, which, in addition, contains x and is contained in U_1. Let $\mathcal{B}$ be a countable base of neighborhoods of the set H in the space X. If B is a member of $\mathcal{B}$, then $F_B = \{y \in V_2 : f(B \times \{y\}) \subseteq W'\}$.

Clearly, the sets F_B, $B \in \mathcal{B}$, cover the whole set V_2. So one of them, say F_U, is dense in a non-empty open subset V' of V_2. In consequence, $f(U \times V') \subseteq clW' \subseteq W$, which shows that $U \times V' \subseteq \omega(f; Q_n)$.

For $x \in X$, let D_x consist of all those points y from Y such that (x, y) belongs to each set $\omega(f; \mathcal{Q}_n)$. The sets D_x are dense G_δ in the space Y. Let us show that f is jointly continuous at each point (x, y), where $y \in D_x$.

Let W be an arbitrary open set in Z containing a point $f(x, y)$, $y \in D_x$. For each n there exist a $W_n \in \mathcal{Q}_n$ and an open neighborhood N_n of the point (x, y) such that $f(N_n) \subseteq W_n$. Since $\{\mathcal{Q}_n\}$ is a development in the space Z, $\{W_n\}$ is a local base at the point $f(x, y)$. Hence $W_m \subseteq W$ for some m and therefore $f(N_m) \subseteq W$. □

Theorems of similar type as our Theorem 6 were also obtained by A.Bouziad (cf.[B1; Proposition 3.6]) and by N.Martin and Z.Piotrowski (cf.[P]). The following example was suggested by the referee to show some limitations of our Theorem 8. The example itself is attributed to J.B.Brown.

Example 1. Let $X = \bigoplus_{x \in [0,1]} [0,1] \times \{x\}$ be the disjoint union of unit segments, $Y = [0,1]$, and let f be any real functions on $X \times Y$ such that f is separately continuous on the square $[0,1] \times \{x\} \times [0,1]$ and has a point of discontinuity in $[0,1] \times \{x\} \times \{x\}$, for each $x \in [0,1]$. Then f is separately continuous on the locally compact metric space $X \times Y$ but there is no a dense G_δ subset A of Y such that $X \times A \subseteq C(f)$. □

Corollary 7. *(A.Bouziad [B2]). Let $(G;+)$ be an algebraic group endowed with a topology which is Baire and Moore and such that the group operation $+ : G \times G \to G$ is separately continuous. Then the group operation $+$ is jointly continuous. That is, any semitopological group that is Baire and Moore is a paratopological group.*

Let $(G;\cdot)$ be a multiplicative algebraic group. The group G is said to be *acting* on a set Y if there is a function $A : G \times Y \to Y$, called *action*, so that the following conditions are satisfied:

(a) $A(1, y) = y$ for each $y \in Y$;

(b) $A(g \cdot h, y) = A(g, A(h, y))$ for all g, h in G and y in Y;

(c) If y, z are in Y, then there exists a g in G such that $A(g, y) = z$.

Corollary 8. *(A.Bouziad [B3]). Let an abelian group G act on the set Y and let A be an action. Suppose G and Y are endowed with topologies such that:*

(i) G is a Baire *strict* $p-space$,
(ii) the group operation $\cdot$ is separately continuous,
(iii) Y is a Baire Moore space,
(iv) the action $A : G \times Y \to Y$ is separately continuous.
Then the action A is jointly continuous.

3. Namioka Theorem

In Proposition 5, we showed that $fuzzy\ \alpha - favorable$ spaces are $\sigma-well\ \beta-unfavorable$. Thus $fuzzy\ \alpha-favorable$ spaces are Namioka by Saint Raymond's theorem. Nevertheless, we want to offer a direct proof that $fuzzy\ \alpha - favorable$ spaces are Namioka because our proof is quite elementary.

Theorem 9. *Let $f : X \times Y \to M$ be a separately continuous function from the product of a $fuzzy\ \alpha - favorable$ space X and a compact space Y into a metric space M. Then there exists a dense G_δ subset A of X such that the function f is jointly continuous at each point of the set $A \times Y$.*

This is a generalization of the original Namioka theorem to the class of $fuzzy\ \alpha - favorable$ spaces. The basic idea of our proof is similar to Namioka's one (cf. [N]).

We need some elementary facts first. For the sake of completeness we include their proofs.

Lemma 10. *Let F be a closed subset of a product $X \times Y$. If Y is a compact space and V is an open subset of Y, then the set $V_F = \{x \in X : F_x \subseteq V\}$ is an open subset of X.*

Proof. The set $E = F - X \times V$ is closed in $X \times Y$. Since Y is compact, $\pi_X(E)$ is closed in X. Hence the set V_F is open in X being the complement of the set $\pi_X(E)$. □

Lemma 11. *Let $f : X \times Y \to M$ be a function into the metric space (M, d) such that f^y is continuous for each $y \in Y$.*

Let δ be a positive number such that the set $D = \{x \in X : diam(f(\{x\} \times Y)) \leq \delta\}$ is dense in the space X. Then for each $\varepsilon > 0$ and for each $(x, y) \in X \times Y$ there exists a neighborhood W of the point (x, y) such that $diam f(W) < \delta + \varepsilon$.

Proof. Suppose to the contrary that there exist (a, b) and $\varepsilon > 0$ such that $diam f(W) \geq \delta + \varepsilon$ for each neighborhood W of the point (a, b). Since the function f^y is continuous, there exists an open neighborhood U of the point a such that $d(f(a, b), f(x, b)) < \frac{\varepsilon}{3}$ for each $x \in U$. There exists a point $(p, q) \in U \times Y$ such that $d(f(a, b), f(p, q)) > \delta + \frac{2\varepsilon}{3}$. There exists an open neighborhood V of the point p such that $V \subseteq U$ and $d(f(p, q), f(x, q)) < \frac{\varepsilon}{3}$ for each $x \in V$. Let $z \in D \cap V$. Then $\delta + \frac{2\varepsilon}{3} < d(f(a, b), f(p, q)) \leq d(f(a, b), f(z, b)) + d(f(z, b), f(p, q)) \leq \frac{\varepsilon}{3} + d(f(z, b), f(z, q)) + d(f(z, q), f(p, q)) \leq \delta + \frac{2\varepsilon}{3}$; a contradiction. □

Lemma 12. *Let $f : X \times Y \to M$ be a function into the metric space (M, d) such that f^y is continuous for each $y \in Y$. Let $\delta > 0$, $F \subseteq Y$, and $a \in X$ be such that $diam f(\{a\} \times F)) \geq \delta$. If $\varepsilon < \delta$, then the set $\{x \in X : diam f(\{x\} \times F)) \leq \varepsilon\}$ does not contain the point a in its closure.*

Proof. Suppose to the contrary that it is not the case. There exist u, v in F such that $d(f(a, u), f(a, v)) > \delta - \frac{\delta-\varepsilon}{3}$. There exists a neighborhood U of the point a such that $d(f(a, u), f(x, u)) < \frac{\delta-\varepsilon}{3}$ and $d(f(a, v), f(x, v)) < \frac{\delta-\varepsilon}{3}$ for each $x \in U$. By our assumption, there exists $z \in U$ such that $diam(f(\{z\} \times F)) \leq \varepsilon$. Hence $d(f(a, u), f(a, v)) \leq d(f(a, u), f(z, u)) + d(f(z, u), f(z, v)) + d(f(z, v), f(z, v)) < \frac{\delta-\varepsilon}{3} + \varepsilon + \frac{\delta-\varepsilon}{3} = \frac{2\delta+\varepsilon}{3}$. On the other hand, $d(f(a, u), f(a, v)) \geq \delta - \frac{\delta-\varepsilon}{3} = \frac{2\delta+\varepsilon}{3}$; a contradiction. □

Proof of the Theorem. Let $\Omega_\varepsilon(f) = \{p \in X : \omega(f; p) \geq \varepsilon\}$. In order to prove our theorem it is enough to show that for each positive number ε the set $\pi_X(\Omega_\varepsilon(f))$ is nowhere dense in

X for, then, $\omega(f;p) = 0$ for each
$p \in \left(X - \bigcup\{\pi_X(\Omega_{\frac{1}{n}}(f)) : n = 1, 2, ..\}\right) \times Y$.

Assume to the contrary that for some $r > 0$ the set $\pi_X(\Omega_r(f))$ is not nowhere dense in X. Since Y is compact and $\Omega_r(f)$ is closed in $X \times Y$, the projection $\pi_X : X \times Y \to X$ is a perfect map as is the restriction of π_X to the set $\Omega_r(f)$. Hence $\Omega_r(f)$ contains a closed subset, say F, such that $\pi_X|F$ is irreducible and there exists a non-empty open set $U \subseteq \pi_X(F)$. It follows that

(*) if $W \subseteq X \times Y$ is open and $U \cap \pi_X(W \cap F) \neq \emptyset$, then the interior of that set is non-empty too.

We will use (*) in a description of moves of player β in our game. Let us fix δ and ε so that $0 < \varepsilon < \delta < r$.

The first set player β starts with is the set U. Then player α responds, using its winning strategy, by choosing open sets $V_0(x)$ for each $x \in U$. As $x \in V_0(x) \subseteq U$, there exists $y_x \in Y$ such that $(x, y_x) \in F$. By continuity of the function f_x, there exists an open neighborhood $W_0(x)$ of the point y_x such that $diam(f(\{x\} \times W_0(x))) < \varepsilon$. By (*), there exists a non-empty open set $G_0(x) \subseteq \pi_X(F \cap (V_0(x) \times W_0(x)))$. For each $z \in G_0(x)$ there exists $y_z \in W_0(x)$ such that $\omega(f; (z, y_z)) \geq r$. Therefore the set $L = \{z \in G_0(x) : diam(f(\{z\} \times W_0(x))) \leq \delta\}$ is nowhere dense, by virtue of Lemma 12. A set $U_1(x)$ player β chooses in the set $V_0(x)$ is the one that is non-empty, open, contained in $G_0(x)$ and disjoint from the set L.

For the next purposes let us display explicitly a property the chosen set $U_1(x)$ possesses. Namely,

(**) there exists an open set $W_0(x)$ in Y such that for each z in $U_1(x)$ we can find y_z in $W_0(x)$ such that $(z, y_z) \in F$ and yet for each $z \in U_1(x)$, $diam(f(\{z\} \times W_0(x))) > \delta$.

Suppose U_n is one of open sets player β has chosen on the $n - th$ stage. Along with this set, there has also been defined an open set W_{n-1} in Y that satisfies (**). Player α, using its winning strategy, chooses $V_n(x)$ for each $x \in U_n$. If

$x \in U_n$, then let $y_x \in W_{n-1}$ be such that $(x, y_x) \in F$. By the continuity of the function f_x, there exists an open neighborhood $W_n(x)$ of the point y_x such that $clW_n(x) \subseteq W_{n-1}$ and $diam(f(\{x\} \times W_n(x))) < \varepsilon$. By (*), there exists a non-empty open set $G_n(x) \subseteq \pi_X (F \cap (V_n(x) \times W_n(x)))$. Since for each $z \in G_n(x)$ there exists $y_z \in W_n(x)$ such that $\omega(f; (z, y_z)) \geq r$, the set

$L = \{z \in G_n(x) : diam(f(\{z\} \times W_n(x))) \leq \delta\}$ is a nowhere dense subset of the space X, by virtue of Lemma 12. The set $U_n(x)$ player β chooses in the set $V_n(x)$ is the one that is non-empty, open, contained in $G_n(x)$ and disjoint from the set L. Note that the set $W_n(x)$ together with $U_{n+1}(x)$ satisfies the property (**).

Because player α has used its winning strategy in the game, there exists a sequence $\{x_n : n = 1, 2, ..\}$ such that $x_1 \in U$ and $x_{n+1} \in U_n(x_n)$ for each n, and it has an accumulation point, say p. Let us consider also the sequence$\{W_n(x_n)\}$ associated with the sequence $\{U_{n+1}(x_{n+1})\}$. If $W = \bigcap\{W_n(x_n) : n = 1, 2, ..\}$, then, by virtue of Lemma 13, $diam(f(\{p\} \times W)) \leq \varepsilon$. From the other hand side, since p is a point of the set $\bigcap\{U_n(x_n) : n = 1, 2, ..\}$, $diam(f(\{p\} \times W_n(x_n))) \geq \delta$ for each n. Since Y is compact and $clW_{n+1}(x_{n+1}) \subseteq W_n(x_n)$, $diam(f(\{p\} \times W)) \geq \delta$, which contradicts the preceding inequality.

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