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PERIODIC POINTS FROM PERIODIC PRIME ENDS

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ABSTRACT. This paper generalizes a theorem of Barge and Gillette asserting that if an orientation preserving plane homeomorphism F has a fixed prime end associated with an invariant continuum Δ which separates the plane into exactly two domains, then F has a fixed point in Δ. The generalization goes in two directions. The Barge-Gillette theorem is proved for a continuum with more than two complementary domains if for all but one complementary domains U, F has a fixed prime end in U. The other generalization addresses the existence of periodic points with least period q provided certain conditions concerning F^q and periodic prime ends with the same least period are met.

1. Introduction

M. Barge and R. Gillette proved in [3] that if an orientation preserving plane homeomorphism F has a fixed prime end associated with an invariant continuum Δ that separates the plane into exactly two domains, then F has a fixed point in Δ. We

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expand on the ideas in [3] and extend the theorem of Barge and Gillette to the case when Δ has finitely many invariant complementary domains and F has a fixed prime end in all but possibly one of the complementary domains. We also provide an answer to the question posed in [3] as to whether periodic prime ends imply periodic points of the same least period.

Similarly as in [3], throughout this paper, F is an orientation preserving homeomorphism of the plane $\mathbb{R}^2$ and Δ is a continuum contained in $\mathbb{R}^2$ that is invariant under F. Contrary to [3], we do not assume that F separates $\mathbb{R}^2$ into exactly two components.

We refer the reader to [3] for the description of the following well-known notions: a prime end, the prime end compactification, the principal set of a prime end, a fixed prime end, and the index $\mathrm{i}(G;\gamma)$ of an orientation preserving homeomorphism G of $\mathbb{R}^2$ around a simple closed curve γ on which the homeomorphism is fixed point free. We also rely on the methods of [3], in particular on the **Proof of Theorem 1** on page 209 from which the following lemma can be extracted:

Lemma 1. *Suppose that F is fixed point free on Δ and U is an invariant bounded complementary domain of Δ. Then there is a simple closed curve $\gamma \subset U$ such that F has no fixed points between γ and Δ or on γ, and*

1. *if F has no fixed prime ends associated with Δ in U, then $\mathrm{i}(F;\gamma) \leq 1$;*
2. *if F has a fixed prime end associated with Δ in U and the intersection of the principal sets of all fixed prime ends in U is non-empty, then $\mathrm{i}(F;\gamma) \leq 0$.*

Remark 1. *A similar conclusion can be drawn for the unbounded complementary domain considered as a bounded domain in the one-point compactification of the plane.*

The results in [3] and in this paper strongly depend on the Cartwright-Littlewood theorem asserting that if an orientation preserving homeomorphism of the plane maps a non-separating plane continuum onto itself, then the homeomorphism possesses a fixed point in this continuum, see [5]. H. Bell [4] proved the Cartwright-Littlewood theorem for orientation reversing homeomorphism of the plane.

2. Fixed prime ends

Theorem 1. *If Δ has finitely many complementary domains $U_1, \dots, U_n$ and F has a fixed prime end in each of the domains $U_1, \dots, U_{n-1}$, then F has a fixed point in Δ.*

Proof. If $n = 1$, the above is the Cartwright-Littlewood theorem; if $n = 2$, it is the Barge-Gillette theorem.

Let $\mathcal{X}$ be the set of all pairs (X, G), where G is an orientation preserving homeomorphism of $\mathbb{R}^2$ and X is a continuum invariant under G with finitely many complementary domains such that

1. G has a fixed prime end in all but possibly one of the complementary domains;
2. G has no fixed point in X.

Suppose that $\mathcal{X}$ is non-empty. Let $(X', G') \in \mathcal{X}$ be a pair with the minimum number of complementary domains $V_1, \dots, V_k$, and G' has a fixed prime end in each of the domains $V_1, \dots, V_{k-1}$.

Let K be an invariant subcontinuum of X' and let W be a complementary domain of K intersecting a domain V_i for some $i = 1, \dots, k$. Then $X' - W$ is an invariant continuum with no more than k complementary domains. To see that $X' - W$ is connected, note that $\mathrm{Bd}(W)$ is a connected subset of $X' - W$, and it intersects every component of $X' - W$. Since V_i and K are invariant, so are W and $X' - W$. Each complementary domain of $X' - W$ contains at least one of the sets $V_1, \dots, V_k$. So the number of complementary domains of $X' - W$ does not exceed k. Moreover, the set of the complementary domains of $X' - W$ consists of the domain containing W and some of the sets $V_1, \dots, V_k$.

Every set $\mathrm{Bd}(V_j)$ is an invariant continuum. For $j = 1, \dots, k - 1$, let W_j be the complementary domain of $\mathrm{Bd}(V_j)$ which intersects V_k. Let $Y_j = X' - W_j$. Similarly as in Lemma 3 in [3], we can show that for every fixed prime end P in V_j, the principal set $\mathrm{Pr}(P)$ contains the boundary of V_j. If $\mathrm{Bd}(V_j)$ is not contained in $\mathrm{Pr}(P)$, then $\mathrm{Pr}(P)$ does not separate V_j from V_k. Let W' be the complementary domain of $\mathrm{Pr}(P)$ that intersects V_k as well as V_j. Then $(X' - W', G')$ is an element of $\mathcal{X}$ with $X' - W'$ having fewer than k complementary domains. Therefore all principal sets of fixed prime ends in V_j intersect. By Lemma 1, there are simple closed curves $\gamma_j \subset V_j$ such that $\mathrm{i}(G'; \gamma_k) \leq 1$, $\mathrm{i}(G'; \gamma_j) \leq 0$ for $j = 1, \dots, k-1$, and G' has no fixed points between γ_j's and X', or on γ_j's. Hence

$$\sum_{j=1}^{k} \mathrm{i}(G'; \gamma_j) \leq 1$$

which is a contradiction. The above sum should be 2, the Euler characteristic of S^2. □

The restriction that Δ has finitely many complementary domains all of which are invariant is immaterial. The bounded non-invariant complementary domains as well as the invariant complementary domains in which F has no fixed points can be added to Δ without changing the number of fixed points in the continuum. If F has a fixed point in infinitely many (invariant) complementary domains of Δ, then F has a fixed point in Δ. Hence Theorem 1 yields the following:

Corollary 1. *If F has a fixed prime end in all but one of the complementary domains of Δ containing a fixed point of F or unbounded, then F has a fixed point in Δ.*

In [6] and [7], the author gives estimates for the number of fixed points in a plane continuum invariant under an orientation reversing homeomorphism of $\mathbb{R}^2$. One could ask a similar question for orientation preserving homeomorphisms:

Question 1. *Under what conditions does an orientation preserving homeomorphism of the plane possess more than one fixed point in a given invariant continuum ?*

Definition 1. *Plane continua $X_1, ..., X_n$ are separated, if there are pairwise disjoint disks, $D_1, ..., D_n$ with $X_i \subset D_i$.*

The following corollary of Theorem 1 provides a rather trivial answer to Question 1:

Corollary 2. *If F has a fixed prime end in all bounded complementary domains of Δ which contain a fixed point, and there are k mutually separated invariant continua in Δ, then F has at least k fixed points in Δ.*

Remark 2. *An easy example of a continuum consisting of n simple closed curves with one common point a, and a plane homeomorphism under which each of the simple closed curves is invariant, but having only one fixed point, the point a, shows that the assumption of "separated" is essential.*

3. Periodic prime ends

Definition 2. *Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism and let X be a continuum invariant under G. A prime end P associated with X is periodic, if there is a positive integer $q \geq 1$ such that P is a fixed prime end associated with X of G^q. The smallest such positive integer q is the least period of P. A periodic prime end P associated with X is in the complementary domain U of X, if U is invariant under G, and for some integer q, P is a fixed prime end associated with X of G^q in U.*

Under similar conditions as in Theorem 1 or Corollary 1, a periodic prime end associated with Δ, clearly implies a periodic point in Δ. In [3], the authors ask whether the existence of a periodic prime end of least period q implies the existence of a periodic point of least period q.

Theorem 2. *Suppose that Δ has finitely many complementary domains $U_1, \ldots, U_n$ and $n \geq 2$. If*

1. *F has a periodic prime end associated with Δ in U_1 of least period q, and*

2. *for* $1 < i < n$*, if* U_i *is invariant under* F^q*, then* F^q *has a fixed prime end associated* Δ *in* U_i*,*
3. U_n *is invariant,*

then F *has a periodic point in* Δ *of least period* q*.*

Proof. F can be modified away from the continuum Δ so that F has a fixed point in each of the invariant complementary domains. Without loss of generality, we may assume that $F(a) = a$ for some $a \in U_1$ and that U_n is the unbounded component. Let $\widehat{U}_1$ be the prime end compactification of U_1 and let $\psi : U_1 - \{a\} \to \widehat{U}_1 - \{a\}$ be the natural embedding. There is a map $\widehat{F} : \widehat{U}_1 - \{a\} \to \widehat{U}_1 - \{a\}$ so that for $x \in U_1 - \{a\}$, we have $\psi(F(x)) = \widehat{F}(\psi(x))$. The periodic prime end P is a point in $\widehat{U}_1 - \{a\}$ of least period q. $\widehat{F}$ restricted to the boundary circle of $\widehat{U}_1 - \{a\}$ has the rotation number $\frac{p}{q} > 0$, where p and q are relatively prime. (More about the rotation number and periodic prime ends can be found in [2].)

Let $\widetilde{A}$ be the q-fold covering space of the annulus $A = \mathbb{R}^2 - \{a\}$ and let $\pi : \widetilde{A} \to A$ be the projection. Similarly, let $\widetilde{B}$ be the q-fold covering space of $\widehat{U}_1 - \{a\}$, and let $\widehat{\pi} : \widetilde{B} \to \widehat{U}_1 - \{a\}$ be the projection. The embedding $\psi : U_1 - \{a\} \to \widehat{U}_1 - \{a\}$ lifts to an embedding $\widetilde{\psi} : p^{-1}(U_1 - \{a\}) \to \widetilde{B}$. Thus the diagram

$$\begin{array}{ccccc} \pi^{-1}(U_1 - \{a\}) & \xrightarrow{\pi|\pi^{-1}(U_1 - \{a\})} & U_1 - \{a\} & \xrightarrow{F} & U_1 - \{a\} \\ \downarrow{\scriptstyle \widetilde{\psi}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \widetilde{B} & \xrightarrow{\widehat{\pi}} & \widehat{U}_1 - \{a\} & \xrightarrow{\widehat{F}} & \widehat{U}_1 - \{a\} \end{array}$$

commutes.

The map $\widehat{F}$ lifts to a map $\Gamma : \widetilde{B} \to \widetilde{B}$, whose rotation number on $\partial\widetilde{B}$ is $\frac{p}{q^2}$. Let T and $\widehat{T}$ be deck transformations of $\widetilde{A}$ and $\widetilde{B}$ respectively, with the rotation numbers of $T|\partial\widetilde{A}$ and $\widehat{T}|\partial\widetilde{B}$ equal $\frac{1}{q}$. Hence the map $\widehat{T}^{-p} \circ \Gamma^q$ has a fixed point P'.

Let C be the plane obtained from $\widetilde{A}$ by adding a point $\tilde{a}$ which compactifies the closure (in $\widetilde{A}$) of the set $\pi^{-1}(U_1 - \{a\})$. Let $H : C \to C$ be the extension of $T^{-p} \circ \widetilde{F}^q : \widetilde{A} \to \widetilde{A}$ to C. Thus P' is a fixed prime end of the homeomorphism H in the domain $V = \pi^{-1}(U_1 - \{a\}) \cup \{\tilde{a}\}$. If W is an invariant bounded complementary domain of $\pi^{-1}(\Delta)$ different from V, then $\pi(W) = U_i$ for some $2 \le i \le n-1$. By assumption 2, H has a fixed prime end in W. By Theorem 1, H has a fixed point c in $\pi^{-1}(\Delta)$. Let $b = \pi(c)$. Clearly, $F^q(b) = b$. It remains to show that if $F^r(b) = b$ for some divisor r of q, then $r = q$.

Suppose that $x \neq a$ is a periodic point of F with least period r, where $q = rs$, and $\tilde{x} \in \pi^{-1}(x)$ is a fixed point of H. We have $\widetilde{F}^r(\tilde{x}) \in \pi^{-1}(x)$ or equivalently $\widetilde{F}^r(\tilde{x}) = T^k(\tilde{x})$ for some integer $k \ge 0$. Then $\widetilde{F}^{rs}(\tilde{x}) = T^{ks}(\tilde{x})$. If $T^{-p}(T^{ks}(\tilde{x})) = \tilde{x}$, then $T^{ks}(\tilde{x}) = T^p(\tilde{x})$ and $ks \bmod q = p$. So p and q have a common divisor s, and since p and q are relatively prime, then $s = 1$. Hence $r = q$ is the least period of b. □

Assumption 2 of Theorem 2 can be relaxed to the consideration of fixed prime ends of $T^{-p} \circ F^q$ in the components of $\pi^{-1}(U_i)$ invariant under $T^{-p} \circ F^q$. In addition, Theorem 2 can be generalized to Δ with infinitely many complementary domains in a similar fashion as in Corollary 1. Therefore, if T is defined as in the above proof, we have:

Corollary 3. *Suppose that Δ has at least two invariant complementary domains U and V and F possesses a periodic prime end associated with Δ in U of least period q. If for all of the complementary domains W, different from U and V, the invariance of a component of $\pi^{-1}(W)$ under $T^{-p} \circ F^q$ implies the existence of a fixed prime end associated with Δ of $T^{-p} \circ F^q$ in this component, then F has a periodic point in Δ of least period q.*

The examples below illustrate the necessity of the assumptions of Theorem 2.

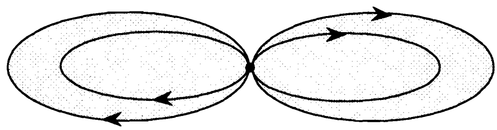

FIGURE 1. Non-separating continuum.

The example in Figure 1 shows that the invariant complementary U_n, $n \neq 1$ is needed. The pictured continuum Δ does not separate the plane. The homeomorphism F is composition of a rotation about the center point x of Δ through π and a movement along the ellipses tangent at x so that F^2 has only one fixed point x. There are two periodic prime ends of least period 2, but there are no periodic points of least period 2.

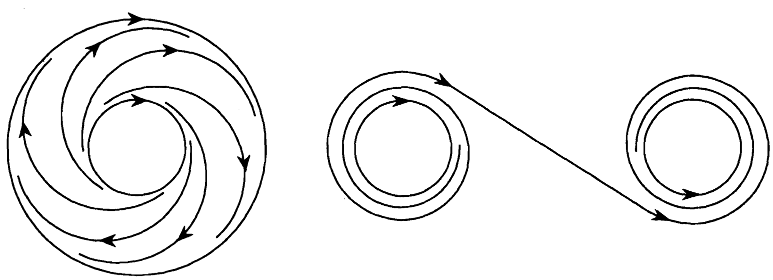

FIGURE 2. No periodic points.

Figure 2 shows two examples without periodic points. The continuum to the left consists of two concentric circles and k spirals winding on the two circles. A homeomorphism rotating the circles, whose rotation number is irrational on each of the circles and periodically permuting the domains between the two circles, has periodic prime ends. If in addition the homeomorphism moves the points on the spirals towards the outside circle, then it has no periodic points in the continuum. Assumption 1 of Theorem 2 is not satisfied, there are no periodic prime ends in any of the invariant complementary domains. The continuum to the right has three complementary domains:

two disks and the unbounded domain. The disks rotate about their centers through an irrational angle. The continuum consists of the two boundaries of the disks and a spiral whose ends wind around the disks. The example is similar to the example described in [3]. There are two fixed prime ends in the unbounded domain, but there are no periodic points in the continuum.

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