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A UNICOHERENT CONTINUUM WHOSE SECOND SYMMETRIC PRODUCT IS NOT UNICOHERENT

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ABSTRACT. In this paper we show an example of a unicoherent continuum whose second symmetric product is not unicoherent. This answers a question by A. García-Máynez and A. Illanes.

1. INTRODUCTION

A connected topological space X is said to be *unicoherent* provided that whenever A and B are closed, connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected. A metric, compact, connected space is a *continuum.* For a given positive integer n and a continuum X, denote by $F_n(X)$ the space consisting of all nonempty subsets of X, having at most n elements, with the Vietoris topology (See [6, 0.12]). This is the so called n^{th}*-symmetric product* of X. Symmetric products were introduced by K. Borsuk and S. Ulam in [1]. They posed the following question: If X is a Peano unicoherent continuum, is it true that $F_n(X)$ is unicoherent? This question

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was answered affirmatively by T. Ganea in [2] who proved, the following theorem.

Theorem 1.1. [2] *If X is a connected, locally connected, unicoherent Hausdorff space, Then $F_n(X)$ is unicoherent for all n.*

For each topological space Y, we define

$$b_0(Y) = (\text{number of components of } Y) - 1$$

if this number is finite, and $b_0(Y) = \infty$ otherwise. The *multicoherence degree, $r(X)$,* of a connected space X is defined by:

$$r(X) = sup\{b_0(H \cap K) : H \text{ and } K \text{ are closed connected subsets of } X \text{ and } X = H \cup K\}$$

Observe that X is unicoherent if and only if $r(X) = 0$. A continuum is said *multicoherent* provided that $r(X) \neq 0$.
In [4] A. Illanes proved the following results.

Theorem 1.2. [4, thm. 2.6] *If X is pathwise connected, locally connected and Hausdorff, then $F_n(X)$ is unicoherent for all $n \geq 3$.*

Theorem 1.3. [4, thm. 1.6] *If $F_2(X)$ is normal and X is Hausdorff, connected, locally connected and multicoherent, then $r(F_2(X)) = 1$.*

Recently S. Macías in [5] proved the following results.

Theorem 1.4. [5, thm. 8] *If X is a continuum, then $F_n(X)$ is unicoherent for each $n \geq 3$.*

Theorem 1.5. [5] *If X is a continuum, then $r(F_2(X)) \leq 1$*

In [3, prob. 4] A. García-Máynez and A. Illanes, posed the following question: If X is a unicoherent continuum, is it true that $F_2(X)$ is unicoherent? In this paper we give an example which answers this question in the negative. We construct a unicoherent plane continuum X such that $F_2(X)$ is not unicoherent.

2. The Example

In this section we give a unicoherent plane continuum X such that $F_2(X)$ is not unicoherent.

Example 2.1. *There exists a unicoherent plane continuum X such that $F_2(X)$ is not unicoherent.*

Define

$$S_1 = \{e^{it} : t \in \mathbb{R}\}, \quad S_2 = \{3e^{it} : t \in \mathbb{R}\}$$

and

$$Y = \{(\frac{t}{1+ \mid t \mid} + 2)e^{it} : t \in \mathbb{R}\}.$$

Let $X = S_1 \cup Y \cup S_2$ (X is the union of two circles and a spiral which surrounds them asymptotically). Observe that X is a unicoherent continuum. We shall prove that $F_2(X)$ is not unicoherent. Let

$$\mathcal{A} = \{\{z,w\} \in F_2(X) : \text{Im}(zw) \geq 0\} \quad \text{and}$$
$$\mathcal{B} = \{\{z,w\} \in F_2(X) : \text{Im}(zw) \leq 0\},$$

where $\text{Im}(zw)$ is the imaginary part of the complex number zw. It is easy to check that $\mathcal{A}$ and $\mathcal{B}$ are closed. It is clear that $\mathcal{A} \cup \mathcal{B} = F_2(X)$.

We will prove that $\mathcal{A}$ is connected.

Let $\mathcal{C} = \mathcal{D} \cup \mathcal{E}$, where

$$\mathcal{D} = \{A \in \mathcal{A} : A \subset Y\} \quad \text{and}$$
$$\mathcal{E} = \{\{z,w\} \in \mathcal{A} : z \in S_1 \quad \text{and} \quad w \in S_2\}$$

Claim 1. $\mathcal{C}$ is a connected subset of $\mathcal{A}$.

Let $P = \{e^{iu}, 3e^{iv}\} \in \mathcal{E}$ then $\text{Im}(3e^{i(u+v)}) \geq 0$. In order to see that $\mathcal{E}$ is connected, it is enough to show that there exists a connected subset $\mathcal{L}$ of $\mathcal{E}$ such that P and $\{1,3\} \in \mathcal{L}$. Suppose, for example, that $0 \leq u \leq v \leq 2\pi$. Then $0 \leq u+v \leq 4\pi$. Since $\text{Im}(3e^{i(u+v)}) \geq 0$, $0 \leq u+v \leq \pi$ or $2\pi \leq u+v \leq 3\pi$. We consider the following two cases:

Case I. If $0 \leq u+v \leq \pi$, then define the set

$$\mathcal{L} = \{\{e^{i(u+r)}, 3e^{i(v-r)}\} : r \in [0,(v-u)/2]\} \cup \{\{e^{ir}, 3e^{ir}\} :$$

$$r \in [0,(u+v)/2]\}.$$

Notice that $\mathcal{L}$ contains the elements P and $\{1,3\}$. Observe also that $\mathcal{L}$ is the union of two connected sets which have the common point

$$\{e^{i(u+v)/2}, 3e^{i(u+v)/2}\}.$$

Then $\mathcal{L}$ is connected. To see that $\mathcal{L} \subset \mathcal{E}$. Take a point $Q \in \mathcal{L}$. Then we have two possibilities:

a) if $Q = \{e^{i(u+r)}, 3e^{i(v-r)}\}$, then

$$\operatorname{Im}((e^{i(u+r)})(3e^{i(v-r)})) = \operatorname{Im}(3e^{i(u+v)}) \geq 0.$$

b) if $Q = \{e^{ir}, 3e^{ir}\}$ with $r \in [0,(u+v)/2]$, then $0 \leq r \leq \pi/2$. Thus

$$\operatorname{Im}((e^{ir})(3e^{ir})) = \operatorname{Im}(3e^{2ir}) \geq 0.$$

Case II. If $2\pi \leq u+v \leq 3\pi$. Let $l = ((u+v)/2 - \pi)$. Then $0 \leq l \leq \pi/2$. Notice that $0 \leq 2\pi - v \leq (u-v)/2 + \pi$. Define the set

$$\mathcal{L} = \{\{e^{i(u-r)}, 3e^{i(v+r)}\} : r \in [0,(u-v)/2+\pi]\} \cup \{\{e^{ir}, 3e^{ir}\} : r \in [0,l]\}.$$

Notice that $\mathcal{L}$ contains the elements P and $\{1,3\}$. Notice also that $\mathcal{L}$ is the union of two connected sets which have the common point

$$\{e^{il}, 3e^{i(l+2\pi)}\} = \{e^{il}, 3e^{il}\}.$$

Then $\mathcal{L}$ is connected. The proof that $\mathcal{L} \subset \mathcal{E}$ is similar to the one in case I. Therefore, in both cases $\mathcal{L}$ is connected, it lies in $\mathcal{E}$ and it contains P and $\{1,3\}$. Therefore $\mathcal{E}$ is connected.

For each $t \in I\!\!R$, let

$$g(t) = (\frac{t}{1+\mid t\mid} + 2)e^{it}.$$

To complete the proof that $\mathcal{C}$ is connected, take an arbitrary point $A \in \mathcal{D}$. We will show that there exists a connected subset $\mathcal{F}_0$ of $\mathcal{C}$ such that $A \in \mathcal{F}_0$ and $\mathcal{E} \cap \mathcal{F}_0 \neq \emptyset$. Notice that A has the form $\{g(t), g(s)\}$, where $t, s \in I\!\!R$. We may assume that $t \leq s$. Let $m = (t+s)/2$. Define

$$\mathcal{F} = \{\{g(t-r), g(s+r)\} : r \in I\!\!R\}$$

Since g is continuous, $\mathcal{F}$ is connected and $\mathcal{F} \subset \mathcal{D}$. Consider the sequence

$$B_n = \{g(t + s - m - 2\pi n), g(t + s - m + 2\pi n)\}.$$

Then B_n converges to $\{e^{im}, 3e^{im}\}$. This proves that $M = \{e^{im}, 3e^{im}\}$ belongs to the closure of $\mathcal{F}$. Then $\mathcal{F}_0 = \mathcal{F} \cup \{M\}$ is connected, $A \in \mathcal{F}_0$ and $\mathcal{E} \cap \mathcal{F}_0 \neq \emptyset$. This concludes the proof of Claim 1.

Claim 2. $\mathcal{A} = \overline{\mathcal{C}}$

Let $B \in \mathcal{A}$, we will show a sequence $\{B_n\}_{n=1}^{\infty}$ in $\mathcal{C}$ such that B_n converges to B. We consider several cases.

a) if $B = \{e^{it}, e^{is}\} \subset S_1 \times S_1$. Let $B_n = \{g(t - 2\pi n), g(s - 2\pi n)\}$,
b) if $B = \{3e^{it}, 3e^{is}\} \subset S_2 \times S_2$. Let $B_n = \{g(t + 2\pi n), g(s + 2\pi n)\}$,
c) if $B = \{e^{it}, g(s)\} \subset S_1 \times Y$. Let $B_n = \{g(t - 2\pi n), g(s)\}$, and
d) if $B = \{g(t), 3e^{is}\} \subset Y \times S_2$. Let $B_n = \{g(t), g(s + 2\pi n)\}$.

This completes the proof of Claim 2. By claims 1 and 2, we conclude that $\mathcal{A}$ is connected.

Define $h : X \longrightarrow X$ by

$$h(z) = \begin{cases} e^{i(t+\pi/2)} & \text{if } z = e^{it} \in S_1, \\ 3e^{i(t+\pi/2)} & \text{if } z = 3e^{it} \in S_2, \\ g(t + \pi/2) & \text{if } z = g(t) \in Y \end{cases}$$

It is easy so show that h is continuous. Define $H : F_2(X) \longrightarrow F_2(X)$ by $H(\{w, z\}) = \{h(w), h(z)\}$. Then H is continuous and $H(\mathcal{A}) = \mathcal{B}$. Thus $\mathcal{B}$ is also connected.

Claim 3. $\mathcal{A} \cap \mathcal{B}$ is not connected.

Let

$$\mathcal{H} = \{\{z, w\} : zw \in \mathbb{R} \text{ and } zw \geq 0\} \text{ and}$$

$$\mathcal{K} = \{\{z, w\} : zw \in \mathbb{R} \text{ and } zw \leq 0\}.$$

Then $\mathcal{A} \cap \mathcal{B} = \mathcal{H} \cup \mathcal{K}$. Since $\mathcal{H}$ and $\mathcal{K}$ are two disjoint closed subsets of $F_2(X)$, we conclude that $\mathcal{A} \cap \mathcal{B}$ is not connected. This completes the proof, showing that $F_2(X)$ is not unicoherent. ■

We conclude with the following problems.

Problem 1. Does there exist an indecomposable continuum X such that $F_2(X)$ is not unicoherent?

Problem 2. Does there exist an hereditarily unicoherent continuum X such that $F_2(X)$ is not unicoherent?

Problem 3. (J.J. Charatonik) Does there exist an hereditarily unicoherent, hereditarily decomposable continuum X such that $F_2(X)$ is not unicoherent?

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