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PROPERTIES OF n-BUBBLES IN n-DIMENSIONAL COMPACTA AND THE EXISTENCE OF $(n-1)$-BUBBLES IN n-DIMENSIONAL clc^n COMPACTA [1]

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ABSTRACT. An n-dimensional compact metric space X is called an n-bubble if the Alexander-Spanier cohomology with compact supports of X with integer coefficients, denoted by $H^n(X)$, is non-zero, but $H^n(A) = 0$ for every proper closed subset A of X. Under the setting that X is an n-dimensional compact metric space and $f: X \to X$ is homotopic to the identity, we show that every n-bubble in X is contained in its image.

We give a positive partial solution to a question of W. Kuperberg [9] by showing that if X is an n-dimensional clc^n compact metric space such that $H^n(V)$ is finitely generated for every connected open subset V of X, then X contains an $(n-1)$-bubble.

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1. Introduction

We first give some preliminary definitions [13]. By a *compactum* we mean a compact metric space. The various metrics and distances will be designated by the letter d. The diameter of a subset A of a metric space will be denoted $diam(A)$. If X is a space and $A \subset X$, then $\overline{A}$ will denote the closure of the set A, $int(A)$ its interior. By a *map* or *mapping* we mean a continuous function.

For cohomology we will use the Alexander-Spanier cohomology groups with compact supports and the notation of Massey's book [10]. The q-dimensional cohomology group with compact supports of a locally compact Hausdorff space X with integer coefficients will be denoted by $H^q(X)$. Following Massey we denote the homomorphism from $H^q(U)$ to $H^q(X)$ associated with an open subset U of X by $\tau_{U,X}$ or simply τ when no confusion could occur (see [10] for the definition). For a compact Hausdorff space Alexander-Spanier cohomology with compact supports is naturally isomorphic to Čech cohomology [11], and for any locally compact Hausdorff space X and for any integer $q > 0$, $H^q(X) \cong H^q(X^+)$, where X^+ is the one point compactification of X. Thus one can interpret the results of this paper in terms of Čech cohomology.

By a *compact ANR* we mean a compact absolute neighborhood retract [2]. If f is a map from X to Y, $f \simeq 0$ means that f is homotopic to a constant map, $f_A\colon A \to fA$ is the map defined by $f_A(x) = f(x)$, and $f^*\colon H^q(Y) \to H^q(X)$ will denote the induced homomorphism of f. The map $f\colon X \to Y$ is an $\epsilon - map$ if for every $y \in Y$ the diameter $diam(f^{-1}y) \leq \epsilon$. The group of integers will be denoted by $\mathbf{Z}$. By the *dimension of* X we mean the covering dimension of X. The following definition is essentially stated in Borel [1].

Definition 1. *The cohomological dimension* $dim_{\mathbf{Z}}X$ *of a space* X *with respect to the group* $\mathbf{Z}$ *is defined to be the least integer* n *(or* ∞*) such that* $H^q(U) = 0$ *for every open subset* U *of* X *and* $q > n$.

The following two definitions are cohomology versions of two definitions given by W. Kuperberg [9].

Definition 2. *A compactum X is said to be "n-cyclic" if $H^n(X) \neq 0$.*

Definition 3. *An n-dimensional compactum X is called an n-dimensional closed Cantor manifold or an "n-bubble" if it is n-cyclic and $H^n(A) = 0$ for every proper closed subset A of X.*

The next definition is given by Bredon [3].

Definition 4. *X is "clc^n" (cohomologically locally n-connected) if for each $q \leq n$, $x \in X$ and each closed neighborhood N of x, there is a closed neighborhood $M \subset N$ of x such that $0 = i^*\colon H^q(N) \to H^q(M)$.*

In 1972 W. Kuperberg [9] raised a question "Does every n-dimensional compactum contain an $(n-1)$-bubble?"

In this paper we give a positive partial solution to the question by showing that if X is a clc^n compactum such that $H^n(V)$ is finitely generated for every connected open subset V of X, then X contains an $(n-1)$-bubble.

We show some properties of n-bubbles in an n-dimensional compactum. In particular we also show that if X is an n-dimensional compact metric space such that $H^n(X)$ is finitely generated but X contains infinitely many distinct n-bubbles then X contains an infinite sequence of distinct n-bubbles such that the limit of the sequence in the Hausdorff metric is the closure of the union of all the n-bubbles in the sequence.

2. Preliminaries

In this section, we show a new approach to the problem of the existence of n-bubbles in an n-dimensional compactum X with finitely generated $H^n(X)$. These theorems are just cohomological versions of known results of W. Kuperberg's, but we show alternative proofs.

Definition 5. *Let X be a compactum, a an element of $H^n(X)$, and A a closed subset of X. A is said to be "a carrier of a" provided $i^*(a) \neq 0$, where $i^*: H^n(X) \to H^n(A)$ is induced by the inclusion. A carrier A of a is said to be "irreducible" if no proper subset of A is a carrier of a.*

Clearly, every n-bubble is an irreducible carrier of an element of $H^n(X)$. Also, by the continuity of the Alexander-Spanier cohomology with compact supports [11], every carrier A of an element $a \in H^q(X)$ contains an irreducible carrier of a. But unlike the homology case [9], even when A_1 and A_2 are carriers of an element $a \in H^n(X)$, $A_1 \cap A_2$ doesn't have to be a carrier of a. Instead we have the following lemma. The proof of this lemma is straightforward so we omit it.

Lemma 1. *Let X be a compactum and a be an element of $H^n(X)$. Suppose that $a = n_1a_1 + n_2a_2 + \cdots + n_ra_r$, where $a_k \in H^n(X)$ and $0 \neq n_k \in \mathbf{Z}$ for $k = 1, \ldots, r$; then every carrier of a is a carrier of at least one of $a_1, \ldots, a_r$.*

The following is the cohomological version of a theorem of W. Kuperberg [9]. It can be proved by translating Kuperberg's proof into cohomology. We will show another proof of this theorem in Section 3.

Theorem 1. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated. Let $X_1 \supset X_2 \supset \ldots$ be a decreasing sequence of closed subsets of X. Then the intersection $X_0 = \bigcap_{k=1}^{\infty} X_k$ is n-cyclic if and only if every X_k is n-cyclic.*

Definition 6. *Let $(\mathcal{F}, \leq)$ be a partially ordered set and let a be an element of $\mathcal{F}$. Then a is said to be "a minimal element" in $\mathcal{F}$ if for any $b \in \mathcal{F}$, $b \leq a$ implies $a = b$.*

The following also is the cohomological version of another theorem of W. Kuperberg [9]. We could use Theorem 1 to prove the first part of it, but we provide an alternative proof. We have no similar proof to Kuperberg's for the second part.

Theorem 2. *Every n-dimensional, n-cyclic compactum X for which $H^n(X)$ is finitely generated contains an n-bubble. Moreover, the number of n-bubbles contained in X is at most countable.*

Proof of the first part of the theorem: Let $\{a_1, a_2, \ldots, a_r\}$ be a finite set of generators for $H^n(X)$. Let $\mathcal{F}_k$ be the set of all irreducible carriers of a_k and let $\mathcal{F} = \cup \mathcal{F}_k$. Then $\mathcal{F}$ is partially ordered by inclusion. By Lemma 1 combined with the fact that every carrier contains an irreducible carrier, it is easy to see that A is an n-bubble in X if and only if A is a minimal element of $\mathcal{F}$. Since any two different irreducible carriers of an element a_k have no inclusion between them, every chain in $\mathcal{F}$ has at most r elements. Therefore $\mathcal{F}$ has a maximal chain and therefore contains a minimal element.

To prove the second part of the theorem, we will need the following lemma.

Lemma 2. *Let X be an n-dimensional, n-cyclic compactum with two distinct n-bubbles A and B. Then neither kernel of i_A^* and i_B^* is contained in the other, where $i_A^* : H^n(X) \to H^n(A)$ and $i_B^* : H^n(X) \to H^n(A)$ are the homomorphisms induced by the inclusions $i_A \colon A \hookrightarrow X$ and $i_B \colon B \hookrightarrow X$.*

Proof: Since $A \neq B$, $A \cap B$ is a proper closed subset of A and B and therefore $H^n(A \cap B) = 0$. Hence, by the Mayor-Vietoris sequence,

$$H^n(A \cup B) \to H^n(A) \oplus H^n(B)$$

is onto. Let $j_A \colon A \hookrightarrow A \cup B$, $j_B \colon B \hookrightarrow A \cup B$, and $h \colon A \cup B \hookrightarrow X$. Then there exists an element $a \in H^n(A \cup B)$ such that $j_A^*(a) \neq 0$ but $j_B^*(a) = 0$. Also since $h^* \colon H^n(X) \to H^n(A \cup B)$ is an epimorphism, there exists an element $b \in H^n(X)$ such that $h^*(b) = a$. Hence $i_A^*(b) = j_A^* h^*(b) = j_A^*(a) \neq 0$, but $i_B^*(b) = j_B^* h^*(b) = j_B^*(a) = 0$. Thus $b \in Ker i_B^*$ but $b \notin Ker i_A^*$. Therefore $Ker\ i_B^* \not\subset Ker\ i_A^*$. The same argument shows that there is a non-zero element $c \in H^n(X)$ such that $i_A^*(c) = 0$ but $i_B^*(c) \neq 0$. Therefore $Ker\ i_A^* \not\subset Ker\ i_B^*$.

Proof of the second part of Theorem 2: It follows from Lemma 2 that the number of n-bubbles in X is at most the number of subgroups of $H^n(X)$. But $H^n(X)$ can have at most countably many subgroups.

3. Properties of n-bubbles in n-dimensional compacta

In this section, we are mainly concerned with the properties of n-bubbles in an n-dimensional compactum. We start with one of our major tools.

Lemma 3. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated. If A is a closed subset of X, then there exists a closed neighborhood N of A such that $i^*\colon H^n(N) \to H^n(A)$ is an isomorphism, where $i\colon A \hookrightarrow N$.*

Proof: Consider the following long exact sequence:

$$\cdots \longrightarrow H^n(X \setminus A) \xrightarrow{\tau} H^n(X) \xrightarrow{j^*} H^n(A) \longrightarrow 0$$

Since $H^n(X)$ is finitely generated, $Im\ \tau = Ker\ j^*$ is finitely generated. Let $\{\xi_k\}_{k=1}^r$ be the set of generators of $Im\ \tau = Ker\ j^*$. Then for each k there is a corresponding $\eta_k \in H^n(X \setminus A)$ such that $\tau(\eta_k) = \xi_k$. Also for each η_k there is an open set W_k whose closure is compact and is contained in $X \setminus A$; furthermore there is $\eta'_k \in H^n(W_k)$ with $\tau_k(\eta'_k) = \eta_k$ where $\tau_k\colon H^n(W_k) \to H^n(X \setminus A)$. Let $W = \cup_{k=1}^r W_k$. Then for each k there is $\hat{\eta}_k \in H^n(W)$ such that $\tau'(\hat{\eta}_k) = \eta_k$, where $\tau'\colon H^n(W) \to H^n(X \setminus A)$. Let $N = X \setminus W$; then $int(N) = X \setminus \overline{W} \supset A$ since $\overline{W} \subset X \setminus A$. Hence N is a closed neighborhood of A. We have the following commutative diagram:

$$\begin{array}{ccccc} H^n(X\setminus A) & \xrightarrow{\ \tau\ } & H^n(X) & \xrightarrow{\ j^*\ } & H^n(A) \\ \uparrow{\scriptstyle \tau'} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle i^*} \\ H^n(W) & \xrightarrow{\ \tau''\ } & H^n(X) & \xrightarrow{\ h^*\ } & H^n(X\setminus W) = H^n(N) \end{array}$$

Now we show that i^* is an isomorphism. Clearly i^* is an epimorphism. To prove i^* is a monomorphism, let $a \in H^n(N)$ be such that $i^*(a) = 0$. Since h^* is an epimorphism, there is $b \in H^n(X)$ such that $h^*(b) = a$. Since $i^*h^* = j^*$, $j^*(b) = 0$. Hence there is a $b' \in H^n(X \setminus A)$ such that $\tau(b') = b$, but b' is in the subgroup generated by $\{\eta_k\}_{k=1}^{r}$ so that there is $c \in H^n(W)$ such that $\tau'(c) = b'$. Since $\tau'' = \tau\tau'$, $\tau''(c) = b$. Therefore $a = h^*(b) = h^*\tau''(c) = 0$.

The following corollary of the Lemma is the theorem of W. Kuperberg that we referred to as Theorem 1 in Section 2. Here we give another proof.

Corollary 1. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated. Let $X_1 \supset X_2 \supset \ldots$ be a decreasing sequence of closed subsets of X. Then the intersection $X_0 = \bigcap_{k=1}^{\infty} X_k$ is n-cyclic whenever all the X_k are n-cyclic.*

Proof: Let N be a closed neighborhood of X_0 such that $i^*\colon H^n(N) \to H^n(X_0)$ is an isomorphism, where $i\colon X_0 \hookrightarrow N$. Then there exists a number k_0 such that for all $k \geq k_0$ $X_k \subset N$. If we let $i_k\colon X_k \hookrightarrow N$ be the inclusion for $k \geq k_0$, $i_k^*\colon H^n(N) \to H^n(X_k)$ is an epimorphism and therefore $H^n(N) \neq 0$. Thus $H^n(X_0) \neq 0$.

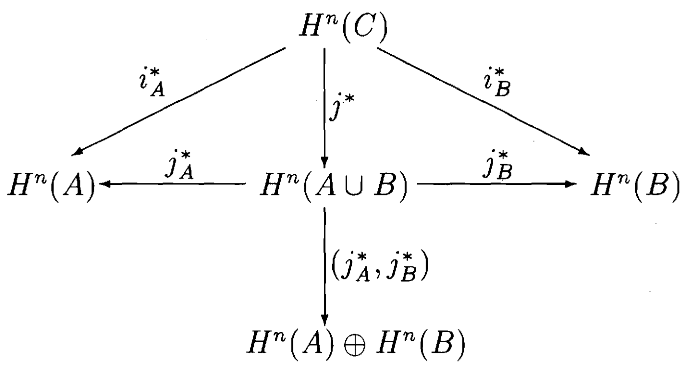
Theorem 3. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated, A is an n-bubble in X, B is an n-dimensional closed subset of X with $H^n(B) \neq 0$, and C is a closed subset of X such that $C \supset A \cup B$. If either i_A^* or i_B^* is an isomorphism, then $B \supset A$, where $i_A^*\colon H^n(C) \to H^n(A)$ and $i_B^*\colon H^n(C) \to H^n(B)$ are the homomorphisms induced by the inclusions.*

Proof: Suppose that $B \not\supset A$. Then $B \cap A$ is a proper closed subset of A. Thus $H^n(A \cap B) = 0$.

Thus, in the following Mayor-Vietoris sequence

$$\cdots \to H^n(A{\cup}B) \xrightarrow{(j_A^*, j_B^*)} H^n(A){\oplus}H^n(B) \longrightarrow H^n(A{\cap}B) \to \ldots$$

(j_A^*, j_B^*) is an epimorphism. Now consider the following diagram:



Since either i_A^* or i_B^* is an isomorphism, j^* is a monomorphism. But every set is n-dimensional and hence every homomorphism induced by inclusion is an epimorphism. Thus j^* is an epimorphism and therefore an isomorphism. This implies that either j_A^* or j_B^* is an isomorphism. That is, that i_A^* and j^* are isomorphisms implies j_A^* is an isomorphism and that i_B^* and j^* are isomorphisms implies j_B^* is an isomorphism.

Case I. j_A^* is an isomorphism. Since (j_A^*, j_B^*) is an epimorphism and $H^n(A) \neq 0, H^n(B) \neq 0$, there exists an element $b \in H^n(A \cup B)$ such that $j_A^*(b) = 0$ but $j_B^*(b) \neq 0$. But since j_A^* is an isomorphism, we have that $b = 0$, which is a contradiction.

Case II. j_B^* is an isomorphism. Again there exists an element $b \in H^n(A{\cup}B)$ such that $j_A^*(b) \neq 0$, but $j_B^*(b) = 0$. But since j_B^* is an isomorphism we have that $b = 0$, which is a contradiction.

Therefore in either case, $B \supset A$.

The following is a special case of Theorem 3.

Corollary 2. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated, and that A is an n-bubble in X, and B and C are n-dimensional closed subsets of X such that $C \supset A \cup B$ and $i^*\colon H^n(C) \to H^n(A)$ is an isomorphism, where $i\colon A \hookrightarrow C$. Then $H^n(B) \neq 0$ if and only if $B \supset A$.*

We examine an n-dimensional compactum that has finitely generated n-th cohomology but has infinitely many distinct n-bubbles.

Theorem 4. *Let X be an n-dimensional compactum such that $H^n(X)$ is finitely generated. Suppose that $< X_k >$ is a sequence of distinct n-bubbles in X. Then $< X_k >$ has a convergent subsequence $< X_{s_k} >$ such that in the Hausdorff metric* $\lim < X_{s_k} >= \overline{\cup_{k=1}^{\infty} X_{s_k}}$.

Proof: Since $< X_k >$ has a convergent subsequence, we may assume without loss of generality that $< X_k >$ is convergent.

Let $X_0 = \lim X_k$. Then $X_0 = \cap_{i=1}^{\infty} \overline{\cup_{k=i}^{\infty} X_k}$. Then, by Lemma 3, there exists a closed neighborhood N of X_0 such that $i^*\colon H^n(N) \to H^n(X_0)$ is an isomorphism, where $i\colon X_0 \hookrightarrow N$. Since X_0 is the limit of the sequence $< X_k >$, there exists an integer k_0 such that if $k \geq k_0$ then $X_k \subset N$.

Hence, by Theorem 3, $X_k \subset X_0$ for all $k \geq k_0$. Let $s_k = k + k_0 - 1$; then $\lim < X_{s_k} > \supset \overline{\cup_{k=1}^{\infty} X_{s_k}}$. Clearly $\lim < X_{s_k} > \subset \overline{\cup_{k=1}^{\infty} X_{s_k}}$. Therefore $\lim < X_{s_k} >= \overline{\cup_{k=1}^{\infty} X_{s_k}}$.

Corollary 3. *Suppose that X is an n-dimensional compactum such that $H^n(X)$ is finitely generated and X has infinitely many distinct n-bubbles. If X has no proper closed subset that contains infinitely many distinct n-bubbles, then $X = \lim < X_k >$ where $< X_k >$ is a convergent sequence of infinitely many distinct n-bubbles.*

Proof: Let $< X_k >$ be a convergent sequence of infinitely many distinct n-bubbles. Then by the proof of Theorem 4 there exists an integer k_0 such that $\lim < X_k >= \overline{\cup_{k=k_0}^{\infty} X_k}$.

Hence $\lim < X_k >$ is a closed subset of X and contains infinitely many distinct n-bubbles. Therefore $\lim < X_k >= X$.

The following lemma is another important tool in this section.

Lemma 4. *Let A be a q-bubble in a compactum X. Then for any closed subset B of X, $B \supset A$ if and only if* $\operatorname{Ker} i_B^* \leq \operatorname{Ker} i_A^*$, *where $i_B^*\colon H^q(A \cup B) \to H^q(B)$ and $i_A^*\colon H^q(A \cup B) \to H^q(A)$ are induced by the inclusions.*

Proof: Clearly $B \supset A$ implies that $Ker\ i_B^* \leq Ker\ i_A^*$.
Suppose that $B \not\supset A$. Then $A \cap B$ is a proper subset of A, and so $H^q(A \cap B) = 0$. Consider the following commutative diagram:

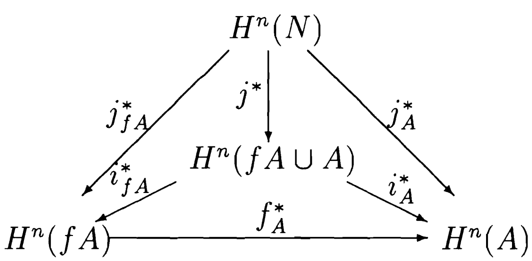
$$
\begin{array}{ccc}
H^q((B \cup A) \setminus (B \cap A)) & \xrightarrow{\tau_1} & H^q(B \cup A) \\
\downarrow i_{B1}^* & & \downarrow i_B^* \\
H^q(B \setminus (B \cap A)) & \xrightarrow{\tau_2} & H^q(B)
\end{array}
\qquad
\begin{array}{ccc}
H^q((B \cup A) \setminus (B \cap A)) & \xrightarrow{\tau_1} & H^q(B \cup A) \\
\downarrow i_{A1}^* & & \downarrow i_A^* \\
H^q(A \setminus (B \cap A)) & \xrightarrow{\tau_3} & H^q(A)
\end{array}
$$

Since $H^q(B \cap A) = 0$, we have that τ_k is an epimorphism for $k = 1$, 2 and 3. We also note that $H^q((B \cup A) \setminus (B \cap A)) \cong H^q(B \setminus (B \cap A)) \oplus H^q(A \setminus (B \cap A))$. Since $H^q(A) \neq 0$ and τ_3 is an epimorphism, we can find an element $a \in H^q(A \setminus (B \cap A))$ such that $\tau_3(a) \neq 0$. Since (i_{B1}^*, i_{A1}^*) is an isomorphism, there exists an element b in $H^q((B \cup A) \setminus (B \cap A))$ such that $i_{A1}^*(b) = a$ and $i_{B1}^*(b) = 0$. Then $i_B^* \tau_1(b) = \tau_2 i_{B1}^*(b) = 0$ and $i_A^* \tau_1(b) = \tau_3 i_{A1}^*(b) \neq 0$. Therefore $\tau_1(b) \in Ker\ i_B^*$, but $\tau_1(b) \notin Ker\ i_A^*$. Thus $Ker\ i_B^* \leq Ker\ i_A^*$ implies $B \supset A$.

In view of Lemma 3 and Corollary 2, for each n-bubble A in an n-dimensional compactum X such that $H^n(X)$ is finitely generated, there is an positive ϵ such that if $f\colon X \to X$ is ϵ-homotopic to identity on X then $A \subset f(A)$. But we show this is true regardless of ϵ. In fact we prove the following.

Theorem 5. *Suppose that X is an n-dimensional compactum and that $f\colon X \to X$ is a map homotopic to the identity. Then for every n-bubble A in X, $fA \supset A$.*

Proof: By Lemma 4, it suffices to show that $Ker\ i^*_{fA} \leq Ker\ i^*_A$, where $i^*_{fA}\colon H^n(fA \cup A) \to H^n(fA)$ and $i^*_A\colon H^n(fA \cup A) \to H^n(A)$ are induced by the inclusions. Let $f_A\colon A \to fA$ be a map induced by f. We now show that $f^*_A i^*_{fA} = i^*_A$. Let F be a homotopy from $X \times I$ to X such that $F(x,0) = x$ and $F(x,1) = f(x)$. Let $N = F(A \times I)$. Then $(fA \cup A) \subset N$ and $f_A \simeq id_A$ on N. Therefore if we let $j_{fA}\colon fA \hookrightarrow N$ and $j_A\colon A \hookrightarrow N$ then $j_{fA} f_A \simeq j_A$. We have the following diagram:



where $j\colon (fA \cup A) \hookrightarrow N$.

Let $a \in H^n(fA \cup A)$. Since j^* is an epimorphism, there exists an element b in $H^n(N)$ such that $j^*(b) = a$. Then $i^*_A(a) = i^*_A j^*(b) = j^*_A(b) = f^*_A j^*_{fA}(b) = f^*_A i^*_{fA} j^*(b) = f^*_A i^*_{fA}(a)$. Therefore $i^*_A = f^*_A i^*_{fA}$ and hence $Ker\ i^*_{fA} \leq Ker\ i^*_A$.

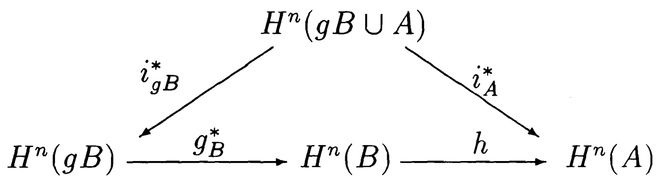
Under the condition that $f\colon X \to K$ and $g\colon K \to X$ are maps such that $gf \simeq id_X$, if A is an n-bubble, then Theorem 5 shows that $gfA \supset A$, but if B is an n-bubble in K then gB doesn't have to contain an n-bubble. But we proved the following theorem.

Theorem 6. *Let X be an n-dimensional compactum and K an n-dimensional polyhedron with a fixed triangulation. Assume that $f\colon X \to K$ and $g\colon K \to X$ are maps such that $gf \simeq id_X$. Let A be an n-bubble in X and B a closed subset of*

fA such that Ker $i^* \leq$ Ker f_A^*, *where* $i^*\colon H^n(fA) \to H^n(B)$ *is induced by inclusion and* f_A *is induced by* f *as defined earlier. Then* $gB \supset A$.

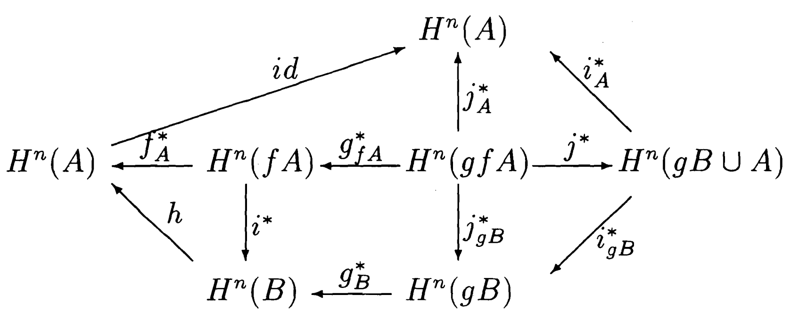
Proof: Since $Ker\ i^* \leq Ker\ f_A^*$ and i^* is an epimorphism, there is a homomorphism $h\colon H^n(B) \to H^n(A)$ such that $hi^* = f_A^*$.

We now prove that the following diagram commutes



where $g_B\colon B \to gB$ is induced by g.

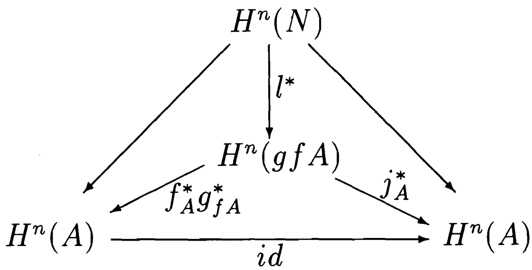
Consider the following diagram:



All homomorphisms except f_A^*, g_{fA}^*, g_B^* and h are induced by inclusions.

By Theorem 5, $gfA \supset A$, and hence $gfA \supset (A \cup gB)$. To show that the above diagrams commute, it suffices to show that the top-left triangle diagram commutes. Let F be a homotopy from $X \times I$ to X such that $F(x,0) = x$ and $F(x,1) = gf(x)$.

Let $N = F(A \times I)$. Then we have the following diagram:



where $l\colon gfA \hookrightarrow N$.

Clearly $lg_{fA}f_A \simeq lj_A$ and l^* is an epimorphism, and hence $f_A^* g_{fA}^* = j_A^*$. Thus the top-left triangle diagram commutes, and therefore the whole diagram commutes.

Let $a \in H^n(gB \cup A)$. Then, since j^* is an epimorphism, there is an element b in $H^n(gfA)$ such that $j^*(b) = a$. So $i_{gB}^*(a) = i_{gB}^* j^*(b) = j_{gB}^*(b)$. Hence $hg_B^* i_{gB}^*(a) = hg_B^* j_{gB}^*(b) = hi^* g_{fA}^*(b) = f_A^* g_{fA}^*(b) = j_A^*(b) = i_A^* j^*(b) = i_A^*(a)$. Therefore $hg_B^* i_{gB}^* = i_A^*$ and hence $Ker\ i_{gB}^* \leq Ker\ i_A^*$. By Lemma 4, we have $gB \supset A$.

Corollary 4. *Let X be an n-dimensional compactum and K an n-dimensional polyhedron with a fixed triangulation. Assume that $f\colon X \to K$ and $g\colon K \to X$ are maps such that $gf \simeq id_X$. Let A be an n-bubble in X and $B = \cup\{\sigma \in K \mid \sigma$ is an $n-simplex$ such that $\sigma \subset fA\}$. Then $gB \supset A$.*

Proof: Clearly $B \subset fA$. Let $j\colon (B \cup K^{n-1}) \hookrightarrow (fA \cup K^{n-1})$, $i\colon B \hookrightarrow fA$, $i_{B,K}\colon B \hookrightarrow (B \cup K^{n-1})$ and $i_{A,K}\colon fA \hookrightarrow (fA \cup K^{n-1})$ be inclusion maps, where K^{n-1} is the $(n-1)$-skeleton of K. Then, by the long exact sequence, $i_{A,K}^*\colon H^n(fA \cup K^{n-1}) \to H^n(fA)$ and $i_{B,K}^*\colon H^n(B \cup K^{n-1}) \to H^n(B)$ are isomorphisms.

We now show that $j^*\colon H^n(fA \cup K^{n-1}) \to H^n(B \cup K^{n-1})$ is an isomorphism and hence that $i^*\colon H^n(fA) \to H^n(B)$ is an isomorphism. Let $N = \cup\{\sigma \in K \mid \sigma$ is an n-simplex such that $\sigma \cap fA \neq \emptyset\}$ and $N' = \{\sigma \in K \mid \sigma$ is an n-simplex such that $\sigma \subset N$ but $\sigma \not\subset B\}$. Then N' is finite, say $N' = \{\sigma_1, \ldots \sigma_s\}$.

For each $1 \leq k \leq s$, we have that $\sigma_k \cap fA$ is a non-empty compact set, but $\sigma_k \not\subset fA$. Therefore there is an open n-ball B_k^n such that $B_k^n \subset (\sigma_k \setminus fA)$. Hence we have a strong deformation retraction F_k of $(\sigma_k \setminus B_k^n)$ to σ_k^{n-1}, where σ_k^{n-1} is the $(n-1)$-skeleton of σ_k.

Define $F\colon [(N \setminus (B_1^n \cup \cdots \cup B_s^n)) \cup K^{n-1}] \times I \to (N \setminus (B_1^n \cup \cdots \cup B_s^n)) \cup K^{n-1}$ by

$$F(x,t) = \begin{cases} x, & \text{if } x \in (B \cup K^{n-1}); \\ F_k(x,t), & \text{if } x \in (\sigma_k \setminus B_k^n) \text{for some} 1 \leq k \leq s. \end{cases}$$

Then F is a strong deformation retraction of $(N \setminus (B_1^n \cup \cdots \cup B_s^n)) \cup K^{n-1}$ to $B \cup K^{n-1}$. Therefore the map $h^*\colon H^n((N \setminus (B_1^n \cup \cdots \cup B_s^n)) \cup K^{n-1}) \to H^n(B \cup K^{n-1})$ induced by inclusion is an isomorphism. Since $fA \cup K^{n-1} \subset (N \setminus (B_1^n \cup \cdots \cup B_s^n)) \cup K^{n-1}$, we have that $j^*\colon H^n(fA \cup K^{n-1}) \to H^n(B \cup K^{n-1})$ also is an isomorphism. Hence $i^*\colon H^n(fA) \to H^n(B)$ is an isomorphism. Therefore, by Theorem 6, $gB \supset A$.

Corollary 5. *Let X be an n-dimensional compactum and K an n-dimensional polyhedron with a fixed triangulation. Assume that $f\colon X \to K$ and $g\colon K \to X$ are maps such that $gf \simeq id_X$. Let A be an n-bubble in X and let B be a closed subset of fA such that there is a map $\varphi\colon A \to K$ with $\varphi A \subset B$ and $f|A \simeq \varphi$. Then $gB \supset A$.*

Proof: Let $f_A\colon A \to fA$ be the map induced by f and let $\varphi_1\colon A \to B$ be the map induced by φ and let $i\colon B \hookrightarrow fA$, $i_B\colon B \hookrightarrow K$ and $i_{fA}\colon fA \hookrightarrow K$ be inclusion maps. Then $i_B\varphi_1 \simeq i_{fA}f_A$. Hence $i_{fA}f_A \simeq i_B\varphi_1 = i_{fA}i\varphi_1$. Thus $f_A^* i_{fA}^* = \varphi_1^* i^* i_{fA}^*$. Since i_{fA}^* is an epimorphism, $f_A^* = \varphi_1^* i^*$. Therefore $Ker\ i^* \leq Ker\ f_A^*$ and hence, by Theorem 6, $gB \supset A$.

The following lemma is widely known (cf. [6]).

Lemma 5. *If X is a compactum and $a \in H^q(X)$, then there exists a positive ϵ such that for every ϵ-map f from X onto a compactum Y there exists $b \in H^q(Y)$ such that $f^*(b) = a$.*

The following theorem is independent of the number of distinct n-bubbles the space has.

Theorem 7. *If X is an n-dimensional compactum such that $H^n(X)$ is finitely generated, then there exists a positive ϵ such that for every ϵ-map f from X onto a compactum Y, $H^n(fA) \neq 0$ for every n-bubble A in X.*

Proof: Let $\{a_1, \ldots, a_s\}$ be the set of generators of $H^n(X)$. Then by Lemma 5 there exist positive $\epsilon_1, \ldots, \epsilon_s$ such that for each k if f_k is an ϵ_k-map from X onto a compactum Y then there exists $b_k \in H^n(Y)$ such that $f_k^*(b_k) = a_k$. Let $\epsilon = min(\epsilon_1, \ldots, \epsilon_s)$ and let f be an ϵ-map from X onto a compactum Y and A an n-bubble in X. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^n(X) & \xleftarrow{\quad f^* \quad} & H^n(Y) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle j^*} \\ H^n(A) & \xleftarrow{\quad f_A^* \quad} & H^n(fA) \end{array}$$

Since $H^n(A) \neq 0$ and i^* is an epimorphism, there exist a non-zero element $a \in H^n(A)$ and integers $n_1, \ldots, n_s$ such that $i^*(n_1a_1 + \cdots + n_sa_s) = a$. Since f is ϵ_k-map for each k, there exists $b \in H^n(Y)$ such that $f^*(b) = n_1a_1 + \cdots + n_sa_s$. Hence $i^*f^*(b) = a$. Therefore $f_A^*j^*(b) = i^*f^*(b) = a$. Thus $j^*(b) \in H^n(fA)$.

4. The existence of $(n-1)$-bubble in n-dimensional clc^n compacta

In this section we examine the existence of $(n-1)$-bubbles in n-dimensional clc^n compacta. We start with the following

Lemma 6. *If H and K are two subcontinua of a continuum X and x_0, x_1, and x_2 are three points in X such that for $0 \leq i, j \leq 2$ and $i \neq j$, each of subcontinua H and K separate x_i from x_j, then $H \cap K \neq \emptyset$.*

Proof: Suppose that $H \cap K = \emptyset$. It follows easily from the definition that there exist disjoint nonempty open subsets C_0, C_1, and C_2 of $X \setminus H$ such that $C_0 \cup C_1 \cup C_2 = X \setminus H$ and C_k contains x_k for $k = 0$, 1, and 2. Since K is a continuum and $K \subset X \setminus H$, K is in one of the C_k's, say C_0. Then $C_1 \cap K = \emptyset = C_2 \cap K$.

Then $C_1 \cup C_2 \cup H$ is connected, and therefore it is in one of the components of $X \setminus K$, but it contains x_1 and x_2. This contradicts the fact that K separates x_1 from x_2.

Theorem 8. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an uncountable collection of mutually disjoint continua in a locally connected continuum X, each of which has the property that $X \setminus X_\lambda$ has more than one component. Then there exists a $\lambda \in \Lambda$ such that $X \setminus X_\lambda$ has exactly two components.*

Proof: Suppose that $X \setminus X_\lambda$ has more than two components for every $\lambda \in \Lambda$.

For every $\lambda \in \Lambda$ there exist a positive ϵ_λ and a point x_λ in X such that $B_\lambda = \{x \in X \mid d(x, x_\lambda) \leq \epsilon_\lambda\}$ lies inside C_λ, one of the components of $X \setminus X_\lambda$. Consider $\{\epsilon_\lambda \mid \lambda \in \Lambda\}$. Since Λ is uncountable, there exists a positive number ϵ_1 so that $\epsilon_1 < \epsilon_\lambda$ for uncountably many λ. Then there is a point x_1 in X such that $B_1 = \{x \in X \mid d(x_1, x) \leq \frac{\epsilon_1}{2}\}$ contains uncountably many x_λ's which satisfy $\epsilon_1 < \epsilon_\lambda$. Thus if $x_\lambda \in B_1$ then $B_1 \subset B_\lambda$. Let $\Lambda_1 = \{\lambda \in \Lambda \mid B_1 \subset B_\lambda\}$.

For every $\lambda \in \Lambda_1$ there exist a positive ϵ'_λ and a point x'_λ in X such that $B'_\lambda = \{x \in X \mid d(x, x'_\lambda) \leq \epsilon'_\lambda\}$ lies inside C'_λ, one of the components of $X \setminus X_\lambda$ but different from C_λ. Also, there exist a positive ϵ_2 and a point x_2 in X such that $B_2 = \{x \in X \mid d(x_2, x) \leq \frac{\epsilon_2}{2}\} \subset B'_\lambda$ for uncountably many $\lambda \in \Lambda_1$.

Let $\Lambda_2 = \{\lambda \in \Lambda_1 \mid B_2 \subset B'_\lambda\}$. Since for every $\lambda \in \Lambda_2$, $X \setminus X_\lambda$ has at least three components, there exist a positive ϵ''_λ and a point x''_λ in X such that $B''_\lambda = \{x \in X \mid d(x, x''_\lambda) \leq \epsilon''_\lambda\}$ lies inside C''_λ, one of the components of $X \setminus X_\lambda$ but different from C_λ and C'_λ. Therefore there exist a positive ϵ_3 and $x_3 \in X$

such that $B_3 = \{x \in X \mid d(x_3, x) \leq \frac{\epsilon_3}{2}\} \subset B''_\lambda$ for uncountably many $\lambda \in \Lambda_2$.

Let $\Lambda_3 = \{\lambda \in \Lambda_2 \mid B_3 \subset B''_\lambda\}$. Then Λ_3 is still uncountable. Let $\mu, \nu \in \Lambda_3$. Then $x_1 \in C_\mu \cap C_\nu$, $x_2 \in C'_\mu \cap C'_\nu$, and $x_3 \in C''_\mu \cap C''_\nu$. Therefore, by Lemma 6, $X_\mu \cap X_\nu \neq \emptyset$. This is a contradiction.

The following is a generalization of Sieklucki's Theorem. (See [4] or [6])

Theorem 9. *Suppose that X is a clc^n compactum with $dim_{\mathbf{Z}} X = n$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an uncountable collection of compacta in X with $dim_{\mathbf{Z}} X_\lambda = n$. Then there are two distinct indices μ, $\lambda \in \Lambda$ such that $dim_{\mathbf{Z}}(X_\mu \cap X_\lambda) = n$.*

A proof of the following lemma can be found in Bredon [3].

Lemma 7. *Let X be a clc^n compactum. Then $H^q(X)$ is finitely generated for $0 \leq q \leq n$.*

The following theorem can be found in Wilder ([13] pp. 100).

Lemma 8. *If A is a compact component of a locally compact Hausdorff space X, and P is an open set containing A, then X is the union of disjoint open sets U, V such that $A \subset U \subset P$.*

Lemma 9. *Suppose that X is a compact Hausdorff space and $0 \neq a \in H^q(X)$. Then there is a component Y of X such that $i^*(a) \neq 0$, where $i^*\colon H^q(X) \to H^q(Y)$ is induced by the inclusion.*

Proof: Let $\{Y_\mu \mid \mu \in \Lambda\}$ be the set of all components of X and let $i_\mu\colon Y_\mu \hookrightarrow X$.

Suppose that $i^*_\mu(a) = 0$ for every $\mu \in \Lambda$. By the weak continuity of Alexander-Spanier cohomology with compact supports, for every component Y_μ of X there exists a closed neighborhood M_μ of Y_μ such that $j^*_\mu(a) = 0$, where $j^*_\mu\colon H^q(X) \to H^q(M_\mu)$ is induced by the inclusion. By Lemma 8, there exists an open and closed subset N_μ such that $Y_\mu \subset N_\mu \subset M_\mu$. Then $\{N_\mu\}$ is an open covering of X. Since X is compact, there is a finite subcovering $\{N_1, \ldots, N_r\}$ of $\{N_\mu\}$.

Let $P_1 = N_1$ and $P_k = N_k \backslash (N_1 \cup \cdots \cup N_{k-1})$ for $k = 2, \ldots, r$. Since N_k is open and closed for each k, P_k also is open and closed and therefore X is the disjoint union of $P_1, \ldots, P_r$. Thus $(h_1^*, \ldots, h_r^*)\colon H^q(X) \to H^q(P_1) \oplus \cdots \oplus H^q(P_r)$ is an isomorphism, where $h_k^*\colon H^q(X) \to H^q(P_k)$ is induced by the inclusion. Therefore there exists k such that $h_k^*(a) \neq 0$. Let $l\colon P_k \hookrightarrow M_k$ be the inclusion map. Then $0 \neq h_k^*(a) = l^* j_k^*(a) = 0$. This is a contradiction.

Theorem 10. *If X is an n-dimensional clc^n compactum such that $H^n(V)$ is finitely generated for every connected open subset V of X, then X has an $(n-1)$-bubble.*

Proof: Since X is an n-dimensional compactum, by the characterization of dimension by mappings into spheres, there exist a closed subset C of X and a map $g\colon C \to S^{n-1}$ such that g can not be extended over X. Since S^{n-1} is a compact ANR, there exists an open neighborhood U of C such that g has an extension over U. Let $\epsilon = d(C, X \setminus U)$.

For each $0 < \mu < \epsilon$ let $X_\mu = \{x \in X \mid d(x, C) = \mu\}$. Then $H^{n-1}(X_\mu) \neq 0$ by Hopf's Extension Theorem. By Theorem 9, there is an uncountable subset Λ of the interval $(0, \epsilon)$ such that for each $\mu \in \Lambda$, X_μ is $(n-1)$-dimensional.

For each $\mu \in \Lambda$, by Lemma 9, there is a component Y_μ of X_μ such that $H^{n-1}(Y_\mu) \neq 0$. By Theorem 8, there exists a μ such that $X \setminus Y_\mu$ has at most two components. If $X \setminus Y_\mu$ has one component then, by hypothesis, $H^n(X \setminus Y_\mu)$ is finitely generated. If $X \setminus Y_\mu$ has two components, U and V, then $H^n(X \setminus Y_\mu) \cong H^n(U) \oplus H^n(V)$, which is finitely generated by the hypothesis. We have the following long exact sequence

$$\cdots \longrightarrow H^{n-1}(X) \longrightarrow H^{n-1}(Y_\mu) \longrightarrow H^n(X \setminus Y_\mu) \longrightarrow \cdots$$

Since $H^{n-1}(X)$ and $H^n(X \setminus Y_\mu)$ are finitely generated, $H^{n-1}(Y_\mu)$ is finitely generated. Therefore, by Theorem 2 of W. Kuperberg, Y_μ has an $(n-1)$-bubble and so does X.

Definition 7. [3] *The space X is called an "n-dimensional cohomology manifold over $\mathbf{Z}$" (denoted $n-cm$) if X has locally* *constant cohomology groups, locally equivalent to* $\mathbf{Z}$ *in degree* n*, and to zero in degrees other than* n*, and if* $dim_{\mathbf{Z}} X < \infty$*.*

Definition 8. *Let* X *be a compactum and* Π *a class of spaces. Then* X *is said to "have a factorization through* Π*" provided for every* $\epsilon > 0$ *there exist a space* $Y \in \Pi$*, a surjective map* $f_\epsilon \colon X \to Y$*, and a map* $g_\epsilon \colon Y \to X$ *such that* $d(g_\epsilon f_\epsilon, id_X) < \epsilon$*.*

Theorem 11. ([5] or [6]) *Let* X *be an* n*-dimensional connected and locally connected compactum that has a factorization through the class of orientable* $n - cm$ *compacta. If* U *is a connected open subset of* X*, then*

$$\tau_{U,X} : H^n(U) \to H^n(X)$$

is an isomorphism.

Corollary 6. *Let* X *be an* n*-dimensional connected and locally connected compactum that has a factorization through the class of orientable* $n - cm$ *compacta. Then* X *is an* n*-bubble. If in addition* X *is* clc^n *then* X *has an* $(n-1)$*-bubble.*

Proof: By Theorem 11, for every open subset U of X $\tau_{U,X} : H^n(U) \to H^n(X)$ is an epimorphism. Hence $H^n(A) = 0$ for every proper closed subset A of X. To show that X is an n-bubble, it suffices to show that X is n-cyclic. Since X is n-dimensional, there exists an open subset U such that $H^n(U) \neq 0$. Consider the set of all components V_μ of U. Since X is locally connected, V_μ is open for every μ. Thus $H^n(U) \cong \oplus H^n(V_\mu)$ and therefore $H^n(V_\mu) \neq 0$ for some μ. Since V_μ is connected, by Theorem 11, $H^n(X) \cong H^n(V_\mu) \neq 0$. Thus X is an n-bubble.

If in addition X is clc^n then, by Lemma 7, $H^n(X)$ is finitely generated. Thus, by Theorem 11, $H^n(U)$ is finitely generated for every connected open subset U of X. Therefore, by Theorem 10, X has an $(n-1)$-bubble.

Remark. If X is an n-dimensional $n - cm$ compactum, then X has an $(n-1)$-bubble.

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