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PROPERTIES OF COMPACTA THAT ARE PRESERVED BY FACTORIZATION AND GENERALIZATIONS OF THEOREMS OF A. DELEANU[1]

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ABSTRACT. Under the condition that Π is a class of compacta and a compactum X has the property that for every $\epsilon > 0$ there exist a space $Y \in \Pi$, a surjective map $f_\epsilon : X \to Y$, and a map $g_\epsilon : Y \to X$ such that $d(g_\epsilon f_\epsilon, id_X) < \epsilon$, we show that (1) if for every $Y \in \Pi$, $H^q(Y)$ is finitely generated and the number of generators of $H^q(Y)$ is less than a fixed number n, then $H^q(X)$ is finitely generated and the number of generators of $H^q(X)$ is less than n. (2) if Π is the class of orientable $n - cm$ compacta and X is in addition n-dimensional connected and locally connected then X has the property that if U is a connected open subset of X then $\tau_{U,X} : H^n(U) \to H^n(X)$ is an isomorphism. The latter generalizes a theorem of A.Deleanu.

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[1]Parts of this paper consist of parts of author's doctoral dissertation under the supervision of G. Kozlowski.

1. Introduction

By a *compactum* we mean a compact metric space. The various metrics and distances will be designated by the letter d. If X is a space and $A \subset X$, then $int(A)$ will denote the interior of the set A. By a *map* or *mapping* we mean a continuous function. If f is a map then $Ker\ f$ will denote the kernel of f and $Im\ f$ the image of f. A compactum X is said to have a factorization through a class Π of spaces if for every $\epsilon > 0$ there exist a space $Y \in \Pi$, a surjective map $f_\epsilon \colon X \to Y$, and a map $g_\epsilon \colon Y \to X$ such that $d(g_\epsilon f_\epsilon, id_X) < \epsilon$.

For cohomology we will use the Alexander-Spanier cohomology groups with compact supports and the notation of Massey's book [8]. The q-dimensional cohomology group with compact supports of a locally compact Hausdorff space X with integer coefficients will be denoted by $H^q(X)$. The homomorphism associated with the open subset U of X will be denoted by $\tau_{U,X}$ or simply τ when no confusion could occur. For a compact Hausdorff space Alexander-Spanier cohomology with compact supports is naturally isomorphic to Čech cohomology [10], and for any locally compact Hausdorff space X and for any integer $q > 0$, $H^q(X) \cong H^q(X^+)$, where X^+ is the one point compactification of X. Thus one can interpret the results of this paper in terms of Čech cohomology.

By a *compact ANR* we mean a compact absolute neighborhood retract [2]. If f is a map from X to Y, $f \simeq 0$ means that f is homotopic to a constant map, and $f^* \colon H^q(Y) \to H^q(X)$ will denote the induced homomorphism of f. The group of integers will be denoted by $\mathbf{Z}$. If A and B are groups then $A \leq B$ will mean that A is a subgroup of B. By the *dimension of* X we mean the covering dimension of X, and it will be denoted by $dim\ X$. The following definition is essentially stated in Borel [1].

Definition 1. *The cohomological dimension $dim_{\mathbf{Z}} X$ of a space X with respect to the group $\mathbf{Z}$ is defined to be the least integer*

n (or ∞) such that $H^q(U) = 0$ for every open subset U of X and $q > n$.

In 1962 A. Deleanu [4] published the following:

Let X be an n-dimensional connected compact ANR such that for every $\epsilon > 0$ there exists an ϵ-map of X onto a closed n-dimensional orientable manifold (depending on ϵ). Let U be a non-empty connected open subset of X. Then the homomorphism

$$\tau_{U,X} : H^n(U) \to H^n(X)$$

is an isomorphism.

In this paper we weaken the condition on X replacing compact ANR by locally connected compactum and the existence of ϵ-map onto a closed n-dimensional orientable manifold by the existence of ϵ-map onto an orientable $n - cm$ compactum. As an application of this generalization we could generalize the second theorem of A.Deleanu [4] in the same way.

We also show a sufficient condition for a compactum X to have finitely generated q-cohomology as a property that is preserved by factorization through a class of spaces.

2. Preliminaries

In this section we will discuss the definition of $n - cm$ compactum and an equivalent condition.

The following definitions are given by Bredon ([3] pp 281, 349, and 374).

Definition 2. *A precosheaf $\mathcal{A}$ on X is a covariant functor from the category of open subsets of X to the category of abelian groups and homomorphisms. If V and U are open subsets of X with $V \subset U$, the homomorphism corresponding to the inclusion is $i_{U,V} : \mathcal{A}(V) \to \mathcal{A}(U)$. We call the homomorphism $i_{U,V}$ the structure maps of the precosheaf $\mathcal{A}$. A homomorphism $h : \mathcal{A} \to \mathcal{B}$ of precosheaves is the family of homomorphisms $h_V : \mathcal{A}(V) \to \mathcal{B}(V)$ commuting with the structure maps of $\mathcal{A}$ and $\mathcal{B}$; that is, h is a natural transformation of functors.*

The constant precosheaf L is the precosheaf taking the value L on each U with the identity structure map for every pair of open subsets.

Definition 3. *A precosheaf $\mathcal{A}$ on X is said to be "locally zero" if for any open set $U \subset X$ and $y \in U$ there is a neighborhood $V \subset U$ of y with $i_{U,V}\colon \mathcal{A}(V) \to \mathcal{A}(U)$ trivial. A homomorphism $h\colon \mathcal{A} \to \mathcal{B}$ of precosheaves is said to be a "local isomorphism" if the precosheaves* Ker h *and* $\mathcal{B}$ /Im h *are both locally zero.*

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be precosheaves on a first countable space X and $h\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then h is a local isomorphism if and only if for each $x \in X$ there is a decreasing sequence $\{V_k\}$ of open neighborhoods of x such that $\{V_k\}$ is a fundamental system of open neighborhoods of x and in the following commutative diagram*

$$\begin{array}{ccccc}
\mathcal{A}(V_{k-1}) & \xleftarrow{\;i_{k-1}\;} & \mathcal{A}(V_k) & \longleftarrow & \mathcal{A}(V_{k+1}) \\
\downarrow & & \downarrow{\scriptstyle h_k} & & \downarrow \\
\mathcal{B}(V_{k-1}) & \longleftarrow & \mathcal{B}(V_k) & \xleftarrow{\;j_k\;} & \mathcal{B}(V_{k+1})
\end{array}$$

Ker $h_k \subset$ Ker i_{k-1} *and* Im $j_k \subset$ Im h_k.

The proof of the above lemma is straightforward so we omit it.

Remark: If $\{V_k\}$ satisfies the condition of the Lemma 1 then every subsequence of $\{V_k\}$ also satisfies the same condition.

Definition 4. [3] *Precosheaves $\mathcal{A}$ and $\mathcal{B}$ on X are said to be "equivalent" if $\mathcal{A}$ and $\mathcal{B}$ are equivalent under the smallest equivalence relation containing the relation of local isomorphism.*

The following can be found in Bredon ([3], pp. 411 and 422)

Lemma 2. *1. Precosheaves $\mathcal{A}$ and $\mathcal{B}$ on X are equivalent if and only if there exists a precosheaf $\mathcal{C}$ and local isomorphisms $\mathcal{C} \to \mathcal{A}$ and $\mathcal{C} \to \mathcal{B}$.*
2. Composites of local isoorphisms are local isomorphisms.

The following are also given by Bredon ([3] pp. 374)

Definition 5. *If $\mathcal{A}$ is a precosheaf on X and U is an open subset of X, then $\mathcal{A}|U$ is the precosheaf on U defined by using the group $\mathcal{A}(V)$ for $V \subset U$ and structure maps $i_{V,W} \colon \mathcal{A}(W) \to \mathcal{A}(V)$ for open sets $W \subset V \subset U$. A precosheaf $\mathcal{A}$ will be said to be "locally constant" if each point $x \in X$ has a neighborhood U such that the precosheaf $\mathcal{A}|U$ on U is equivalent to a constant precosheaf. If this is the constant precosheaf M, where M is an abelian group, then $\mathcal{A}$ is said to be "locally equivalent to M." The space X will be said to be possess "locally constant cohomology groups over $\mathbf{Z}$ locally equivalent to M^*," where M^* is a graded abelian group, if the precosheaf $\mathcal{H}^q \colon U \to H^q(U)$ is locally equivalent to M^q for all q.*

Bredon used the notation $\mathcal{H}^q(X)$ for the precosheaf $(U \to H^q(U))$ but we omit X on the notation since we use this precosheaf for only one space X.

Definition 6. *The space X is called an "n-dimensional cohomology manifold over $\mathbf{Z}$" (denoted $n - cm$) if X has locally constant cohomology groups, locally equivalent to $\mathbf{Z}$ in degree n, and to zero in degrees other than n, and if $dim_{\mathbf{Z}} X < \infty$.*

Proposition 1. *Let X be a compactum with $dim_Z X < \infty$. Then X is an $n - cm$ iff for each $x \in X$ there exists a decreasing sequence $\{V_k\}$ of open neighborhoods of x and homomorphisms $g_k \colon \mathbf{Z} \to H^n(V_k)$ and $f_k \colon H^n(V_k) \to \mathbf{Z}$ such that $\{V_k\}$ is a fundamental system of open neighborhoods of x, $\tau_k \colon H^q(V_{k+1}) \to H^q(V_k)$ is 0 for $q \neq n$ and for each k, in the following diagram*

$$\begin{array}{ccc} H^n(V_k) & \xleftarrow{\ \tau_k\ } & H^n(V_{k+1}) \\ {\scriptstyle g_k}\nwarrow\searrow{\scriptstyle f_k} & & {\scriptstyle f_{k+1}}\swarrow\nearrow{\scriptstyle g_{k+1}} \\ & \mathbf{Z} & \end{array}$$

$g_k f_{k+1} = \tau_k$ and $f_k g_k = id_{\mathbf{Z}}$.

Proof: $(\Rightarrow)$ By the definition of $n - cm$, the precosheaf $\mathcal{H}^n(U \to H^n(U))$ is locally equivalent to $\mathbf{Z}$. Let $x \in X$. Then there exists an open neighborhood U of x such that there exist a precosheaf $\mathcal{B}$ and local isomorphisms $h\colon \mathcal{H}^n \,|\, U \to \mathcal{B}$ and $l\colon \mathbf{Z} \to \mathcal{B}$. By Lemma 1 we can find a decreasing sequence $\{V_k\}$ of open neighborhoods of x in U such that $\{V_k\}$ is a fundamental system of open neighborhoods of x and $Ker\ h_k \subset Ker\ \tau_{k-1}$, $Im\ i_k \subset Im\ h_k \cap Im\ l_k$ and $Ker\ l_k \subset Ker\ j_{k-1}$ are true in the following diagram:

$$\begin{array}{ccccccccc} H^n(U) & \leftarrow \cdots \leftarrow & H^n(V_{k-1}) & \xleftarrow{\ \tau_{k-1}\ } & H^n(V_k) & \xleftarrow{\ \tau_k\ } & H^n(V_{k+1}) & \leftarrow \cdots \\ \downarrow & & \downarrow{\scriptstyle h_{k-1}} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_{k+1}} & \\ B_U & \leftarrow \cdots \leftarrow & B_{V_{K-1}} & \xleftarrow{\ i_{k-1}\ } & B_{V_k} & \xleftarrow{\ i_k\ } & B_{V_{k+1}} & \leftarrow \cdots \\ \uparrow & & \uparrow{\scriptstyle l_{k-1}} & & \uparrow{\scriptstyle l_k} & & \uparrow{\scriptstyle l_{k+1}} & \\ Z_U & \leftarrow \cdots \leftarrow & Z_{k-1} & \xleftarrow{\ j_{k-1}\ } & Z_k & \xleftarrow{\ j_k\ } & Z_{k+1} & \leftarrow \cdots \end{array}$$

where $B_{V_k} = \mathcal{B}(V_k)$ and $Z_k = \mathbf{Z}$ for all k and $j_k = id_{\mathbf{Z}}$ for all k.

Since $Ker\ j_{k-1} = 0$ and $Ker\ l_k \subset Ker\ j_{k-1}$ for all k, l_k is a monomorphism for all k. Also, since $Im\ i_{k-1} \subset Im\ l_{k-1}$, for each element $a \in B_{V_k}$ there is an unique element $a' \in Z_k$ such that $i_{k-1}(a) = l_{k-1} j_{k-1}(a')$. Define $\varphi_k\colon B_{V_k} \to Z_k$ by $\varphi_k(a) = a'$. Then φ_k is a homomorphism. Since $i_{k-1} l_k = l_{k-1} j_{k-1}$ and l_k is a monomorphism, φ_k is an epimorphism.

We now show that $\varphi\colon \mathcal{B} \to Z$ is a local isomorphism. By Lemma 1 it suffices to show that $Ker\,\varphi_k \subset Ker\, i_{k-1}$ and $Im\, j_k \subset Im\, \varphi_k$. Since φ_k is an epimorphism, $Im\, j_k \subset Im\, \varphi_k$ holds. Also if $\varphi_k(a) = 0$ then by the definition of φ_k we have that $i_{k-1}(a) = l_{k-1}j_{k-1}(0) = 0$. Thus $a \in Ker\, i_{k-1}$. Therefore φ is a local isomorphism. Thus, by Lemma 2, $\varphi \circ h$ is a local isomorphism.

We now show that $\varphi_k \circ h_k$ is an epimorphism. Let $a_k \in Z_k$, then there is $a_{k+1} \in Z_{k+1}$ such that $j_k(a_{k+1}) = a_k$. Since φ_{k+1} is an epimorphism, there is an element $b_{k+1} \in B_{V_{k+1}}$ such that $\varphi_{k+1}(b_{k+1}) = a_{k+1}$. Thus $i_k(b_{k+1}) \in B_{V_k}$ and $\varphi_k i_k(b_{k+1}) = j_k\varphi_{k+1}(b_{k+1}) = a_k$. Since $Im\, i_k \subset Im\, h_k$ there is an element $c_k \in H^n(V_k)$ such that $h_k(c_k) = i_k(b_{k+1})$. Therefore $\varphi_k h_k(c_k) = \varphi_k i_k(b_{k+1}) = a_k$.

Note that the relationships $Ker\, \varphi_k h_k \subset Ker\, \tau_{k-1}$ and $Im\, j_k \subset Im\, \varphi_k h_k$ are not necessarily true but we have a subsequence of $\{V_k\}$ which satisfies the above condition. Abusing notation by using the same notation for this subsequence, we have the following diagram:

$$\begin{array}{ccccc}
H^n(V_{k-1}) & \xleftarrow{\tau_{k-1}} & H^n(V_k) & \xleftarrow{\tau_k} & H^n(V_{k+1}) \\
\downarrow{\scriptstyle f_{k-1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k+1}} \\
Z_{k-1} & \xleftarrow{j_{k-1}} & Z_k & \xleftarrow{j_k} & Z_{k+1}
\end{array}$$

with the properties that for each k, f_k is an epimorphism, $Ker\, f_k \subset Ker\, \tau_{k-1}$ and $Im\, j_k \subset Im\, f_k$.

Since, for each k, f_k is an epimorphism and $Z_k = \mathbf{Z}$ is a free abelian group, we have $H^n(V_k) \cong Ker\, f_k \oplus Z_k$. We also have an isomorphism $H^n(V_k)/Ker\, f_k \cong Z_k$; call it f'_k. Since $Ker\, f_{k+1} \subset Ker\, \tau_k$, we have a homomorphism

$$\tau'_k\colon H^n(V_{k+1})/Ker\, f_{k+1} \to H^n(V_k)$$

such that $\tau_k = \tau'_k p_{k+1}$, where

$$p_{k+1} : H^n(V_{k+1}) \to H^n(V_{k+1})/Ker\, f_{k+1}$$

is the natural homomorphism.

Define $g_k\colon Z_k \to H^n(V_k)$ by $g_k = \tau_k' \circ {f_{k+1}'}^{-1} \circ j_k^{-1}$. Then

$$f_k g_k = f_k \tau_k' {f_{k+1}'}^{-1} j_k^{-1} = j_k f_{k+1}' {f_{k+1}'}^{-1} j_k^{-1} = id_{Z_k}$$

and

$$\tau_k = \tau_k' p_{k+1} = \tau_k' {f_{k+1}'}^{-1} f_{k+1} = \tau_k' {f_{k+1}'}^{-1} j_k^{-1} j_k f_{k+1} = g_k j_k f_{k+1}.$$

Therefore if we let $j_k = id_Z$, we have the desired diagram.

For $q \neq n$, by Lemma 2, there exist a precosheaf $\mathcal{C}$ and local isomorphisms $h\colon \mathcal{C} \to \mathcal{H}^q \,|\, U$ and $l\colon \mathcal{C} \to 0$. Then we have a subsequence of $\{V_k\}$ (abusing notation we use the same notation again) such that in the following diagram

$$\begin{array}{ccccc}
H^q(V_{k-1}) & \xleftarrow{\tau_{k-1}} & H^q(V_k) & \xleftarrow{\tau_k} & H^q(V_{k+1}) \\
\uparrow{\scriptstyle h_{k-1}} & & \uparrow{\scriptstyle h_k} & & \uparrow{\scriptstyle h_{k+1}} \\
C_{V_{k-1}} & \xleftarrow{i_{k-1}} & C_{V_k} & \xleftarrow{i_k} & C_{V_{k+1}} \\
\downarrow{\scriptstyle l_{k-1}} & & \downarrow{\scriptstyle l_k} & & \downarrow{\scriptstyle l_{k+1}} \\
0_{k-1} & \xleftarrow{j_{k-1}} & 0_k & \xleftarrow{j_k} & 0_{k+1}
\end{array}$$

for each k, $Ker\ h_k \cup Ker\ l_k \subset Ker\ i_{k-1}$, $Im\ \tau_k \subset Im\ h_k$, and $Im\ j_k \subset Im\ l_k$. Where $0_k = 0$ for each k. Thus $i_k = 0$ for each k. If we let $\varphi_k\colon 0_k \to C_{V_k}$ be the zero homomorphism then, since $i_k = 0$, φ is a local isomorphism by Lemma 1. Hence $h \circ \varphi$ is a local isomorphism. Therefore there is a subsequence of $\{V_k\}$ (again we use the same notation for this subsequence) which has the property that $Im\ \tau_k \subset Im\ (h_k \circ \varphi_k) = 0$. Hence $\tau_k = 0$.

($\Leftarrow$) By Lemma 1, we have local isomorphisms $f\colon \mathcal{H}^n|U \to \mathbf{Z}$ and $\mathcal{H}^q \,|\, U \to 0$ for $q \neq n$. For any $V \subset U$ let $\mathcal{B}(V) = H^n(V)/Ker\ f_V$, and let $h_V\colon H^n(V) \to H^n(V)/Ker\ f_V$ be the natural homomorphism. Then it is easy to see that $h\colon \mathcal{H}^n|U \to \mathcal{B}$ is a local isomorphism. Let $g_V\colon \mathbf{Z} \to H^n(V)/Ker\ f_V$ be the isomorphism such that $g_V^{-1} h_V = f_V$. Then $g\colon \mathbf{Z} \to \mathcal{B}$ is a local isomorphism. Therefore, the precosheaf $\mathcal{H}^n : U \to H^n(U)$ is locally equivalent to $\mathbf{Z}$. Hence X is an $n - cm$.

3. Properties of compacta that are preserved by factorization through a class of spaces

Definition 7. *Let X be a compactum and Π a class of spaces. Then X is said to "have a factorization through Π" provided for every $\epsilon > 0$ there exist a space $Y \in \Pi$, a surjective map $f_\epsilon : X \to Y$, and a map $g_\epsilon : Y \to X$ such that $d(g_\epsilon f_\epsilon, id_X) < \epsilon$. X is said to "have an approximate factorization through Π" provided for every $\epsilon > 0$ there exist a space $Y \in \Pi$ and maps $f_\epsilon : X \to Y$ and $g_\epsilon : Y \to X$ such that $d(g_\epsilon f_\epsilon, id_X) < \epsilon$.*

Clearly, If X has a factorization through a class Π of spaces, then X has an approximate factorization through Π.

Lemma 3. *Let X and Y be compacta and $f : X \to Y$ a map. Suppose that $a \in H^q(X)$ and $b \in H^q(Y)$ such that $f^*(b) = a$. Then there exists a positive ϵ such that if $g : X \to Y$ is a map with $d(f, g) < \epsilon$, then $g^*(b) = a$.*

Proof: Consider Y as a closed subset of Hilbert cube Q. By the weak continuity of the Alexander-Spanier cohomology with compact supports, there exist a closed neighborhood P of Y in Q and $c \in H^q(P)$ such that $i^*(c) = b$, where $i^* : H^q(P) \to H^q(Y)$ is induced by inclusion. Let $\epsilon = d(Y, Q \setminus intP)$.

If $g : X \to Y$ is a map with $d(f, g) < \epsilon$, then $d(if, ig) < \epsilon$. Thus $if \simeq ig$ in P and therefore $(if)^*(c) = (ig)^*(c)$. Hence $g^*(b) = g^* i^*(c) = f^* i^*(c) = f^*(b) = a$.

Theorem 1. *Suppose that X is a compactum, n and n' are positive integers, and Π is a class of compacta such that for each $Y \in \Pi$ the q-th cohomology group $H^q(Y)$ is finitely generated and the number of generators and torsion elements of $H^q(Y)$ is less than n and n', respectively. Suppose also that X has an approximate factorization through Π. Then $H^q(X)$ is finitely generated and the number of generators and torsion elements of $H^q(X)$ is less than n and n', respectively.*

Proof: By Lemma 3 combined with the hypothesis, for each $a \in H^q(X)$ there exists a positive number ϵ_a such that if $\epsilon \leq \epsilon_a$ then $(g_\epsilon f_\epsilon)^*(a) = f_\epsilon^*(g_\epsilon^*(a)) = a$, where $f_\epsilon \colon X \to Y_\epsilon$, and $g_\epsilon \colon Y_\epsilon \to X$ are maps such that $d(g_\epsilon f_\epsilon, id_X) < \epsilon$.

We first show that the number of linearly independent elements in $H^q(X)$ is less than n. Suppose that $\{a_1, a_2, \dots, a_s\}$ is a linearly independent set in $H^q(X)$. If $\epsilon \leq \min(\epsilon_{a_1}, \dots, \epsilon_{a_s})$, then we have that $f_\epsilon^* g_\epsilon^*(a_k) = a_k$ for each k. It is easy to see that $\{g_\epsilon^*(a_k)\}_{k=1}^s$ is a linearly independent set in $H^q(Y_\epsilon)$. Thus $s < n$.

We now show that the number of torsion elements of $H^q(X)$ is less than n'. Let $\{a_1, a_2, \dots, a_s\}$ be a set of distinct torsion elements of $H^q(X)$. If $\epsilon \leq \min(\epsilon_{a_1}, \dots, \epsilon_{a_s})$, then for each $1 \leq k \leq s$ $f_\epsilon^* g_\epsilon^*(a_k) = a_k$ and $g_\epsilon^*(a_k)$ is a torsion element of $H^q(Y_\epsilon)$. Thus $\{g_\epsilon^*(a_1), \dots, g_\epsilon^*(a_s)\}$ is a set of distinct torsion elements of $H^q(Y_\epsilon)$. Thus $s < n'$.

Let $T(X)$ and $T(Y_\epsilon)$ be torsion subgroups of $H^q(X)$ and $H^q(Y_\epsilon)$, respectively. Then f_ϵ^* and g_ϵ^* induce homomorphisms

$$\bar{f}_\epsilon^* \colon H^q(Y_\epsilon)/T(Y_\epsilon) \to H^q(X)/T(X)$$

and

$$\bar{g}_\epsilon^* \colon H^q(X)/T(X) \to H^q(Y_\epsilon)/T(Y_\epsilon).$$

Let $\bar{H}^q(X)$ and $\bar{H}^q(Y_\epsilon)$ denote $H^q(X)/T(X)$ and $H^q(Y_\epsilon)/T(Y_\epsilon)$, respectively. For each $a \in H^q(X)$ let $\bar{a} = a + T(X)$. Since $T(X)$ is finite, in order to prove $H^q(X)$ is finitely generated, it suffices to show that $\bar{H}^q(X)$ is a finitely generated free abelian group.

Let K be a free subgroup of $\bar{H}^q(X)$ with maximum rank. Then K is a subgroup generated by $\bar{a}_1, \dots, \bar{a}_s$ with $s < n$. If $\bar{a} \in \bar{H}^q(X)$ then $\{\bar{a}, \bar{a}_1, \dots, \bar{a}_s\}$ is not a linearly independent set. Thus there is a positive integer m such that $m\bar{a} \in K$. Therefore $\bar{H}^q(X)/K$ is a torsion group.

Choose a positive ϵ such that $\epsilon \leq \min(\epsilon_{a_1}, \dots, \epsilon_{a_s})$. Then $\bar{H}^q(X) \cong Ker\ \bar{g}_\epsilon^* \oplus Im\ \bar{g}_\epsilon^*$ and $Im\ \bar{g}_\epsilon^*$ is free. If $\bar{a} \in K \cap Ker\ \bar{g}_\epsilon^*$, then $\bar{a} = \sum_{k=1}^{s} m_k \bar{a}_k$ and so

$$0 = \bar{f}_\epsilon^* \bar{g}_\epsilon^*(\bar{a}) = \bar{f}_\epsilon^* \bar{g}_\epsilon^*(\sum m_k \bar{a}_k) = \sum m_k \bar{f}_\epsilon^* \bar{g}_\epsilon^*(\bar{a}_k) = \sum m_k \bar{a}_k = \bar{a}.$$

Thus $Ker\ \bar{g}_\epsilon^* \cong (Ker\ \bar{g}_\epsilon^* + K)/K \leq \bar{H}^q(X)/K$.

But $Ker\ \bar{g}_\epsilon^*$ is torsion-free and $\bar{H}^q(X)/K$ is torsion and hence $Ker\ \bar{g}_\epsilon^* = 0$. Therefore $\bar{H}^q(X) \cong Im\ \bar{g}_\epsilon^*$ which is free.

Lemma 4. *Let X be a locally connected compactum that has a factorization through a class Π of locally connected compacta. Then if A and C are proper closed subsets of X such that $A \subset int(C)$, there exists a positive η such that for any $\epsilon \leq \eta$ there exist a proper closed subset B of a locally connected compactum Y in Π and two maps of pairs*

$$(X, A) \xrightarrow{f_\epsilon} (Y, B) \xrightarrow{g_\epsilon} (X, C)$$

such that $d(g_\epsilon f_\epsilon, h) < \epsilon$, where $h\colon (X, A) \hookrightarrow (X, C)$. Moreover, if $X \setminus A$ is connected, $Y \setminus B$ may also be assumed to be connected.

It is a generalization of a result of Ganea [6], but his argument works here too. So we omit the proof.

We have the following properties of $n - cm$ [1].

Lemma 5. *Let X be a connected locally compact Hausdorff $n - cm$. Then*
(1) $H^n(A) = 0$ for every proper closed subset A of X.
(2) If X is orientable and U is connected, then $\tau_{U,X}$ is an isomorphism.

Theorem 2. *Let X be a connected and locally connected compactum that has a factorization through the class of orientable $n - cm$ compacta. If U is a connected open subset of X, then*

$$\tau_{U,X} : H^n(U) \to H^n(X)$$

is an isomorphism.

Proof: Let $A = X \setminus U$. We first show that the map $i^*\colon H^{n-1}(X) \to H^{n-1}(A)$ induced by the inclusion is an epimorphism. Let $a \in H^{n-1}(A)$. By the weak continuity of the

Alexander-Spanier cohomology with compact supports, there exists a closed neighborhood C of A such that there is $c \in H^{n-1}(C)$ with $h_1^*(c) = a$, where $h_1^*\colon H^{n-1}(C) \to H^{n-1}(A)$ is induced by inclusion. By Lemma 3, there exists a positive ϵ_1 such that for any map $f\colon A \to C$ with $d(f, h_1) < \epsilon_1$, we have $f^*(c) = a$. Since X has a factorization through the class of $n - cm$ compacta, by Lemma 4, there exist a positive $\epsilon(< \epsilon_1)$, a proper closed subset B of an $n - cm$ compactum Y such that $Y \setminus B$ is connected, and two maps of pairs

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (X, C)$$

such that $d(gf, h) < \epsilon$, where $h\colon (X, A) \hookrightarrow (X, C)$.

Let $f_1\colon X \to Y, f_2\colon A \to B, g_1\colon Y \to X,$ and $g_2\colon B \to C$ be the maps induced by f and g, respectively. Then $f_2^* g_2^*(c) = h_1^*(c) = a$.

Following the argument of A. Deleanu [4] and Ganea [6], it is easy to show that a is in the image of i^* and therefore i^* is an epimorphism. This implies that $\delta_{X,A}\colon H^{n-1}(A) \to H^n(U)$ in the long exact sequence is 0, and so $\tau_{U,X}$ is a monomorphism.

The same argument also shows that $H^n(A) = 0$ and hence $\tau_{U,X}$ is an epimorphism.

Remark. If we use $\mathbf{Z}/2$ coefficients, then Theorem 2 is true even when X has a factorization through the class of $n - cm$ compacta.

Lemma 6. [6] *Suppose that X is a compact ANR such that for every $\epsilon > 0$ there exists an ϵ-map of X onto a closed n-dimensional orientable manifold (depending on ϵ). Then X has a factorization through the class of closed n-dimensional orientable manifolds.*

Corollary 1. *Let X be an n-dimensional connected compact ANR such that for every $\epsilon > 0$ there exists an ϵ-map of X onto a closed n-dimensional orientable manifold (depending on ϵ). Let U be a non-empty connected open subset of X. Then the* *homomorphism*

$$\tau_{U,X} : H^n(U) \to H^n(X)$$

is an isomorphism.

Proof: By Lemma 6 and Theorem 2, it is clear.

The above corollary is a theorem of A. Deleanu [4]. So our Theorem 2 is a generalization of Deleanu's.

Corollary 2. *Let X be an n-dimensional space satisfying the conditions stated in Theorem 2. Then $H^n(X) \neq 0$.*

Ganea's argument [6] works here too. So we omit the proof.

Theorem 3. *Let X be an n-dimensional connected and locally connected compactum that has a factorization through the class of orientable $n-cm$ compacta. Let A be a closed subset of X and $x \in A$. A necessary and sufficient condition for x to be an interior point of A is that*

$$H^n(U) \neq 0$$

for all suficiently small neighborhoods U of x in A.

The above theorem is a generalization of the second theorem of A. Deleanu [4], but his argument works here too. So we omit the proof. The following is the second theorem of A.Deleanu [4];

Corollary 3. *Let X be an n-dimensional connected compact ANR such that for every $\epsilon > 0$ there exists an ϵ-map of X onto a closed n-dimensional orientable manifold (depending on ϵ). Let A be a closed subset of X and $x \in A$. A necessary and sufficient condition for x to be an interior point of A is that*

$$H^n(U) \neq 0$$

for all sufficiently small neighborhoods U of x in A.

Corollary 4. (Invariance of domain). *Let X be an n-dimensional connected and locally connected compactum that* *has a factorization through the class of orientable $n-cm$ compacta. Let G_1 and G_2 be homeomorphic subsets of X. If G_1 is open, then G_2 is also open.*

Deleanu's argument [4] works here. so we omit the proof.

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