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ON BALANCED TOPOLOGICAL GROUPS

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ABSTRACT. A brief survey is given of the following question concerning topological groups: Does the class of balanced (SIN) groups and the class of functionally balanced (FSIN) groups coincide? A topological group G is a balanced group if the left and right uniform structures on G coincide and it is a functionally balanced group (FSIN) if the class of bounded right uniformly continuous real-valued functions on G coincides with the class of bounded left uniformly continuous real-valued functions. During the period 1988-92 the question was answered in the affirmative by a number of authors for the locally compact case. At present, extensions of these results have been obtained by various authors .The survey mentions these and gives a new characterization of balance suggested by these extensions.

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1. Introduction

It is well known that there are two natural uniformities on a topologicalgroup G. These are the left uniformity $\mathcal{U}_l$ and the right uniformity $\mathcal{U}_r$. If $L_U = \{(x,y) \in G \times G \mid x^{-1}y \in U \in \mathcal{U}\}$ and $R_U = \{(x,y) \in G \times G \mid yx^{-1} \in U \in \mathcal{U}\}$, where $\mathcal{U}$ is a base for the open sets at the neutral element (or the identity) of G, then $\mathcal{U}_l$ is generated by the sets $\{L_U \mid U \in \mathcal{U}\}$ and $\mathcal{U}_r$ is generated by the sets $\{R_U \mid U \in \mathcal{U}\}$. When studying these uniformities, it is probably better from an intuitional viewpoint, to think of $\mathcal{U}_l$ as generated by the sets $\{xU \mid x \in G$ and $U \in \mathcal{U}\}$ and $\mathcal{U}_r$ as generated by the sets $\{Ux \mid x \in G$ and $U \in \mathcal{U}\}$ [Note that if $x^{-1}y \in U$, then $y \in xU$, and if $yx^{-1} \in U$, then $y \in Ux$.]

Definition 1.1. A function $\phi : G \to G$ is uniformly continuous for the pair of structures $(\mathcal{U}_l,\mathcal{U}_r)$ if for each $V \in \mathcal{U}$ there is $U \in \mathcal{U}$ such that $(\phi(x),\phi(y)) \in L_V$ for all $(x,y) \in L_U$. (Equivalently, if $y \in xU$ then $\phi(y) \in V\phi(x)$.) Similar definitions hold for the structure pairs $(\mathcal{U}_r,\mathcal{U}_l)$, $(\mathcal{U}_l,\mathcal{U}_l)$ and $(\mathcal{U}_r,\mathcal{U}_r)$.

Definition 1.2. [10] The two structures, $\mathcal{U}_l$ and $\mathcal{U}_r$, are said to be equivalent if the identity map $i : G \to G$ is uniformly continuous for the structure pairs $(\mathcal{U}_l,\mathcal{U}_r)$ and$(\mathcal{U}_r,\mathcal{U}_l)$. When this happens we say that the group G is balanced.

It is clear that a topological group G is balanced iff given a neighborhood $U \in \mathcal{U}$ there is a neighborhood $V \in \mathcal{U}$ such that $x^{-1}Vx \subset U$, for all $x \in G$. From this, it is easy to see that G is balanced iff $\bigcap\{xUx^{-1} \mid x \in G\}$ is a neighborhood of the neutral element e for each neighborhood $U \in \mathcal{U}$. (See Note 1.20 for the equivalent concept of a SIN group.)

In this paper all topological groups are T_0 and therefore completely regular. If G is a topological group then e will denote the neutral element or identity in G. $\mathbb{N}$ will denote the positive integers, and $\mathbb{R}$ will denote the real numbers. Finally, if A is a subset of a space X, then A^c is the complement of A in X.

The author was introduced to the question of the relation between a balanced topological group and a functionally balanced topological group in the following manner. In 1970, C. Chou asked him the following question: Is it possible to construct a continuous real valued function on a noncompact and nondiscrete locally compact topological group which is not uniformly continuous. At the time, I recalled the following example that I had seen in graduate school. If one lets $f(x) = \Sigma_{n\in\mathbb{N}} f_n(x)$, where:

$$f_n(x) = \begin{cases} 0 & x < n + \frac{n}{2n+2} \\ (2n+2)x - (2n^2+3n) & n + \frac{n}{2n+2} \leq x < n + \frac{1}{2} \\ -(2n+2)x + 2n^2 + 3n + 2 & n + \frac{1}{2} \leq x < n + \frac{n+2}{2n+2} \\ 0 & n + \frac{n+2}{2n+2} \leq x, \end{cases}$$

then f is continuous but not uniformly continuous on the real numbers $\mathbb{R}$. Visually, the graph of f_n is the triangular spike centered at $n+\frac{1}{2}$, of height one above the x-axis, and with base of length $\frac{1}{n+1}$. This led naturally to the following construction.

Construction 1.3. [5] Select first a neighborhood U of the identity e in a locally compact group G with compact closure and a sequence $\{x_i\}$ such that the sequence of neighborhoods $\{x_i U\}$ are pairwise disjoint. Next select a nested sequence of symmetric neighborhoods $V_i \subset U$ whose Haar measures go to 0. Since G is completely regular there are continuous real valued functions f_i satisfying $f_i(x_i) = 1$ and $f_i([x_i V_i]^c) \subset \{0\}$. The function $f(x) = \Sigma_{i\in\mathbb{N}} f_i(x)$ is continuous but not (left) uniformly continuous (the measures of the bases of the spikes go to 0 so $\bigcap_{i\in\mathbb{N}} V_i$ cannot be open).

Since a compact uniform space has only one uniformity giving its topology, this construction gives a simplified proof of the following theorem.

Theorem 1.4. *Let G be a nondiscrete locally compact group G, then the following are equivalent:*

(a) *G is compact.*
(b) *Every continuous bounded real valued function on G is left (right) uniformly continuous.*
(c) *G is uniformly balanced.*

Definition 1.5. A T_0 topological group G is *uniformly balanced* if the classes of bounded real valued continuous functions, bounded real valued left uniformly continuous functions, and bounded real valued right uniformly continuous functions on G coincide.

At this point a natural question to ask is 'how does the balance or nonbalance of a topological group affect the classes of bounded uniformly continuous real valued functions on a group G?' This leads to the following definition.

Definition 1.6. [12] A topological group G is *functionally balanced* if the class of bounded real-valued left uniformly continuous functions coincides with the class of bounded real-valued right uniformly continuous functions on G.

An elementary fact about a topological group G is the following: For each $x \in G$ and $U \in \mathcal{U}$, $V = xUx^{-1}$ is also in $\mathcal{U}$. This fact and the construction in the proof of Theorem 1.4 led the author to consider the following question.

Question 1.7. Let $\{x_i\}$ be a sequence of points in a locally compact group G, and suppose that for each $i \in \mathbb{N}$, $x_iU = V_ix_i$. Under what conditions might $\bigcap_{i\in\mathbb{N}} V_i = \bigcap_{i\in\mathbb{N}} x_iUx_i^{-1}$ be non-open?

If the answer to the above question is that such conditions exist, then such an answer would show the existence of a class of left uniformly continuous real valued functions that are not right uniformly continuous functions on a locally compact group G.

For the next example we recall the definition of the left modular function Δ_l. This function is a continuous homomorphism of the locally compact group G, with left Haar measure m, into the multiplicative group of positive real numbers $\mathbb{R}^* = \{x \in \mathbb{R} \mid x > 0\}$.

Definition 1.8. $\Delta_l(s) = \int f(s^{-1}x)\,dm(x)/\int f(x)\,dm(x)$, where $f \in C_{00}+$, and $s \in G$. Here $C_{00}+$ are the continuous functions with compact support on G. It is well known that Δ_l does not depend on the choice of $f \in C_{00}+$. Furthermore if G is not unimodular, Δ_l is not identically one and so there is a sequence $\{x_i\}$ in G for which $\Delta_l(x_i) \to \infty$ (since Δ_l is a homomorphism).

Example 1.9. **[5]** Let G be a locally compact non-Abelian group. If U is a neighborhood with compact closure of $e \in G$, m is left Haar measure on G, Δ_l is the left modular function, and if the sequence $\{x_i\} \subset G$ then we get a sequence of neighborhoods$\{V_i\}$ of e such that $x_iU = V_ix_i$ and $m(U) = m(x_iU) = m(V_ix_i) = \Delta_l(x_i)m(V_i)$. If a locally compact group is not unimodular then we can select the sequence $\{x_i\}$ so that the sequence $\{x_iU\}$ consists of pairwise disjoint subsets and so that $\Delta_l(x_i) \to \infty$ and $m(Vi) \to 0$ [4]. This implies $\bigcap_{i\in\mathbb{N}} V_i$ cannot be open. Since G is completely regular we can define a sequence of left uniformly continuous functions $\{f_i\}$ such that $f_i(x_i) = 1$, $f_i([x_iU]^c) \subset \{0\}$ and $0 \le f_i \le 1$. Then the function $f = \Sigma_{i\in\mathbb{N}} f_i$ is left uniformly continuous but not right uniformly continuous.

Lemma 1.10. (Folklore Lemma) ([4,8.18]) *Let G be a nondiscrete metric group. G is balanced iff given $x_n \to e$ and $\{y_n\}$ any sequence in G then $(y_n)^{-1}x_ny_n \to e$.*

Example 1.11. **[5]** Suppose G is a locally compact metric group and not balanced. Then there is a neighborhood U with compact closure, a sequence $\{y_n\}$, and a sequence $z_n \to e$ such that $(y_n)^{-1}z_ny_n \notin U$ for each n. This sequence $\{y_n\}$ is

not contained in any compact subset of G (see [5, Th.4.9]). Let U be a neighborhood of e with compact closure. Then passing to subsequences if necessary we can pick a sequence $\{x_n\}$ satisfying
$x_n \notin \bigcup\{\overline{U}x_i\overline{U} \mid i = 1, 2, ..., n-1\}$. (See [5] for the argument.) Let V be a symmetric neighborhood of e such that $V \subset U$ and $V^2 \subset U$, then V is uniformly discrete in the sense that $Vx_iV \bigcap \{Vx_jV = \emptyset$ if $i \neq j$. Since G is completely regular we can define as before a sequence of left uniformly continuous functions $\{f_i\}$ such that $f_i(x_i) = 1$ and $f_i([x_iV]^c) \subset \{0\}$. Then the function $f = \Sigma_{i\in\mathbb{N}} f_i$ is left uniformly continuous but not right uniformly continuous. The sequence of sets $\{V_n\}$ defined by $V_n = x_nV[x_n]^{-1}$ satisfies the condition that $\bigcap_n V_n$ contains no open set.

Observation 1.12. In both Examples 1.9 and 1.11 a sequence of neighborhoods of e of the form $V_n = x_nV[x_n]^{-1}$ was found that satisfy the condition "$\bigcap_n V_n$ contains no open sets". This leads naturally to the following questions.

Question 1.13. Is it true for topological (locally compact) groups that functionally balanced group coincides with balanced group?

Question 1.14. Is it true that a (locally compact) group G is balanced iff for each left (or right) uniformly discrete sequence $\{x_n\}$ in G and for each neighborhood V of e, $\bigcap_n x_nV[x_n]^{-1}$ is a neighborhood of e?

The argument in Example 1.11 shows that a positive answer to Question 1.14 implies a positive answer to Question 1.13.

Definition 1.15. [6] A set B in a topological group G is *right uniformly discrete* if there is a neighborhood U of e such that $Ux \cap Uy = \emptyset$ if $x \neq y$ and $x, y \in B$. A similar definition holds for left uniformly discrete sets.

In an investigation of Question 1.14, the following result was shown in [6] to hold.

Definition 1.16. A topological group is *α-compact* if it can be written as a union of α compact sets but not as a union of *beta* compact sets where $\beta < \alpha$. (α, β are infinite cardinals).

Theorem 1.17. ([6], 1976) *Let G be a locally compact α-compact group. Then G is balanced iff for every right uniformly discrete set B satisfying $|B| \leq \alpha$ and for each neighborhood U of e, $\bigcap_{x \in B} xUx^{-1}$ is a neighborhood of e.*

Note 1.18. This theorem answers Question 1.14 for all σ-compact locally compact groups in the affirmative and leads to the following question.

Question 1.19. Is it true for a locally compact group G that G is balanced iff every open σ-compact subgroup of G is balanced? A positive answer gives a positive answer to Questions 1.13 and 1.14 in the case where G is locally compact.

Note 1.20. Traditionally, balanced groups were called SIN (=small invariant neighborhoods) groups. That is, there is a base $\mathcal{U}$ of neighborhoods of the neutral element such that if $U \in \mathcal{U}$ then $xUx^{-1} = U$ for every $x \in G$. (If G is balanced and if $U = \bigcap_{x \in G} xVx^{-1}$, where V is a neighborhood of e, then for each $y \in G$, $yUy^{-1} = y(\bigcap_{x \in G} xVx^{-1})y^{-1} = \bigcap_{x \in G} yxVx^{-1}y^{-1} = \bigcap_{z \in G} zVz^{-1} = U \subset V$. This means that G has a base of small invariant neighborhoods.)

In the period 1988-92 a number of researchers [including Hansell andTroallic (France) [3], Itzkowitz, Rothman, Strassberg, and Wu [7], Milnes (Canada) [9], Pestov (Russia) [11], and Protasov (Russia) [12]] came up with independent affirmative solutions in the locally compact case to these questions. In fact Hansell, Troallic, and Protasov actually generalized the results of the locally compact case in several directions. (See Sections 2 and 3 for a survey of some of these results). The solution, that the author participated in, made use of Lie group structure theory and approximation by Lie groups (projective limits). This approach became possible after the following had been shown.

Definition 1.21. A locally compact group G satisfies the G_δ-*condition* iff each neighborhood of the identity contains a compact normal G_δ subgroup.

Theorem 1.22. [7] *Let G be a locally compact group satisfying the G_δ-condition. Then G is balanced iff it is functionally balanced.*

At this point, Wu noted that Theorem 1.22 suggested that use of approximation by Lie groups could lead to a solution of the problem in the locally compact case. In fact, using this approach the following theorem was obtained.

Theorem 1.23. [7] *Let G be a locally compact group then the following are equivalent:*

(a) *G is balanced.*
(b) *Every open σ-compact subgroup of G is balanced.*
(c) *For each neighborhood U of e and each right uniformly discrete sequence $\{x_n\} \subset G$ the set $\bigcap_{n\in\mathbb{N}} x_n U (x_n)^{-1}$ is a neighborhood of e.*
(d) *G satisfies the G_δ-condition, and G is the projective limit of balanced Lie groups.*
(e) *G satisfies the G_δ-condition, and each continuous open homomorphic image of G has equal uniformities.*

2. Extensions beyond the locally compact case

Protasov (1991) in his paper, "Functionally balanced groups" [11], actually showed more. His paper was the first to go beyond considering the locally compact situation. He showed that the characterizations of balanced groups hold for almost metrizable groups.

Definition 2.1. (Arhangel'skii, [1]) A topological space X is *of pointwise countable type* if each $x \in X$ lies in a compact subset $K$of X having countable character in X.

Definition 2.2. A subset of X is *of countable character in X* if it contains a countable neighborhood base in X.

Fact 2.3. The class of pointwise countable type topological spaces include metric spaces, locally compact spaces, and Cech-complete topological spaces.

Definition 2.4. A topological group is *almost metrizable* if it is of pointwise countable type.

Theorem 2.5. (Protasov, [12]) *If G is an almost metrizable group then it is balanced iff it is functionally balanced.*

In 1995 Megrelishvili, Nickolas, and Pestov, [8], were able to extend the characterization of balanced groups to locally connected topological groups.

Definition 2.6. A subset A of a topological group G is *left neutral in G* if for each neighborhood V of e in G there is a neighborhood U of e such that $UA \subset AV$. A similar definition holds for right neutral sets. A subset of G that is right and left neutral is called neutral.

The following theorem by these authors has a proof similar in character to the motivating Examples 1.9 and 1.11. Thus for completeness and probable usefulness toward a complete characterization of balanced T_0 groups we will give a slightly modified version of their proof. We note that the complete regularity condition for topological groups is actually a bit stronger than the statement usually used in the definition of a completely regular space. In fact, the proof of [4, Th. 8.2 and 8.4], actually shows that in a T_0 topological group G, given a closed set F and a point a not in F, there is a left uniformly continuous real valued function f satisfying $f(a) = 1$, $f(F) = \{0\}$, and $0 \leq f \leq 1$ on G. (The construction actually yields a left uniformly continuous function ψ satisfying $\psi(a) = 0$, $\psi(F) = \{1\}$, and $0 \leq \psi \leq 1$ on G. This ψ actually satisfies the inequality $|\psi(x) - \psi(y)| \leq 2\sigma(x,y)$, where σ is a left invariant metric constructed in [4,8.2]. $f = 1 - \psi$.) Thus if a point x in G and an open symmetric neighborhood U of x are given, then there is a left uniformly continuous real valued function f on G satisfying $f(x) = 1, f(U^c) = \{0\}$, and $0 \leq f \leq 1$ on G.

Theorem 2.7. **(Megrelishvili, Nickolas, and Pestov [8])** *Every left uniformly discrete subset of a functionally balanced group G is left neutral.*

Proof: Let A be a left uniformly discrete subset of G. By Definition 1.15 there is a symmetric neighborhood W of e satisfying $aW \cap bW = \emptyset$ if $a \neq b$ and $a, b \in A$. Thus there is a symmetric neighborhood V of e satisfying $V^2 \subset W$. [Note that $V \subset W$ and that $aV^2 \cap bV^2 = \emptyset$ for $a \neq b$ and $a, b \in A$.] Let $a \in A$ be fixed. Since G is completely regular we can choose a left uniformly continuous real valued function f_a, satisfying $f_a(a) = 1$, $f_a([aV]^c) = \{0\}$, and $0 \leq f_a \leq 1$ on G. For $b \neq a$ and $b \in A$, let $f_b(x) = f_a(ab^{-1}x)$. Then $f_b(b) = f_a(ab^{-1}b) = f_a(a) = 1$, $f_b([bV]^c) = \{0\}$, and $0 \leq f_b \leq 1$ on G. Let $f = \Sigma_{a \in A} f_a$.From the property of V^2 it is easy to see that the family of open sets aV, $a \in A$, satisfies the condition that if $x_a \in aV$ and $x_b \in bV$ then $x_a V \cap x_b V = \emptyset$. Furthermore, if x in G is arbitrary, then it is easy to check that xV can intersect at most one set of the form aV for $a \in A$. This f is therefore left uniformly continuous and since G is functionally balanced it is also right uniformly continuous. Thus there is a neighborhood of the identity U in G such that if $x, y \in G$ and $xy^{-1} \in U$, then $|f(x) - f(y)| < 1$. Now let $u \in U$ and $a \in A$ be arbitrary. Since $(ua)a^{-1} = u \in U$, it follows that $|f(ua) - f(a)| = |f(ua) - 1| < 1$, so that $ua \in AV$. Thus $Ua \subset AV$ and A is left neutral.□

This theorem was used to prove the following theorem.

Theorem 2.8. **(Megrelishvili, Nickolas, and Pestov [8])** *The following are equivalent for a locally connected topological group G:*

(a) *G is balanced.*
(b) *Every left uniformly discrete subset is left neutral.*
(c) *G is functionally balanced.*

In the next section we will say more about this proof and point out how the method of proof can be used to get another characterization of balanced groups.

Troallic, [14], in 1996 extended the results of the locally compact case to quasi-k-spaces.

Definition 2.9. **[10]** A space X is a *quasi-k-space* if it has the property that $A \subset X$ is closed iff $A \cap K$ is closed in K for every countably compact subset K in X.

Theorem 2.10. **(Troallic [14])** *Let G be a topological group that is also a quasi-k-space. Then the following are equivalent:*

(a) *G is balanced.*
(b) *For each sequence $\{x_n\}$ in G and each neighborhood V of e, $\bigcap_{n \in \mathbb{N}} x_n V (x_n)^{-1}$ is a neighborhood of e.*
(c) *Each countable subgroup of G is balanced.*
(d) *Every right uniformly discrete sequence in G is left uniformly discrete.*
(e) *For each neighborhood V of e in G and for every right uniformly discrete sequence $\{x_n\}$ in G, $\bigcap_{n \in \mathbb{N}} x_n V (x_n)^{-1}$ is a neighborhood of e.*
(f) *G is functionally balanced.*

3. More on Balanced Groups

In Section 2 we discussed several extensions to the original theorem which stated that for locally compact groups the concept of a balanced group and the concept of a functionally balanced group coincide. Note that the two actual generalizations of the locally compact case, namely those of Protasov and of Troallic, involve making use of compactness or countable compactness type conditions in the definition of the class of topological groups discussed. The theorem of Megrelishvili, Nickolas, and Pestov does not involve a compactness type condition, but only the condition of local connectedness. Their method of proof can be used to get another property of T_0 topological groups that is equivalent to the balance property. We begin with a definition.

Definition 3.1. The set A is said to be *strongly neutral in* G if there is an open neighborhood V of e and a neighborhood U of e such that
(a) $a_1V \cap a_2V = \emptyset$, if $a_1 \neq a_2$ and $a_1, a_2 \in A$, and
(b) $Ua \subset aV$, for all $a \in A$.

Note 3.2. Condition (a) says that A is left uniformly discrete and (b) says that A is right uniformly discrete.

The next lemma is proved in [8] and its proof is a transfinite induction.

Lemma 3.3. **(Megrelishvili, Nickolas, Pestov)** *Let V be a neighborhood of e in G, where G is a T_0 group. Then there is a set $A \subset G$ such that*

(a) $aV \cap bV = \emptyset$, *if* $a, b \in A$ *and* $a \neq b$.
(b) $AVV^{-1} = G$.

Theorem 3.4. *Let G be a T_0 group in which every left uniformly discrete set is strongly neutral. Then G is balanced.*

Proof: Let W be an arbitrary open neighborhood of e. Let V be a symmetric neighborhood of e such that $V^5 \subset W$. By Lemma 3.3 there is a subset $A \subset G$ that is left uniformly discrete with $a_1V \cap a_2V = \emptyset$, if $a_1 \neq a_2$ and $a_1, a_2 \in A$, and $AV^2 = G$. Since A is strongly neutral there is a symmetric neighborhood U of e satisfying $Ua \subset aV$ for each $a \in A$. Thus for each $a \in A$, $a^{-1}Ua \subset V$. Now let $g \in G$ be arbitrary. From Lemma 3.3 there is a representation $g = avv'$, where $a \in A$, and $v, v' \in V$. Then $g^{-1}Ug = (avv')^{-1}U(avv') = v'^{-1}v^{-1}(a^{-1}Ua)vv' \subset v'^{-1}v^{-1}Vvv' \subset V^5 \subset W$. Thus $U \subset \bigcap_{g\in G} gWg^{-1}$. Since W is arbitrary, G is balanced.□

Theorem 3.5. *Let G be a T_0 topological group. Then the following are equivalent:*

(a) *G is balanced.*
(b) *Every left uniformly discrete set A in G is strongly neutral.*

(c) *For each left uniformly discrete set A and each neighborhood U of e, $\bigcap_{x\in A} xUx^{-1}$ is a neighborhood of e.*

Proof: (c) $\Rightarrow$ (b): If A is left uniform discrete there is a neighborhood V of e such that $aV \cap bV = \emptyset$, if $a, b \in A$ and $a \neq b$. Now $W = \bigcap_{x\in A} xVx^{-1}$ is a neighborhood of e. Therefore we have $W \subset xVx^{-1}$ for all $x \in A$, so that $Wx \subset xV$ for all $x \in A$. But then A is strongly neutral.

(b) $\Rightarrow$ (a): This is Theorem 3.4. (a) $\Rightarrow$ (c): If G is balanced and U is a neighborhood of e, then $\bigcap_{x\in G} xUx^{-1}$ is a neighborhood of e, so that in particular $\bigcap_{x\in A} xUx^{-1}$ is a neighborhood of e for each left uniformly discrete set A.

Remark 3.6. In their proof of Theorem 2.8, Megrelishvili, Nickolas, and Pestov showed that every left neutral set is strongly neutral if G is locally connected. This follows from the fact that in the case where G is locally connected the neighborhood U of e that satisfies $Ua \subset AV$ can be chosen connected while it is clear that AV is not connected. Thus since $Ua \cap aV \neq \emptyset$, one can conclude from $Ua \subset AV$ that $Ua \subset aV$. Their theorem (our Theorem 2.7) shows that every left discrete set in a functionally balanced group is left neutral. Thus we can combine our Theorem 3.5 with their result as follows.

Theorem 3.7. *Let G be a T_0 group. Then the following are equivalent:*

(a) *G is balanced*

(b) *G is functionally balanced and every left neutral set is strongly neutral.*

Note 3.8. First we note that in the cases of locally compact groups, almost metrizable groups, and quasi-k-groups, the equivalence of balance and functional balance implies the condition that every left neutral set is strongly neutral for those cases. This leads to the following:

Question 3.9. If G is functionally balanced is every left neutral set A strongly neutral? If not, is there an example of a

T_0 group G which is functionally balanced yet contains a left neutral set which is not strongly neutral?

A negative answer to this question is hinted at by the following result of Protasov and Saryev [13,Lemma 3 and Theorem 2].

Theorem 3.10. (Protasov and Saryev) *For a T_0 group G the following are equivalent:*

(a) *G is functionally balanced.*
(b) *For every neighborhood V of e and every subset X of G there is a neighborhood U of e such that $UX \subset XV$.*

Finally, in [4,8.17] the following theorem of Graev [3] is proved: "Let G be a T_0 group having an open basis at e consisting of sets U, such that $xUx^{-1} = U$ for all $x \in G$. Then G is topologically isomorphic with a subgroup of a direct product of topological groups each of which has a two sided invariant metric."

Note that if a group has a two sided invariant metric then it is a balanced metric group. Since a product of balanced groups is balanced, a subgroup of balanced groups is balanced, and the group G is actually balanced (see 1.20), we can state Graev's theorem as follows:

Theorem 3.11. *For a T_0 group G the following are equivalent:*

(a) *G is balanced.*
(b) *G is a subgroup of a product of balanced metric groups.*

Thus to get an example that answers Question 3.9 in the negative one would need an example of a group that cannot be written as a subgroup of a product of metric groups.

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