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ANOTHER CHARACTERIZATION OF THE PSEUDO-ARC

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ABSTRACT. It is known that the pseudo-arc has the property that any two copies of it which are setwise near each other in terms of Hausdorff distance are homeomorphically near each other. We show that it is the only nondegenerate continuum with this property. We also characterize all compacta with this property.

CHARACTERIZATION OF THE PSEUDO-ARC

The pseudo-arc is the simplest nondegenerate hereditarily indecomposable continuum and has been characterized by Bing [2] as, up to homeomorphism, the only nondegenerate hereditarily indecomposable chainable continuum. It is also homogeneous [1, 12], has many interesting mapping properties, and has many other known characterizations. Here we present a characterization in terms of approximating embeddings.

A *compactum* is a compact metric space and a *continuum* is a compact, connected, metric space. A continuum is *indecomposable* if it is not the union of two of its proper subcontinua

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and *hereditarily indecomposable* if each of its subcontinua is indecomposable. A continuum X is *chainable* if for every $\epsilon > 0$ there exists an open cover $\mathcal{C} = \{C_0, C_1, C_2, \ldots, C_n\}$ of X such that $\mathrm{diam}(C_i) < \epsilon$ for each i and $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. (Such a cover is called an *ϵ−chain.*) Each element of a chain is called a *link* of the chain. A chain is *taut* if it satisfies the additional condition that $\overline{C_i} \cap \overline{C_j} \neq \emptyset$ if and only if $|i-j| \leq 1$. A chain $\mathcal{C}$ *properly covers* the space X if $\mathcal{C}$ covers X and every link of $\mathcal{C}$ contains a point not in the closure of any other link of $\mathcal{C}$.

A chain $\{C_0, C_1, \ldots, C_n\}$ is *ϵ−crooked* for some $\epsilon > 0$ if for each $0 \leq i < i + 2 < j \leq n$ there exist indices k and l with $i < k < l < j$ such that $\mathrm{dist}(C_i, C_l) < \epsilon$ and $\mathrm{dist}(C_j, C_k) < \epsilon$. An arc A is *ϵ−crooked* if for any distinct points p and q of A there exist points r and s in A such that either $p < r < s < q$ or $p > r > s > q$ in the relative ordering on A, $\mathrm{dist}(p, s) < \epsilon$, and $\mathrm{dist}(r, q) < \epsilon$. A Peano continuum M is *ϵ−crooked* if every arc in M is ϵ−crooked.

Combining results of Bing [2] yields the following characterization.

Theorem 1. [2] *A nondegenerate continuum X is homeomorphic to a pseudo-arc if and only if for every $\epsilon > 0$ there exists an ϵ−crooked ϵ−chain covering X.*

A *pattern* is a function $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that $|f(i + 1) - f(i)| \leq 1$ for each $0 \leq i < m$. Chain $\mathcal{D} = \{D_0, D_1, \ldots, D_m\}$ *follows the pattern f* in chain $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ if $\mathcal{D}$ closure refines $\mathcal{C}$ and $\overline{D_i} \subset C_{f(i)}$ for each $0 \leq i \leq m$. Sequences of patterns and chains can be used in defining homeomorphisms between chainable continua as given by the following.

Theorem 2. [1, 11] *Let X and Y be chainable continua. For each index i let $\mathcal{C}^i$ and $\mathcal{D}^i$ be chains properly covering X and Y respectively such that:*

1. *$mesh(\mathcal{C}^i) < 1/(i + 1)$ and $mesh(\mathcal{D}^i) < 1/(i + 1)$, and*

2. *there exists a pattern f_i such that chain $\mathcal{C}^{i+1}$ follows the pattern f_i in chain $\mathcal{C}^i$ and chain $\mathcal{D}^{i+1}$ follows the pattern f_i in chain $\mathcal{D}^i$.*

There is a homeomorphism $h : X \to Y$ such that for each $x \in X$ and each index i there is a link C^i_j of $\mathcal{C}^i$ such that $x \in C^i_j$ and $h(x) \in D^i_j$.

The above theorem is a slight variation of theorems by Bing [1] and Moise[11], and is proven in the same manner as their theorems.

A compactum X is said to have *property HN* (for "homeomorphically near") if for any copy X_0 of X in some Euclidean space $\mathbb{R}^n$ (or in the Hilbert cube $\mathbb{Q}$) and any $\epsilon > 0$ there exists $\delta > 0$ such that for any copy X_1 of X in $\mathbb{R}^n$ (or in $\mathbb{Q}$) with $\text{dist}_H(X_0, X_1) < \delta$ there exists a homeomorphism $h : X_0 \to X_1$ such that $\text{dist}(x, h(x)) < \epsilon$ for each $x \in X_0$. (Here $\text{dist}_H(X_0, X_1)$ denotes the Hausdorff distance between X_0 and X_1, i.e. $\text{dist}_H(X_0, X_1) = \inf\{\epsilon | A \subset N(B, \epsilon) \text{ and } B \subset N(A, \epsilon)\}$.)

In 1984 the author [8] published a proof that the pseudo-arc has property *HN*. This result has been known to persons familiar with the pseudo-arc since at least the early 1950's and implicitly referenced if not formally stated. Persons less familiar with the pseudo-arc, however, seemed surprised by references to this fact, prompting the publication of a proof of it. For completeness, we again publish a proof below. We will make use of the following result which is implied by the early work of Bing [1, 2] on the pseudo-arc. A proof of this specific version will also appear in the author's upcoming book [9].

Theorem 3. [1, 2] *If $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ is any chain covering the pseudo-arc P and $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ is any surjective pattern, then there exists a chain $\mathcal{D} = \{D_0, D_1, \ldots, D_m\}$ covering P such that $\mathcal{D}$ follows the pattern f in $\mathcal{C}$.*

Theorem 4. [8] *The pseudo-arc P has property HN.*

Proof: Let P_0 be a pseudo-arc embedded in the metric space X, and let $\epsilon > 0$. Let $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ be a taut $\epsilon-$chain properly covering P_0 consisting of sets which are open in X. There exists $\delta > 0$ such that if P_1 is any other pseudo-arc embedded in X with $\text{dist}_H(P_0, P_1) < \delta$ then $\mathcal{C}$ also properly covers P_1.

By repeated applications of Theorem 3, there exist sequences of chains $\{\mathcal{C}^i\}_{i=0}^{\infty}$ and $\{\mathcal{D}^i\}_{i=0}^{\infty}$ such that, for each i,

1. $\mathcal{C}^i$ properly covers P_0 and $\mathcal{D}^i$ properly covers P_1,
2. $\text{mesh}(\mathcal{C}^i) < 1/(i+1)$ and $\text{mesh}(\mathcal{D}^i) < 1/(i+1)$,
3. there exists a pattern f_i such that $\mathcal{C}^{i+1}$ follows f_i in $\mathcal{C}^i$ and $\mathcal{D}^{i+1}$ follows f_i in $\mathcal{D}^i$, and
4. $\mathcal{C}^0 = \mathcal{D}^0 = \mathcal{C}$.

By Theorem 2, there exists a homeomorphism $h : P_0 \to P_1$ such that for each $p \in P_0$ there exists an index i with $p \in \mathcal{C}_i^0$ and $h(p) \in \mathcal{D}_i^0$. Since $\mathcal{C}^0 = \mathcal{D}^0 = \mathcal{C}$ is an $\epsilon-$chain, we thus have that $\text{dist}(p, h(p)) < \epsilon$ for every $p \in P$. □

The original part of our characterization is the observation that the converse of the above result is true.

Theorem 5. *If the nondegenerate continuum X has property HN, then X is a pseudo-arc.*

Proof: Let X be a nondegenerate continuum with property *HN*, and let X_0 be a copy of X embedded in the Euclidean space $\mathbb{R}^n$ (or in the Hilbert cube $\mathbb{Q}$). Let $\epsilon > 0$, and let $\delta > 0$ be the number guaranteed by property *HN* for $\epsilon/3$ and X_0. (Note that $\delta \leq \epsilon/3$.)

There exists a polygonal arc A in $\mathbb{R}^n$ (or in $\mathbb{Q}$) such that A is $\delta/3-$crooked and $\text{dist}_H(A, X_0) < \delta/2$. Let N be a small tubular neighborhood of A of radius less than $\delta/6$. Notice that N is $\delta-$crooked and $\text{dist}_H(N, X_0) < 2\delta/3$. Since N is either an $n-$cell or a copy of $\mathbb{Q}$ there exists a copy X_1 of X embedded in N such that X_1 intersects both ends of N and thus $\text{dist}_H(X_0, X_1) < \delta$.

Let $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ be an $\epsilon/3-$chain covering X_1 which is $\epsilon/3-$crooked. Such a chain can be formed by subdividing N lengthwise.

By property *HN* there exists a homeomorphism $h : X_0 \to X_1$ such that $\text{dist}(x, h(x)) < \epsilon/3$ for each $x \in X_0$. Let $\mathcal{V} = \{V_0, V_1, \ldots, V_n\}$ where $V_i = h^{-1}(U_i \cap X_1)$ for each $0 \leq i \leq n$. Then $\mathcal{V}$ is an $\epsilon-$crooked $\epsilon-$chain covering X_0. Since such a chain exists for every $\epsilon > 0$, by Theorem 1 X_0 (and hence X) is a pseudo-arc. □

It should be noted that in the proofs of each of Theorems 4 and 5 we actually have a stronger result than that given by the statement of property *HN*. In the proof of Theorem 4 we only need assume that the pseudo-arc is embedded in a metric space, not necessarily in an Euclidean space. In the proof of Theorem 5 it suffices to assume that for any two copies X_0 and X_1 of X embedded in an Euclidean space which are sufficiently close in terms of Hausdorff distance there is a continuous function $f : X_0 \to X_1$ with $\text{dist}(x, f(x)) < \epsilon$ for every $x \in X_0$. It is not necessary to assume that f is a homeomorphism.

Compacta with Property *HN*

While the pseudo-arc is the only nondegenerate continuum with property *HN* it is not the only such compactum. Let C_0 be a Cantor set in the metric space X and let $\epsilon > 0$. If $\mathcal{U}$ is an open cover of C by disjoint sets in X of diameter less than ϵ, there exists $\delta > 0$ such that each Cantor set C_1 in X with $\text{dist}_H(C_0, C_1) < \delta$ is covered by $\mathcal{U}$ and intersects every element of $\mathcal{U}$. There is then a homeomorphism $h : C_0 \to C_1$ with x and $h(x)$ in a common element of $\mathcal{U}$ for each $x \in C_0$. Thus the Cantor set C also has property *HN*.

Each of the following compacta clearly has property *HN:*

- a finite set,
- a Cantor set,
- a finite collection of disjoint pseudo-arcs, and

- the union of finitely many disjoint pseudo-arcs and a finite set.

One can show that the above are the only compacta with property *HN.*

Theorem 6. *If the compactum X has property HN, then X is one of the four types of spaces listed above.*

Proof. Clearly each of these spaces has property *HN.*

Case 1. Suppose that the compactum X has a nondegenerate component and for every $\gamma > 0$ there exist infinitely many components of X each of diameter less than γ. Let X_0 be a copy of X embedded in the Hilbert cube $\mathbb{Q}$.

Let $\epsilon > 0$ be such that some component of X_0 has diameter greater than 3ϵ. Let $0 < \delta < \epsilon$. Let $\{x_0, x_1, \ldots, x_n\}$ be a finite set of points of X_0 such that for each $x \in X_0$ there exists x_i with $\mathrm{dist}(x, x_i) < \delta/2$.

By hypothesis, X_0 has infinitely many components of diameter less than $\delta/2$.

Thus, there exist disjoint nonempty closed sets $C_0, C_1, \ldots, C_n$, and C_{n+1} such that $X_0 = C_0 \cup C_1 \cup \cdots \cup C_n \cup C_{n+1}$ and $\mathrm{diam}(C_i) < \delta/2$ for each $0 \leq i \leq n$. By assumption, $\mathrm{diam}(C_{n+1}) > 3\epsilon$. Let $f : X_0 \to \mathbb{Q}$ be an embedding of X_0 into $\mathbb{Q}$ such that $x_i \in f(C_i)$ for each $0 \leq i \leq n$, $\mathrm{diam}(f(C_i)) < \delta/2$ for each $0 \leq i \leq n+1$, and $\mathrm{dist}(x_0, f(C_{n+1})) < \delta/2$.

Let $X_1 = f(X_0)$. By construction, $\mathrm{dist}_H(X_0, X_1) < \delta$. However, X_0 has a component of diameter greater than 3ϵ while every component of X_1 has diameter less than $\epsilon/2$. Thus, any homeomorphism h of X_0 onto X_1 must satisfy $\mathrm{dist}(x, h(x)) > \epsilon$ for some $x \in X_0$. Hence, the compactum X does not have property *HN.*

Case 2. Let Y be a compactum having infinitely many nondegenerate components. Let Y_0 be a copy of Y embedded in the Hilbert cube $\mathbb{Q}$. By case 1, if Y has property *HN* then Y has only finitely many small nondegenerate components. Thus there exists $\epsilon > 0$ such that no nondegenerate component of Y_0 has diameter less than 3ϵ.

Let $0 < \delta < \epsilon$. Let L be a nondegenerate component of Y_0 which contains a limit of other nondegenerate components of Y_0. Let M be a nondegenerate component of Y_0 distinct from L such that every point of M is a distance less than $\delta/2$ from L. There exist disjoint closed sets D_0 and D_1 such that $Y_0 = D_0 \cup D_1$, $L \subset D_0$, $M \subset D_1$ and every point of D_1 is a distance less than δ from D_0. Let $Y_1 = D_0 \cup D_2$, where D_2 is a homeomorphic copy of D_1 in $\mathbb{Q}$, $\text{diam}(D_2) < \epsilon$, D_2 is contained in the $\delta-$neighborhood of D_0, and $D_0 \cap D_2 = \emptyset$.

Thus Y_1 is a homeomorphic copy of Y in $\mathbb{Q}$ with $\text{dist}_H(Y_0, Y_1) < \delta$. However, if $h : Y_0 \to Y_1$ is any homeomorphism then $\text{dist}(y, h(y)) > \epsilon$ for some $y \in Y_0$, since every nondegenerate component of Y_0 has diameter greater than 3ϵ and Y_1 contains a nondegenerate component with diameter less than ϵ. Thus Y does not have property *HN*.

Case 3. Let Z be a compactum containing infinitely many points, with every component of Z a singleton. Let Z_0 be an embedding of Z in the Hilbert cube $\mathbb{Q}$. Suppose Z_0 contains an isolated point z_0.

Let $\epsilon > 0$ be such that every other point of Z_0 is a distance at least 2ϵ from z_0. Let $0 < \delta < \epsilon$, and let z_1 be a limit point of Z_0.

There exist disjoint closed sets E_0 and E_1 such that $Z_0 = E_0 \cup E_1$, $z_1 \in E_1$, and every point of E_1 is a distance less than δ from E_0. Let $Z_1 = E_0 \cup E_2$, where E_2 is a homeomorphic copy of E_1 in $\mathbb{Q}$, E_2 is contained in the $\delta-$neighborhood of z_0, and $E_0 \cap E_2 = \emptyset$.

Thus Z_1 is a homeomorphic copy of Z in $\mathbb{Q}$ with $\text{dist}_H(Z_0, Z_1) < \delta$. However, if $k : Z_0 \to Z_1$ is any homeomorphism and z is the point of Z_0 with $k(z)$ the point of E_2 corresponding to z_1, then $\text{dist}(z, k(z)) > \epsilon$. Thus Z does not have property *HN*.

By the above three cases we have that any compactum W such that W has property *HN* and W has infinitely many components must have every component a singleton and contain no isolated points. Thus W is homeomorphic to the Cantor set.

Any compactum with only finitely many components which has property *HN* must have every component having proprety *HN* and hence be one of the other three types listed. □

Hereditary Equivalence

Though no formal connection is evident, property *HN* is suggestive of a possible relationship to hereditary equivalence. A continuum X is *hereditarily equivalent* if every nondegenerate subcontinuum of X is homeomorphic to X.

The arc is clearly hereditarily equivalent. The pseudo-arc was constructed by Moise [11] as an example of a nondegenerate indecomposable continuum which was hereditarily equivalent, answering negatively a question of Mazurkiewicz [10].

Jones [6] has asked the following.

Question 7. What effect, if any, does hereditary equivalence have on homogeneity?

The pseudo-arc is currently the only known nondegenerate hereditarily indecomposable continuum which is either hereditarily equivalent or homogeneous.

Henderson [5] has shown that the only nondegenerate decomposable hereditarily equivalent continuum is the arc. Cook [4] has shown that every hereditarily equivalent continuum is tree-like. It is worth briefly recalling the outline of Cook's argument.

Proof: (Outline) Let X be an hereditarily equivalent continuum. Suppose $f : X \rightarrow P$ is an essential map from X to the one-dimensional polyhedron P. There is a minimal subcontinuum Y of X on which f is essential. However, for every $\epsilon > 0$ there is an ϵ−homeomorphism of Y onto a proper subcontinuum Z of Y (since Y is also hereditarily equivalent and the space Y^Y of self-maps of Y is separable).

By choice of Y, f is inessential on Z. This is a contradiction since the space P^Y of all mappings of Y into P is an ANR, and hence an essential map of Y into P cannot be arbitrarily close to an inessential map.

Every continuum of dimension greater than one contains a subcontinuum admitting an essential map onto a one-dimensional polyhedron ([7], p. 271). Since any nondegenerate subcontinuum of X is homeomorphic to X, X is not of dimension greater than one.

Any one-dimensional continuum admitting no essential map onto a one-dimensional polyhedron is tree-like [3]. □

While not specifically considering embeddings, the critical use of the existence of an ϵ—homeomorphism between near subcontinua is suggestive of a possible relationship to property *HN*.

Question 8. Is every hereditarily equivalent continuum chainable?

If the answer to the above question is yes, then the two currently known examples, i.e. the arc and the pseudo-arc, are the only nondegenerate hereditarily equivalent continua.

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