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THE CARDINALITY OF THE COARSEST QUASI-PROXIMITY CLASS OF LOCALLY COMPACT T_2 SPACES

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ABSTRACT. In the class of the locally compact T_2 spaces we prove that the cardinality of the coarsest quasi-proximity class is equal to 1 if and only if either X is compact or X is non-Lindelöf. If the space is non-compact, Lindelöf then we show that this cardinality is at least $2^{2^{\aleph_0}}$. A characterization of the elements of the coarsest compatible quasi-uniformity is also given.

1. INTRODUCTION

In this paper we are going to continue the investigation of the problems arisen in [12], [13] and [8]. In [12] we partly answer the following question: what is the cardinality of the set of all compatible (transitive, non-transitive, totally bounded) quasi-uniformities for a given topological space. We proved that the number of the transitive quasi-uniformities that a topological space admits is either 1 or at least $2^{2^{\aleph_0}}$. In the T_2 case it was

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shown that there are at least $2^{2^{\aleph_0}}$ compatible non-transitive quasi-uniformities if X is infinite. In [8] Künzi proved that there exists a space which admits exactly two compatible totally bounded quasi-uniformities and he also showed that the number of compatible quasi-uniformities on X is less than or equal to $2^{2^{nw(X)}}$ where $nw(X)$ denotes the network weight of X.

In [13] we generalized a theorem of [12] by showing that if δ is a quasi-proximity such that $\mathcal{V}_\delta$ is transitive then the number of the (transitive) quasi-uniformities compatible with δ are either 1 or at least $2^{2^{\aleph_0}}$.

The following question remained open however : what can be said about this cardinality if $\mathcal{V}_\delta$ is not transitive. An important and interesting quasi-proximity of a topological space is the coarsest one if it exists and in most cases it is non-transitive. In this paper we examine the cardinality of the quasi-proximity class of the coarsest compatible quasi-proximity of locally compact T_2 spaces.

In the first section we give some basic results. For example we characterize in two distinct ways when a quasi-uniformity is in $\pi(\delta^0)$. We also verify that $\pi(\delta^0)$ is closed for sup hence it has a finest member. It is shown that if X is compact then the cardinality of the coarsest quasi-proximity class is equal to 1 and in the next paragraph we prove that this is also true when X is non-Lindelöf. To prove this we use a characterization of the elements of $\mathcal{V}^0$. We also prove that if $\mathcal{V} \in \pi(\delta^0)$, $\mathcal{V} \neq \mathcal{V}^0$ then $\mathcal{V}$ is uniformly locally compact. In the last section it is proved that the cardinality of $\pi(\delta^0)$ is at least $2^{2^{\aleph_0}}$ if X is non-compact and Lindelöf.

Now we shall mention some results and definitions which will be frequently used later; for elementary results on quasi-uniform spaces the reader has to consult [3].

We call an entourage V in X open if $\mathrm{int}V = V$ where $(\mathrm{int}V)(x) = \mathrm{int}(V(x))$.

If we say that δ is a quasi-proximity we use it in the sense of [3]1.22. If $\mathcal{V}$ is a quasi-uniformity, then $\delta(\mathcal{V})$ and $\tau(\mathcal{V})$ will always denote the quasi-proximity and the topology induced by $\mathcal{V}$ respectively, $\tau(\delta)$ will denote the topology induced by δ. Let $\pi(\delta) = \{\mathcal{V} : \delta(\mathcal{V}) = \delta\}$. We know from [3]1.33 that for every δ, $\pi(\delta) \neq \emptyset$ moreover there exists a coarsest element of $\pi(\delta)$; it is denoted by $\mathcal{V}_\delta$ and is totally bounded and the only totally bounded member of $\pi(\delta)$.

If a topological space X is given such that there exists the coarsest compatible quasi-proximity then it will be denoted by δ_X^0 or simply δ^0 if there is no danger of misunderstanding. If there exists the coarsest compatible quasi-uniformity then we use the notation $\mathcal{V}^0$ or $\mathcal{V}_X^0$ for this quasi-uniformity.

If X is T_2 then the condition that there is a coarsest compatible quasi-uniformity is equivalent with X being locally compact (see [11]4.6 and [6]). It is easy to check that $\mathcal{V}^0$ is transitive if and only if X is 0-dimensional.

If X is a space, $A, B \subset X$ then we will use the notation $U_{A,B}^X = U_{A,B} = (A \times B) \cup ((X - A) \times X)$. It is known that a totally bounded quasi-uniformity has a subbase consisting of sets of the form $U_{A,B}$ where $A\overline{\delta(\mathcal{V})}(X - B)$.

It is known from [3]1.46 that if X is locally compact T_2 then a subbase for $\mathcal{V}^0$ is $\{U_{K,G} : K \subset G \subset X,\ K$ is compact and G is open$\}$.

We will need a basic definition.

Definition 1.1. *Let (X, τ) be a topological space. Then $N(X)$ or $N(\tau)$ ($T(X)$ or $T(\tau)$) denotes the set of all compatible (transitive) quasi-uniformities on X respectively.*

We emphasize that in this paper each topological space considered is a locally compact T_2 space.

2. The tools

First we give characterizations for $\mathcal{V} \in N(X)$ being in $\pi(\delta^0)$.

Proposition 2.1. *Let X be locally compact T_2, $\mathcal{V} \in N(X)$. Then $\mathcal{V} \in \pi(\delta^0)$ if and only if $V \in \mathcal{V}$, $V(A) \neq X$ $(A \subset X)$ implies $\mathrm{cl}(A)$ is compact and $\mathrm{cl}(A) \subset V(A)$.*

Proof: To prove the necessity let $V \in \mathcal{V} \in \pi(\delta^0)$, $A \subset X$ such that $V(A) \neq X$. Then by [3]1.28 $A\overline{\delta(\mathcal{V})}(X - V(A))$ hence $U_{A,V(A)} \in \mathcal{V}_{\delta(\mathcal{V})} = \mathcal{V}^0$ and there are $K_i \subset G_i \subset X$ such that K_i is compact, G_i is open and $\cap_1^n U_{K_i,G_i} \subset U_{A,V(A)}$. If $x \in A$ then let $J(x) = \{i : x \in K_i\}$. By assumption $J(x) \neq \emptyset$ for every $x \in A$. It is easy to check that

$$A \subset \bigcup_{x \in A} \bigcap_{j \in J(x)} K_j \subset V(A),$$

but $\bigcup_{x\in A} \bigcap_{j \in J(x)} K_j$ is obviously compact, so is $\mathrm{cl}(A)$ and $\mathrm{cl}(A) \subset V(A)$.

To prove the converse we have to verify that $\mathcal{V}_{\delta(\mathcal{V})} \subset \mathcal{V}^0$. Let $U_{A,B} \in \mathcal{V}_{\delta(\mathcal{V})}$ such that $B \neq X$. Since $A\overline{\delta(\mathcal{V})}(X - B)$ then there is an open $V \in \mathcal{V}$ such that $V(A) \subset B$ so by assumption $\mathrm{cl}(A)$ is compact and $\mathrm{cl}(A) \subset V(A)$. One can easily show that $U_{\mathrm{cl}(A),V(A)} \subset U_{A,B}$ which yields that $U_{A,B} \in \mathcal{V}^0$. $\square$

Corollary 2.2. *If $\mathcal{V} \in \pi(\delta^0)$, $V_0, V \in \mathcal{V}$ such that $V_0^2 \subset V$, and $V(B) \neq X$ $(B \subset X)$ then $\mathrm{cl}(V_0(B))$ is compact and $\mathrm{cl}(V_0(B)) \subset V(B)$.*

Proof: Let $A := V_0(B)$, $V := V_0$ in 2.1. $\square$

Corollary 2.3. *If $A \subset X$, $A \neq X$, $V(A) = A$ $(V \in \mathcal{V} \in \pi(\delta^0))$ then A is compact.* $\square$

Corollary 2.4. *If there exists a transitive $\mathcal{V} \in \pi(\delta^0)$ then X is 0-dimensional.* $\square$

Recall that a quasi-uniformity $\mathcal{V}$ is called uniformly regular if for every $V \in \mathcal{V}$ there is $V_0 \in \mathcal{V}$ such that $\forall x \in X$ $\mathrm{cl}(V_0(x)) \subset V(x)$ (see [2]1.10, [1]). 2.2 yields that if $\mathcal{V} \in \pi(\delta^0)$ then $\mathcal{V}$ fulfills a much stronger condition.

Definition 2.5. *Let us call a quasi-uniformity $\mathcal{V}$ strongly uniformly regular if for every $V \in \mathcal{V}$ there is $V_0 \in \mathcal{V}$ such that $\mathrm{cl}(V_0(A)) \subset V(A)$ $(\forall A \subset X)$.*

Obviously every uniformity is strongly uniformly regular and the converse does not hold. A possible counterexample is $\mathcal{V}^0$ of a 0-dimensional, locally compact, non-compact T_2 space (see 2.16).

Remark 2.6. *The two notions, uniform regularity and strong uniform regularity do not coincide.*

Proof: Let $X = \{0; \frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ with the standard topology and let $\mathcal{V} = \mathrm{fil}_{X\times X}\{V_n : n \in \mathbb{N}\}$ where $V_n(x) = \{x\}$ if $x \neq 0$ and $V_n(0) = \{0; \frac{1}{k} : k \geq n\}$. It is obvious that $\mathcal{V} \in T(X)$. It is straightforward to check that $\mathcal{V}$ is uniformly regular. If $A = X - \{0\}$ then $V(A) = A$ and $\mathrm{cl}(V(A)) = X$ for every $V \in \mathcal{V}$ which shows that it is not strongly uniformly regular. □

Proposition 2.7. *If $\mathcal{V} \in \pi(\delta^0)$ then $\mathcal{V}$ is strongly uniformly regular, moreover each $V_0 \in \mathcal{V}$ is suitable which satisfies $V_0^2 \subset V$.*

Proof: 2.2. □

Proposition 2.8. *Let $V \in \mathcal{V} \in N(X)$, $A \subset X$. Then ($V(A) \neq X$ implies $\mathrm{cl}(A) \subset V(A)$) is equivalent with the condition that for every $x \in X$, $V^{-1}(x)$ is a neighbourhood of x ($\iff$ $\mathcal{V}^* = \sup\{\mathcal{V}, \mathcal{V}^{-1}\} \in N(X)$).*

Proof: Let us prove first the necessity. Suppose indirectly that there is a point $x \in X$ which does not satisfy the condition. Let $A = \{y \in X : x \notin V(y)\}$. Obviously $x \notin V(A)$, $x \in \mathrm{cl}(A)$ and $V(A) \neq X$. By assumption we get $\mathrm{cl}(A) \subset V(A)$ hence $x \in V(A)$ which is a contradiction.

To prove the opposite case suppose that there exists a subset A of X such that $V(A) \neq X$ and $\mathrm{cl}(A) \not\subset V(A)$. Now there is $x \in \mathrm{cl}(A)$ such that $x \notin V(A)$, and there is $y \in V^{-1}(x) \cap A$. Obviously $x \in V(y)$ and $x \notin V(y)$. □

Proposition 2.9. *Let $\mathcal{V} \in N(X)$ and $\mathcal{V}'$ be one of its subbases. In this case $\mathcal{V} \in \pi(\delta^0)$ if and only if $\forall x \in X$, $\forall V \in \mathcal{V}'$*

1. *$V^{-1}(x)$ is a neighbourhood of x and*
2. *$X - \mathrm{int}(V^{-1}(x)) = \mathrm{cl}(X - V^{-1}(x))$ is compact.*

Proof: First we prove the necessity. 2.8 implies 1, and let $A = \{y \in X : x \notin V(y)\}$. Obviously $V(A) \neq X$ hence $\mathrm{cl}(A)$ is compact by 2.1 and $A = X - V^{-1}(x)$.

If we want to prove the sufficiency then by 2.1 and 2.8 it is enough to show that $V(A) \neq X$ implies $\mathrm{cl}(A)$ is compact. Let $\cap_1^n V_i \subset V$ where $V_i \in \mathcal{V}'$. Let $x \in X - V(A)$. Then $A \cap V^{-1}(x) = \emptyset$ therefore $A \subset \mathrm{cl}(X - V^{-1}(x)) \subset \cup_1^n \mathrm{cl}(X - V_i^{-1}(x))$ which is compact by assumption. □

We emphasize a trivial but important consequence.

Corollary 2.10. *Let $\mathcal{V} \in N(X)$. Then $\mathcal{V} \in \pi(\delta^0)$ if and only if $\forall x \in X$, $\forall V \in \mathcal{V}$ $V^{-1}(x)$ is a neighbourhood of x and* $\mathrm{cl}(X - V^{-1}(x))$ *is compact.*

Examining the second condition we may say that it means that $V^{-1}(x)$ is a huge set $(V \in \mathcal{V} \in \pi(\delta^0), x \in X)$ in the sense that its complement is contained in a compact set where we can think that compact sets are small (since the set of all compact sets is an ideal in the lattice of all closed sets).

Proposition 2.11. *Let $\mathcal{V} \in N(X)$. In this case $\mathcal{V} \in \pi(\delta^0)$ if and only if $\mathcal{V}^{-1} \in N(\tau')$ where $\tau' = \{N \in \tau : X - N$ is compact $\} \cup \{\emptyset\}$.*

Proof: First observe that τ' is a topology.

Let $\mathcal{V} \in \pi(\delta^0)$ and $x \in X$, $V \in \mathcal{V}$. By 2.10 $x \in \mathrm{int}_\tau V^{-1}(x) \subset V^{-1}(x)$ and $\mathrm{int} V^{-1}(x) \in \tau'$.

Let $x \in G \in \tau'$. Then $K = X - G$ is compact and $x \notin K$. It is known that there exists a compact set K_1 such that $K \subset \mathrm{int} K_1$ and $x \notin K_1$. Obviously $U_{K,K_1} \in \mathcal{V}^0$ hence $U_{K,K_1}^{-1} = U_{X-K_1,X-K} \in (\mathcal{V}^0)^{-1} \subset \mathcal{V}^{-1}$. Now $U_{X-K_1,G}(x) = G$ which yields that $\tau' \subset \tau(\mathcal{V}^{-1})$.

To prove the sufficiency let $x \in X$, $V \in \mathcal{V}$. $V^{-1}(x)$ is a τ'-neighbourhood of x so it is a τ-neighbourhood too and there is $N \in \tau'$ such that $x \in N \subset V^{-1}(x)$. By 2.10, $X - V^{-1}(x) \subset X - N$ yields the required statement. □

Corollary 2.12. *$\pi(\delta^0)$ is closed for* sup.

Proof: Let $\mathcal{V}_i \in \pi(\delta^0)$ $(i \in I)$. By the previous proposition (2.11) $\mathcal{V}_i^{-1} \in N(\tau')$ and so is $(\sup\{\mathcal{V}_i : i \in I\})^{-1}$. □

Corollary 2.13. *$\pi(\delta^0)$ always has a finest member.*

The following fact is known in some ways (see e.g. [3] 1.45) but we give here a simple direct proof.

Proposition 2.14. *If X is compact T_2 then there exists a unique compatible uniformity on X, moreover it equals to the coarsest compatible quasi-uniformity.*

Proof: Let $\mathcal{V}$ be a compatible uniformity and $\mathcal{V}^0$ be the coarsest compatible quasi-uniformity whose subbase is $\{U_{K,G} : K \subset G \subset X,\ K \text{ is compact and } G \text{ is open}\}$. Let $V \in \mathcal{V}$ and $V_0 \in \mathcal{V}$ be symmetric such that $V_0^4 \subset V$. By the compactness of X there are points $x_1, \dots, x_n$ in X such that $X = \cup_1^n V_0(x_i)$. Let $K_i = \mathrm{cl}(V_0(x_i))$, $G_i = \mathrm{int}(V_0(K_i))$. Obviously K_i is compact, G_i is open and $K_i \subset G_i$. In the standard way one can easily prove that $\cap_1^n U_{K_i,G_i} \subset V$ which yields that $V \in \mathcal{V}^0$. □

We are ready to prove the main theorem of this section.

Theorem 2.15. *If X is compact T_2 then $|\pi(\delta^0)| = 1$.*

Proof: By 2.10 it is straightforward that $\mathcal{V}^* = \sup\{\mathcal{V}, \mathcal{V}^{-1}\}$ is a compatible uniformity if $\mathcal{V} \in \pi(\delta^0)$. But $\mathcal{V} \subset \mathcal{V}^* = \mathcal{V}^0$ by the previous observation (2.14). □

Proposition 2.16. *For a locally compact T_2 space X the followings are equivalent: 1. X is compact 2. $\mathcal{V}^0$ is a uniformity 3. $\exists \mathcal{V} \in \pi(\delta^0)$ such that $\mathcal{V}^{-1} \in \pi(\delta^0)$ 4. $\exists \mathcal{V} \in \pi(\delta^0)$ such that $\mathcal{V}^{-1} \in N(X)$ 5. $\exists \mathcal{V} \in \pi(\delta^0)$ such that $\mathcal{V}^* \in \pi(\delta^0)$.*

Proof: 1⇒2: 2.14. The implications 2⇒3, 3⇒4, 2⇒5 are obvious. 5⇒2: If $\mathcal{V}$ is a uniformity then so is $\mathcal{V}_\omega = \mathcal{V}_{\delta(\mathcal{V})}$. 4⇒1: Let $x \in K \subset X$ such that K is a compact neighbourhood of x. Then there is a $V \in \mathcal{V}$ such that $V^{-1}(x) \subset K$. Therefore $X = K \cup (X - \mathrm{int}(V^{-1}(x)))$ but by 2.9 $X - \mathrm{int}(V^{-1}(x))$ is compact. □

Lemma 2.17. *Let $Y \subset X$ be closed, $\mathcal{V} \in \pi(\delta^0_X)$. Then $\mathcal{V}|_Y \in \pi(\delta^0_Y)$.*

Proof: If $V \in \mathcal{V}$ then conditions 1 and 2 hold in 2.9. We have to show that $V|_Y$ also satisfies these conditions for Y. 1 is obvious, and $\mathrm{cl}_Y(Y - V|_Y^{-1}(y)) = \mathrm{cl}_X(Y - V|_Y^{-1}(y)) \cap Y \subset \mathrm{cl}_X(X - V^{-1}(y)) \cap Y$ $(y \in Y)$ which is compact and we get condition 2 holds too. □

3. Non-Lindelöf spaces

The aim of this section is proving the fact that $|\pi(\delta^0)| = 1$ if X is non-Lindelöf.

Definition 3.1. *If V is an entourage on X then let*

$$\mathrm{supp}V = \mathrm{cl}(\{x \in X : V(x) \neq X\}).$$

Proposition 3.2. *Let $V_0, V \in \mathcal{V} \in \pi(\delta^0)$ such that $V_0^3 \subset V$. Then $V_0(\mathrm{supp}V) \subset \mathrm{supp}V_0$. (Hence $\mathrm{supp}V \subset \mathrm{int}(\mathrm{supp}V_0)$.)*

Proof: Let $x \in \mathrm{supp}V$, $y \in V_0(x)$. We have to prove that $y \in \mathrm{supp}V_0$. By 2.9 there exists $z \in V_0^{-1}(x) \cap \{x' : V(x') \neq X\} \neq \emptyset$. Now $V_0(y) \subset V_0^2(x) \subset V_0^3(z) \subset V(z) \neq X$ and we get $V_0(y) \neq X$. □

Proposition 3.3. *Let $V_0, V \in \mathcal{V} \in \pi(\delta^0)$ such that $V_0^2 \subset V$. If $x \in \mathrm{supp}V$ then $V_0(x) \neq X$.*

Proof: By 2.9 there is a $z \in V_0^{-1}(x) \cap \{x' : V(x') \neq X\} \neq \emptyset$. In the usual way we get $V_0(x) \neq X$. □

Lemma 3.4. *Let $V, V_0 \in \mathcal{V} \in \pi(\delta^0)$, $V_0^4 \subset V$. Then if $x \in \mathrm{supp}V$ then $\mathrm{cl}(V_0(x))$ is compact.*

Proof: By 3.3 $V_0^2(x) \neq X$ and then 2.2 gives the statement. □

Theorem 3.5. *Let V be a neighbournet in X. Then $V \in \mathcal{V}^0$ if and only if $\mathrm{supp}V$ is compact, there exists a neighbournet V_0 such that $V_0^2 \subset V$ and $V_0^{-1}(x)$ is a neighbourhood of x for every $x \in \mathrm{supp}V$.*

Proof: If $V \in \mathcal{V}^0$ then there are $K_i \subset G_i \subset X$ such that K_i is compact, G_i is open, and $U = \bigcap_1^n U_{K_i,G_i} \subset V$. If $x \notin \bigcup_1^n K_i$ then $U(x) = X \subset V(x)$. By 2.9 $V_0^{-1}(x)$ is a neighbourhood of x if $x \in X$ whenever $V_0 \in \mathcal{V}^0$.

To prove the sufficiency let $L_x = \operatorname{int}(V_0(x) \cap V_0^{-1}(x))$ and $K_x \subset L_x$ be a compact neighbourhood of x if $x \in K = \operatorname{supp} V$. Then $K \subset \bigcup_{x \in K} K_x$ and K being compact imply that $K \subset \bigcup_{i=1}^n K_{x_i}$. We show that $U = \bigcap_1^n U_{K_{x_i},L_{x_i}} \subset V$ which yields that $V \in \mathcal{V}^0$. If $x \notin K$ then $V(x) = X$. If $x \in K$ then there is i such that $x \in K_{x_i}$. Then $x \in V_0^{-1}(x_i)$ and $V_0(x_i) \subset V(x)$ and $L_{x_i} \subset V_0(x_i)$ so that $U(x) \subset U_{K_{x_i},L_{x_i}}(x) \subset V(x)$. □

We remark that $\operatorname{cl}(X - V^{-1}(x))$ is compact in the previous theorem ($x \in X$); moreover all sets $\operatorname{cl}(X - V^{-1}(x))$ are contained in the same compact set ($\operatorname{supp} V$). (Compare with 2.9.)

Corollary 3.6. *Let $V \in \mathcal{V} \in \pi(\delta^0)$. Then $V \in \mathcal{V}^0$ if and only if* $\operatorname{supp} V$ *is compact.*

The following proposition generalizes the fact that $\operatorname{cl}(X - V^{-1}(x))$ is compact if $V \in \mathcal{V} \in \pi(\delta^0)$.

Proposition 3.7. *If $V \in \mathcal{V} \in \pi(\delta^0)$, $K \subset X$ is compact then* $\operatorname{cl}(X - \bigcap_{x \in K} V^{-1}(x)) = \operatorname{cl}(\{x \in X : K \not\subset V(x)\})$ *is compact.*

Proof: Let $V_0 \in \mathcal{V}$ such that $V_0^2 \subset V$. Then obviously there are $x_i \in X$ ($i = 1, \dots, n$) such that $K \subset \bigcup_1^n V_0(x_i)$. We show that $\bigcap_1^n V_0^{-1}(x_i) \subset \bigcap_{x \in K} V^{-1}(x)$. If $y \in K$ then there is i such that $y \in V_0(x_i)$, hence $x_i \in V_0^{-1}(y)$, $V_0^{-1}(x_i) \subset V^{-1}(y)$ and we get $\bigcap_1^n V_0^{-1}(x_j) \subset \bigcap_{x \in K} V^{-1}(x)$.

Now $\operatorname{cl}(X - \bigcap_{x \in K} V^{-1}(x)) \subset \operatorname{cl}(X - \bigcap_1^n V_0^{-1}(x_i)) = \bigcup_1^n \operatorname{cl}(X - V_0^{-1}(x_i))$ which is compact (2.9). □

Proposition 3.8. *Let $\mathcal{V} \in \pi(\delta^0)$, $V, V_0 \in \mathcal{V}$ such that $V_0^2 \subset V$ and suppose that there exists $x \in X$ such that $X - V_0(x)$ is compact. In this case $V \in \mathcal{V}^0$*

Proof: Let $K_1 = X - V_0(x)$, $K_2 = \operatorname{cl}(X - V_0^{-1}(x))$. We know that both sets are compact. By 3.7, there exists a compact set

K_3 such that if $z \notin K_3$ then $K_1 \subset V(z)$. If $y \notin K_2$ then $y \in V_0^{-1}(x)$, and $X - K_1 \subset V_0(x) \subset V_0^2(y) \subset V(y)$. We get that if $y \notin K_2 \cup K_3$ then $V(y) = X$. In other words supp$V$ is compact and 3.6 is applicable. □

Corollary 3.9. *If* $V_0, V \in \mathcal{V} \in \pi(\delta^0)$ *such that* $V_0^2 \subset V$ *and* $V \notin \mathcal{V}^0$ *then* $\operatorname{supp} V_0 = X$.

Proposition 3.10. *Let* $V_0, V \in \mathcal{V} \in \pi(\delta^0)$ *such that* $V_0^8 \subset V$ *and* $V \notin \mathcal{V}^0$. *In this case* $\mathrm{cl}(V_0(x))$ *is compact for every* $x \in X$.

Proof: By 3.9 $\operatorname{supp} V_0^4 = X$ and by 3.4 $\mathrm{cl}(V_0(x))$ is compact. □

Recall that a quasi-uniform space $(X, \mathcal{V})$ is uniformly locally compact provided that there is a $V \in \mathcal{V}$ such that $\forall x \in X$, $\mathrm{cl}(V(x))$ is compact (see [9]2). Now we can prove the following:

Corollary 3.11. *If* X *is locally compact* T_2, $\mathcal{V} \in \pi(\delta^0)$, $\mathcal{V} \neq \mathcal{V}^0$ *then* $\mathcal{V}$ *is uniformly locally compact. If* X *is non-compact then* $\mathcal{V}^0$ *is not uniformly locally compact.*

Proof: There is $V \in \mathcal{V} - \mathcal{V}^0$ and 3.10 is applicable.

To prove the second part indirectly observe that if $\mathrm{cl}(V(x))$ is compact for every $x \in X$ then $\operatorname{supp} V = X$. But by 3.6 $\operatorname{supp} V$ must be compact as well – a contradiction. □

Now we can already prove the following:

Theorem 3.12. *If* X *is locally compact* T_2, *non-Lindelöf* ($\Longleftrightarrow$ *not* σ*-compact*) *then* $|\pi(\delta^0)| = 1$.

Proof: Suppose the contrary, that there exists $\mathcal{V} \in \pi(\delta^0)$, $\mathcal{V} \neq \mathcal{V}^0$. Then there is $V \in \mathcal{V} - \mathcal{V}^0$. Let $V_0 \in \mathcal{V}$ such that $V_0^8 \subset V$ and let V_0 be open. By 3.10 $\mathrm{cl}(V_0(x))$ is compact ($x \in X$). Then we can find a sequence (x_i) such that $x_{n+1} \notin \cup_1^n V_0(x_i)$. By X being non-Lindelöf there exists $y \in X - \cup_1^\infty V_0(x_i)$ thus $\{x_i : i \in \mathbb{N}\} \subset X - V_0^{-1}(y)$ but the closure of this set is compact by 2.9 so there is a cluster point z of (x_i). By 2.9 $V_0^{-1}(z)$

is a neighbourhood of z so there is $n \in \mathbb{N}$ such that $x_n \in V_0^{-1}(z)$, $z \in V_0(x_n)$. But obviously $\{x_i : i > n\} \subset X - V_0(x_n)$ which is closed. Hence $z \in X - V_0(x_n)$ which is a contradiction. □

4. The Lindelöf, non-compact case

In this section we partly answer the question : what can be said about $|\pi(\delta^0)|$ if X is non-compact, Lindelöf, by proving that $|\pi(\delta^0)| \geq 2^{2^{\aleph_0}}$ in this case.

Lemma 4.1. *Let $f : X \to Y$ be a continuous surjective function such that K being compact in Y implies that $f^{-1}(K)$ is compact. If V is a neighbournet in Y such that 2.9 1 and 2 hold then $(f \times f)^{-1}(V)$ is also a neighbournet in X and satisfies these conditions.*

Proof: If $x \in X$ then $(f \times f)^{-1}(V)(x) = f^{-1}(V(f(x)))$ and similarly $\left((f \times f)^{-1}(V)\right)^{-1}(x) = f^{-1}(V^{-1}(f(x)))$. Condition 1 and $(f \times f)^{-1}(V)$ being a neighbournet are straightforward. If $x \in X, X - f^{-1}(V^{-1}(f(x))) = f^{-1}(Y - V^{-1}(f(x)))$; but $\mathrm{cl}(Y - V^{-1}(f(x)))$ is compact. □

Lemma 4.2. *Let $f : X \to Y$ be a surjective function, and $\mathcal{V}_1, \mathcal{V}_2$ be two distinct quasi-uniformities on Y. Then $f^{-1}(\mathcal{V}_1) \neq f^{-1}(\mathcal{V}_2)$.*

Proof: Suppose that there is a $V \in \mathcal{V}_1 - \mathcal{V}_2$ and there is $V_2 \in \mathcal{V}_2$ such that $(f \times f)^{-1}(V_2) \subset (f \times f)^{-1}(V)$. Obviously $f^{-1}(V_2(f(x))) \subset f^{-1}(V(f(x)))$ $(x \in X)$. By the surjectivity of f, $V_2(f(x)) \subset V(f(x))$ and again by the surjectivity $V_2 \subset V$ and $V \in \mathcal{V}_2$ – a contradiction. □

Theorem 4.3. *Let X and Y be locally compact, non-compact* T_2 *spaces. Let $f : X \to Y$ be continuous, surjective and if K is a compact set then let $f^{-1}(K)$ be compact. In this case $|\pi(\delta_Y^0)| \leq |\pi(\delta_X^0)|$.*

Proof: Let $\mathcal{V}_1, \mathcal{V}_2 \in \pi(\delta_Y^0)$ such that $\mathcal{V}_1 \neq \mathcal{V}_2$. It is enough to prove that $\mathcal{V}_1' = \mathrm{fil}_{X\times X}\{f^{-1}(\mathcal{V}_1), \mathcal{V}^0\} \neq \mathcal{V}_2' = \mathrm{fil}_{X\times X}\{f^{-1}(\mathcal{V}_2), \mathcal{V}^0\}$

since by 4.1 $\mathcal{V}_1', \mathcal{V}_2' \in \pi(\delta_X^0)$. Suppose that there exist $V \in \mathcal{V}_1 - \mathcal{V}_2$, $V_2 \in \mathcal{V}_2$ and $U \in \mathcal{V}^0$ such that $U \cap (f \times f)^{-1}(V_2) \subset (f \times f)^{-1}(V)$. Then for every $x \in X$, $U(x) \cap f^{-1}(V_2(f(x))) \subset f^{-1}(V(f(x)))$. Let $K = \mathrm{supp} U$ and $x \notin K$. There exists such an x by 3.6 and the assumption. Hence $V_2(f(x)) \subset V(f(x))$ $\forall x \in X - K$ and we get $V_2(y) \subset V(y)$ $\forall y \in Y - f(K)$.

Let $V_0 \in \mathcal{V}_1$ such that $V_0^4 \subset V$ and V_0 is open. $f(K) \subset \bigcup_{x \in f(K)}(V_0(x) \cap V_0^{-1}(x))$ and $f(K)$ being compact implies that $f(K) \subset \cup_1^n(V_0(x_i) \cap V_0^{-1}(x_i))$. Let $W = \cap_1^n U_{K_i,G_i}$ where $K_i = \mathrm{cl}(V_0(x_i))$ and $G_i = V_0^2(x_i)$. Since $V \notin \mathcal{V}_2$ then $V \notin \mathcal{V}_Y^0$ and by 3.8 $V_0^2(x_i) \neq Y$ and 3.4 yields that $\mathrm{cl}(V_0(x_i))$ is compact, and $K_i \subset G_i$ holds by 2.2. Hence $W \in \mathcal{V}_Y^0$, we want to prove that $W(y) \subset V(y)$ $\forall y \in f(K)$. If $y \in f(K)$ then there is i such that $y \in V_0(x_i) \cap V_0^{-1}(x_i)$ and $y \in K_i$. Thus $x_i \in V_0(y)$ implies $V_0^2(x_i) \subset V_0^3(y) \subset V(y)$. In other words $U_{K_i,G_i}(y) \subset V(y)$ and $W(y) \subset V(y)$.

Now $W \cap V_2 \subset V$ but $W \in \mathcal{V}_Y^0 \subset \mathcal{V}_2$. Therefore $V \in \mathcal{V}_2$ which is a contradiction. □

Theorem 4.4. *If X is non-compact, Lindelöf and 0-dimensional then* $|\pi(\delta^0) \cap (N(X) - T(X))| \geq 2^{2^{\aleph_0}}$.

Proof: To prove the inequality it is enough to show that each quasi-uniformity $\mathcal{V}_\sigma$ constructed in [12]3.1 is in $\pi(\delta^0)$ if we set $\mathcal{B} = \{\text{compact-open sets}\} \cup \{X, \emptyset\}$ and let (N_i) be a strictly increasing sequence of compact-open sets such that $\cup_1^\infty N_i = X$. It is straightforward that $\mathcal{V}_{\mathcal{B}} = \mathcal{V}^0$. Using the notation of [12], by 2.9 we have to check that each V_i and U_A $(i \in \mathbb{N}, A \in \sigma)$ satisfy conditions 1 and 2. To prove condition 1 let $x \in N_n - N_{n-1}$. Then $N_n - N_{n-1} \subset V_i^{-1}(x) \cap U_A^{-1}(x)$. Condition 2 is a consequence of $X - N_n \subset V_i^{-1}(x) \cap U_A^{-1}(x)$. □

Corollary 4.5. *Let $X = \mathbb{N}$ be equipped with the discrete topology. Then* $|\pi(\delta^0)| = 2^{2^{\aleph_0}}$.

Proof: Obviously $|\pi(\delta^0)| \leq |N(X)| \leq 2^{2^{\aleph_0}}$ and apply 4.4. □

Theorem 4.6. *If $X = [0, +\infty) \subset \mathbb{R}$ is equipped with the standard topology then* $|\pi(\delta^0)| = 2^{2^{\aleph_0}}$.

Proof: We know that $|N(X)| = 2^{2^{\aleph_0}}$ (see [12]4.4 and [8] Corollary 2) so we have to prove that $|\pi(\delta^0)| \geq 2^{2^{\aleph_0}}$.

Let $\mathcal{A} \subset \exp(\mathbb{N})$ be an almost disjoint system such that $|\mathcal{A}| = 2^{\aleph_0}$ and $A \in \mathcal{A}$ implies $|A| = \aleph_0$. (Almost disjoint means that if $A, B \in \mathcal{A}$ then $|A \cap B|$ is finite.) If $A \in \mathcal{A}$, $\epsilon \in \mathbb{R}^+$, $0 < \epsilon \leq 1$ are given then we define $f = f_{A,\epsilon}$. If $n \in \mathbb{N} \cup \{0\}$ then let

$$f(n) = \begin{cases} \epsilon & \text{if } n \notin A \\ \frac{\epsilon}{n} & \text{if } n \in A. \end{cases}$$

If $x \in X, n < x < n+1$ for some $n \in \mathbb{N}$ then let $f(x)$ be the point on the line segment between $(n, f(n))$ and $(n+1, f(n+1))$ which is above x. Obviously f is continuous. Let $V_{A,\epsilon}(x) = V_{f_{A,\epsilon}}(x) = [0, x + f_{A,\epsilon}(x))$.

We prove that $V^2_{f_{A,\epsilon}} \subset V_{f_{A,3\epsilon}}$. Let $f = f_{A,\epsilon}$. Let $x \in X$ be fixed. If $y \in V_f(x)$ then $0 \leq y < x + f(x)$. If $y < x$ then the absolute value of the slope of the segment between $(y, f(y))$ and $(x, f(x))$ is less than 1. Hence $y + f(y) < x + f(x)$ in other words $V_f(y) \subset V_f(x)$. If $x < y < x + f(x)$ holds then we can say the same about the slope of that segment so $f(y) \leq 2f(x)$. We get $y + f(y) < x + 3f(x)$ in other words $V_f(y) \subset V_{f_{A,3\epsilon}}(x)$.

Now let $\mathcal{B} \subset \mathcal{A}$. Let

$$\mathcal{V}_{\mathcal{B}} = \mathrm{fil}_{X \times X}\{\mathcal{V}^0, V_{f_{A,\epsilon}} : A \in \mathcal{B}, 0 < \epsilon \leq 1\}.$$

By the previous observation the set which generates $\mathcal{V}_{\mathcal{B}}$ is a quasi-uniform subbase. It is easy to check that $\mathcal{V}_{\mathcal{B}} \in N(X)$. To prove that $\mathcal{V}_{\mathcal{B}} \in \pi(\delta^0)$ we have to check conditions 1 and 2 in 2.9 for the given subbase. Let $f = f_{A,\epsilon}$, $x \in X$. Obviously $[x, +\infty) \subset V_f^{-1}(x)$ so condition 2 holds. Suppose indirectly that there is a sequence (y_n) such that $y_n \to x$, $y_n < x$ and $x \notin V_f(y_n)$ $(n \in \mathbb{N})$. Then $f(y_n) \to 0$ must hold and we would get that $f(x) = 0$ since f is continuous and this is a contradiction.

It remains to show that if $\mathcal{B}_1 \neq \mathcal{B}_2$ then $\mathcal{V}_{\mathcal{B}_1} \neq \mathcal{V}_{\mathcal{B}_2}$. Suppose for example that there exists $B \in \mathcal{B}_1 - \mathcal{B}_2$. Let $V = V_{f_{B,1}} \in \mathcal{V}_{\mathcal{B}_1}$. If indirectly $V \in \mathcal{V}_{\mathcal{B}_2}$ then there are $\epsilon > 0$, $U \in \mathcal{V}^0$, $A_j \in \mathcal{B}_2$ such that $U \cap \bigcap_{j=1}^m V_{f_{A_j,\epsilon}} \subset V$. $B \notin \mathcal{B}_2$ so it is easy to check that $B - \cup_1^m A_j$ is infinite. Thus there exists $k \in (\mathbb{N} - \text{supp}U) \cap (B - \cup_1^m A_j)$ such that $\frac{1}{k} < \epsilon$ since by 3.6 $\text{supp}U$ is compact. Then $U(k) \cap \bigcap_1^m V_{f_{A_j,\epsilon}}(k) = X \cap [0, k+\epsilon) = [0, k+\epsilon)$ but $V_{f_{B,1}}(k) = [0, k+\frac{1}{k})$ which is a contradiction. □

Proposition 4.7. *If $X \subset [0, +\infty)$ is closed, not bounded then $|\pi(\delta_X^0)| \geq 2^{2^{\aleph_0}}$.*

Proof: There are two cases to consider.

1. There is a sequence (x_n) such that $x_n \to +\infty$ and $x_n \notin X$ $(n \in \mathbb{N})$. We can assume that $x_n < x_{n+1}$, $\inf X < x_1$ and $(x_n, x_{n+1}) \cap X \neq \emptyset$ $(n \in \mathbb{N})$. In this case let $(y_n) \subset X$ be a sequence such that $y_1 < x_1$, $x_{i-1} < y_i < x_i$ $(i \geq 2)$. We define a function $g : X \to \{y_i : i \in \mathbb{N}\}$. If $x \in X$ then let

$$g(x) = \begin{cases} y_1 & \text{if } x < x_1, \\ y_i & \text{if } x_{i-1} < x < x_i \ (i \geq 2). \end{cases}$$

Obviously by the choice of (x_n) and (y_n), g is continuous, surjective, $K \subset \{y_i : i \in \mathbb{N}\}$ being compact implies that $g^{-1}(K)$ is also compact. By 4.3 and 4.5 we get $|\pi(\delta_X^0)| \geq 2^{2^{\aleph_0}}$ since $\{y_i : i \in \mathbb{N}\}$ is homeomorphic to $\mathbb{N}$.

2. There exists no such sequence in other words there is $y \in X$ such that $[y, +\infty) \subset X$. We define $g : X \to [y, +\infty)$. If $x \in X$ then let

$$g(x) = \begin{cases} y & \text{if } x \leq y, \\ x & \text{if } x \geq y. \end{cases}$$

It is straightforward to check that g is continuous, surjective, $K \subset [y, +\infty)$ being compact implies that $g^{-1}(K)$ is compact too. So $|\pi(\delta_X^0)| \geq 2^{2^{\aleph_0}}$ by 4.3 and 4.6. □

Corollary 4.8. *If $X \subset [0, +\infty)$ is closed, not bounded then $|\pi(\delta_X^0)| = 2^{2^{\aleph_0}}$.*

Proof: 4.7 and [8] Corollary 2. □

Lemma 4.9. *If X is locally compact, non-compact T_2, Lindelöf then there exists $f : X \to Y \subset [0,+\infty)$ such that Y is closed, not bounded and f is continuous, surjective and K being compact implies that $f^{-1}(K)$ is also compact.*

Proof: It is easy to construct a sequence of compact sets $K_n \subset X$ $(n \in \mathbb{N})$ such that $K_n \subset \mathrm{int}K_{n+1}$ and $\cup_1^\infty K_i = X$. Then by induction we can assign to each $r \in [1,\infty) \cap \mathbb{Q}$ a compact set K_r in X such that if $r_1 < r_2$ then $K_{r_1} \subset \mathrm{int}K_{r_2}$. We have defined K_r if $r \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and let (r_m) be a sequence such that $\{r_m : m \in \mathbb{N}\} = [n,n+1] \cap \mathbb{Q}$, $r_m \neq r_k$ $(k \neq m)$ and $r_1 = n$, $r_2 = n+1$. If $K_{r_1}, \ldots, K_{r_{k-1}}$ are defined then there are $l, m < k$ such that $d(r_l, r_k) = \min\{d(r_j, r_k) : r_j < r_k,\ j < k\}$ and $d(r_k, r_m) = \min\{d(r_k, r_j) : r_k < r_j,\ j < k\}$. Let K_{r_k} be chosen such that $K_{r_l} \subset \mathrm{int}K_{r_k} \subset K_{r_k} \subset \mathrm{int}K_{r_m}$ and K_{r_k} is compact. We do the same for all $n \in \mathbb{N}$ and we get the sequence K_r for $r \in [1,\infty) \cap \mathbb{Q}$.

Now we define f. If $x \in X$ then let $f(x) = \inf\{r \in [1,\infty) \cap \mathbb{Q} : x \in K_r\}$. We show that this f has all required properties.

To prove that f is continuous let $x \in X$ and $f(x) \in (a,b)$. Let r_1, r_2 be chosen such that $a < r_1 < f(x) < r_2 < b$. Then $x \in \mathrm{int}K_{r_2} - K_{r_1}$ which is open and $f(\mathrm{int}K_{r_2} - K_{r_1}) \subset (a,b)$.

Let $Y = f(X)$. Then Y is not bounded since if $n \in \mathbb{N}$ then there is $x \notin K_n$ hence $f(x) \geq n$.

To show that Y is closed let (y_n) be a convergent sequence such that $y_n \in Y$ and $y_n \to y \in \mathbb{R}$. Then there is a sequence (x_n) in X such that $f(x_n) = y_n$. There exists a $w \in \mathbb{Q}$ such that $y_n < w\ \forall n \in \mathbb{N}$, therefore $x_n \in K_w$ which is compact so there is a cluster point x of (x_n). Obviously $y = f(x)$ because f is continuous.

We show that K being compact implies that $f^{-1}(K)$ is compact. Suppose that $K \subset [1,n]$, for an $n \in \mathbb{N}$. If $x \in f^{-1}(K)$ then $x \in K_{n+1}$. So $f^{-1}(K) \subset K_{n+1}$. But f being continuous implies that $f^{-1}(K)$ is closed. □

Theorem 4.10. *If X is locally compact* T_2*, non-compact, Lindelöf then* $|\pi(\delta^0_X)| \geq 2^{2^{\aleph_0}}$.

Proof: 4.9,4.7 and 4.3. □

Corollary 4.11. $|\pi(\delta^0)| = 1$ *if and only if X is either compact or non-Lindelöf.*

Finally we present a counterexample to the following question. For some time we believed that if X is non-compact and Lindelöf then $|\pi(\delta^0)| = 2^{2^{\aleph_0}}$ holds. Much evidence seemed to support such a conjecture, for example 2.15, 2.17, 2.9 and 3.10. But we are going to show now that it is not correct.

Theorem 4.12. *There exists a space X for which* $|\pi(\delta^0)| > 2^{2^{\aleph_0}}$.

Proof: Let Y be a discrete topological space with $|Y| = c$ where $c = 2^{\aleph_0}$. Let us consider the Čech-Stone compactification βY of Y. If $A \subset Y$, let us use the notation: $\hat{A} = \{\tilde{u} : \tilde{u}$ is an ultrafilter on Y such that $A \in \tilde{u}\}$.

Let $X = \cup_{i=1}^{\infty} X_i$ where X_i is homeomorphic to βY $(i \in \mathbb{N})$ and $X_i \cap X_j = \emptyset$ $(i \neq j)$. Let $\cup_1^{\infty} \tau_i$ be a base for τ, i.e. X is the topological sum of the X_i-s.

We assign for every $A \subset Y$ a transitive neighbournet W_A on X. If $x \in X_n$ then let $W_A(x) = \cup_{i=1}^{n-1} X_i \cup U_{\hat{A}}^{X_n}(x)$ where $U_{\hat{A}}^{X_n}$ denotes the image of the neighbournet $U_{\hat{A}} = U_{\hat{A},\hat{A}} \subset \beta Y \times \beta Y$ by a fixed homeomorphism between βY and X_n. It is easy to see that $W_A^2 = W_A$.

If $\mathcal{A}$ is a filter on Y then let $\mathcal{V}_{\mathcal{A}} = \mathrm{fil}_{X \times X}\{\mathcal{V}_X^0; W_A : A \in \mathcal{A}\}$. We show that $\mathcal{V}_{\mathcal{A}} \in \pi(\delta^0)$. In order to prove this it is enough to check the two conditions of 2.9. Let $x \in X_n$ and $A \in \mathcal{A}$. By the definition of W_A, $\cup_{i=n+1}^{\infty} X_i \subset W_A^{-1}(x)$, hence condition 2 holds. To establish condition 1 it is enough to verify that $W_A^{-1}(x) \cap X_n$ is open, which is equal to $U_{\hat{A}}^{-1}(x)$ and it is open.

Now we prove that if $\mathcal{A}, \mathcal{B}$ are filters on Y, $\mathcal{A} \neq \mathcal{B}$ then $\mathcal{V}_{\mathcal{A}} \neq \mathcal{V}_{\mathcal{B}}$. This obviously implies that $|\pi(\delta^0)| \geq 2^{2^c} > 2^c$.

We can assume that there is $A \in \mathcal{A} - \mathcal{B}$. Then $W_A \in \mathcal{V}_{\mathcal{A}}$ and we show that $W_A \notin \mathcal{V}_{\mathcal{B}}$. Suppose indirectly that

$W_A \in \mathcal{V}_{\mathcal{B}}$. Then there are $U \in \mathcal{V}_X^0$, $B_i \in \mathcal{B}$ such that $U \cap \bigcap_1^n W_{B_i} \subset W_A$. By 3.6 supp$U$ is compact hence there is a $k \in \mathbb{N}$ such that $X_k \cap \mathrm{supp}U = \emptyset$. If we restrict the previous entourages to X_k we get $\cap_1^n U_{\hat{B}_i} \subset U_{\hat{A}}$. Let α be the lattice of subsets of X_k which is generated by the sets $\emptyset, \hat{B}_1, \dots, \hat{B}_n, \beta Y$, and let $\beta = \{\emptyset, \hat{A}, \beta Y\}$. By [11]2.6 we get that $\beta \subset \alpha$ since α and β are l-interior preserving open covers and $U_\alpha = \cap_1^n U_{\hat{B}_i}$, $U_\beta = U_{\hat{A}}$. Hence $\hat{A}$ can be written as a union of finite intersections from $\hat{B}_1, \dots, \hat{B}_n$. By the properties of the operator $\hat{}$, then A is a union of finite intersections from $B_1, \dots, B_n$ but then $A \in \mathcal{B}$ – a contradiction. □

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