Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 23, 1998

PSEUDO EXTERIORS IN HYPERSPACES

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Dedicated to Professor Sam B. Nadler, Jr.

ABSTRACT. Given a continuum X, let $\mathcal{C}(X)$ be the hyperspace of all nonempty subcontinua of X. In this paper we introduce the concept of *pseudo exterior.* Given $A \in \mathcal{C}(X)\setminus\{X\}$, a subcontinuum B of X is in the pseudo exterior of A if $A \cap B \neq \emptyset$ and for every $\varepsilon > 0$, there exists a subcontinuum C of B such that $A \cap C = \emptyset$ and $\mathcal{H}(B, C) < \varepsilon$, where $\mathcal{H}$ denotes the Hausdorff metric for $\mathcal{C}(X)$. Using pseudo exteriors we obtain characterizations of several classes of continua and some other results.

INTRODUCTION.

A. Illanes ([I2]) introduced the concept of semi-boundaries in hyperspaces as follows: Given a continuum X and a proper subcontinuum A of it, the *semi-boundary* of $\mathcal{C}(A)$ (the hyperspace of subcontinua of A) is defined by $SB(A) = \{C \in \mathcal{C}(A) \mid$ there exists a map $\alpha\colon [0,1] \to \mathcal{C}(X)$ such that $\alpha(0) = C$ and $\alpha(t)$ is not contained in A for all $t > 0\}$. This notion has been very useful in proving several important theorems in Hyperspace theory (see [I1]).

1991 *Mathematics Subject Classification.* Primary, 54B20.
Key words and phrases. Arc, composant, continuum, hereditarily indecomposable continuum, hyperspace, indecomposable continuum, order arc.

In this paper we modify this notion by considering the *pseudo exterior* of a proper subcontinuum A of X in $\mathcal{C}(X)$ as the set consisting of all $B \in \mathcal{C}(X)$ such that $B \cap A \neq \emptyset$ and satisfying that for each $\varepsilon > 0$, there exists $C \in \mathcal{C}(B)$ such that the Hausdorff distance between C and B is smaller that ε and $C \cap A = \emptyset$. We use pseudo exteriors to characterize several classes of continua.

Definitions.

If Z is a topological space and $A \subseteq Z$, then the closure of A in Z is denoted by $Cl_Z(A)$, its interior by $Int_Z(A)$, and its boundary by $\partial_Z(A)$. A topological space Z is said to be *pathwise connected* provided that for each pair of points, z_1 and z_2, of Z, there exists a continuous functions $\alpha\colon [0,1] \to Z$ such that $\alpha(0) = z_1$ and $\alpha(1) = z_2$. If the function α can be chosen to be one–to–one, then Z is said to be *arcwise connected.* If (Y, d) is a metric space, then given $A \subseteq Y$ and $\varepsilon > 0$, the open ball around A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$, we write $\mathcal{V}_\varepsilon(y)$ for $\mathcal{V}_\varepsilon(\{y\})$. If A is a subset of Y, then the *diameter of A* is: $\text{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$. The term *map* means a continuous function. The set of positive integers is denoted by $\mathbb{N}$, and the set of reals by $\mathbb{R}$.

A *continuum* is a nonempty, compact, connected, metric space. A continuum X is *decomposable* if $X = A \cup B$, where A and B are proper subcontinua of X. X is *indecomposable* if it is not decomposable. X is *hereditarily indecomposable* if all of its subcontinua are indecomposable. A continuum X is said to be *irreducible* if there exist two points x and y in X such that no proper subcontinuum of X contains both points x and y. The continuum X is *unicoherent* provided that every time $X = A \cup B$, where A and B are subcontinua, $A \cap B$ is connected. A continuum X is *homogeneous* provided that for each pair x and y of its points, there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$. Given a continuum X and a point x in X, then the *composant* of x, κ_x, is the union of all the proper subcontinua of X containing x.

An *arc* is any space homeomorphic to $[0,1]$. The images of 0 and 1 under a homeomorphism are the *end points* of the arc. A *half open arc* is an arc minus one of its end points, and an *arc segment* is an arc with its end points removed. That is, a half open arc is a space homeomorphic to $[0,1)$ and an arc segment is a space homeomorphic to $(0,1)$.

A continuum X is a *triod* if it contains a subcontinuum B such that $X \setminus B$ has at least three components. A *simple triod* is the union of three arcs having one end point in common and disjoint everywhere else. A continuum is *a–triodic* provided that it does not contain triods.

A *chain* is a finite collection $\{U_1, \ldots, U_m\}$ of open sets such that $U_j \cap U_k \neq \emptyset$ if and only if $|j-k| \leq 1$. The elements of a chain are called *links.* For $\varepsilon > 0$ an *ε–chain* is a chain in which each link has diameter less than ε. A continuum is *chainable* if for each $\varepsilon > 0$, it can be covered by an ε–chain. It is known that every subcontinuum of a chainable continuum is chainable (see [C–V], Theorem (9.C.4)). It is also known that every chainable continuum is unicoherent (see [C–V], Theorem (9.C.12)).

A *simple closed curve* is any space homeomorphic to the unit circle $\mathcal{S}^1$. A *circular chain* is a finite collection $\{U_1, \ldots, U_m\}$ of open sets such that $U_j \cap U_k \neq \emptyset$ if and only if $|j-k| \leq 1$ or $j, k \in \{1, m\}$. The elements of a circular chain are called *links.* For $\varepsilon > 0$ an *ε–circular chain* is a circular chain in which each link has diameter less than ε. A continuum is *circularly chainable* if for each $\varepsilon > 0$, it can be covered by an ε–circular chain. A *solenoid* is a circularly chainable homogeneous continuum, each of whose nondegenerate subcontinua is an arc. In particular $\mathcal{S}^1$ is a solenoid.

A *generalized Warsaw circle* is an arcwise connected circularly chainable continuum which is not a simple closed curve. It is known that a continuum X is a generalized Warsaw circle if and only if there exists a one–to–one onto map $f\colon [0,\infty) \to X$ such that $f([0,1]) = Cl_X(f([t,\infty))) \setminus f([t,\infty))$ for each $t > 1$.

Such a map f is called a *rolling map* for X. (see [N1], Theorem 6).

Given a continuum X, the *hyperspace of nonempty subcontinua of* X is:

$$\mathcal{C}(X) = \{A \subseteq X \mid A \text{ is a continuum}\},$$

It is known that $\mathcal{C}(X)$ is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

(see [N2], (0.1)). In fact, $\mathcal{C}(X)$ is an arcwise connected continuum (see [N2], (1.13)). An *order arc* in $\mathcal{C}(X)$ is a subset Γ of it such that if A and B belong to Γ then either $A \subseteq B$ or $B \subseteq A$ (see [N2], (1.2)). By [N2], (1.3), there is a one–to–one map $\Lambda\colon [0,1] \to \mathcal{C}(X)$ such that $\Lambda([0,1]) = \Gamma$ and $\Lambda(s) \subset \Lambda(t)$ if $s < t$. Let us observe that there is an isometric copy of X contained in $\mathcal{C}(X)$, namely: $\mathcal{F}_1(X) = \{\{x\} \mid x \in X\}$.

Let X be a continuum, $\mathcal{C}(X)^*$ denotes the hyperspace of proper subcontinua of X. For each $A \in \mathcal{C}(X)^*$, let $\mathcal{I}(A) = \{B \in \mathcal{C}(X) \mid A \cap B \neq \emptyset\}$, and $\xi_X(A) = \{B \in \mathcal{I}(A) \mid$ For every $\varepsilon > 0$, there exists $C \in \mathcal{C}(B)$ such that $\mathcal{H}(B,C) < \varepsilon$ and $A \cap C = \emptyset$ $\}$. $\xi_X(A)$ is called the *pseudo exterior* of A in X.

Main Theorems.

Let X be a continuum and let us observe the following:

(1) For each $A \in \mathcal{C}(X)^*$, $\xi_X(A) \subseteq \mathcal{I}(A) \setminus \mathcal{C}(A)$.

(2) If $B \in \mathcal{I}(A)$ then $B \in \xi_X(A)$ if and only if there exists a sequence, $\{C_n\}_{n=1}^{\infty}$, of proper subcontinua of B converging to B such that, for every $n \in \mathbb{N}$, $C_n \cap A = \emptyset$.

(3) If $B \in \xi_X(A)$ then $B \cap A \subseteq \partial_X(A)$.

(4) If $X \in \xi_X(A)$ then $Int_X(A) = \emptyset$.

(5) The converse of (4) is not true. To see this, let

$$X = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbb{R}^2 \,\middle|\, 0 < x \leq \frac{2}{\pi}\right\} \cup \{(0,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 2\}.$$

Let $A = \{0\} \times [0,1]$, then $Int_X(A) = \emptyset$ but $X \notin \xi_X(A)$, since for any subcontinuum B of X such that $\mathcal{H}(B,X) < \frac{1}{2}$, $A \subseteq B$.

(6) If $B \in \mathcal{C}(X)^*$ and $A \in \mathcal{C}(B)^*$ then $\xi_B(A) \subseteq \xi_X(A)$.

Theorem 1. *Let X be a continuum. If $A \in \mathcal{C}(X)^*$, then $\xi_X(A) \neq \emptyset$.*

Proof: Let U be a component of $X \setminus A$, then $Cl_X(U)$ is a continuum and $Cl_X(U) \cap A \neq \emptyset$, by the Boundary Bumping Theorem (see [N3], 5.4). Thus $Cl_X(U) \in \mathcal{I}(A)$. Let x be any point of U, and let $\Lambda\colon [0,1] \to \mathcal{C}(X)$ be an order arc such that $\Lambda(0) = \{x\}$ and $\Lambda(1) = Cl_X(U)$. Let $t \in [0,1]$ be such that $\Lambda(t) \cap A \neq \emptyset$ and such that if $0 \leq s < t$ then $\Lambda(s) \cap A = \emptyset$. Thus $\Lambda(t) \in \xi_X(A)$. □

Theorem 2. *The continuum X is indecomposable if and only if $X \in \xi_X(A)$ for every $A \in \mathcal{C}(X)^*$.*

Proof: If X is indecomposable, then X has uncountably many composants (see [H–Y], Theorem 3–46), and they are dense in X (see [K], Theorem 2, p. 209). Thus, using a composant different from the one containing A, we can find subcontinua of X as close to X as we want, and not meeting A. Hence, $X \in \xi_X(A)$ for all $A \in \mathcal{C}(X)^*$.

Suppose that for every $A \in \mathcal{C}(X)^*$, $X \in \xi_X(A)$, then by remark (4), above, we have that for each $A \in \mathcal{C}(X)^*$, $Int_X(A) = \emptyset$. But this implies that X is indecomposable (see [H–Y], Theorem 3–41). □

Theorem 3. *Let X be a continuum. Then X is hereditarily indecomposable if and only if for each $A \in \mathcal{C}(X)^*$, $\xi_X(A) = \mathcal{I}(A) \setminus \mathcal{C}(A)$.*

Proof: Suppose X is hereditarily indecomposable. Let $A \in \mathcal{C}(X)^*$. We already noticed that $\xi_X(A) \subseteq \mathcal{I}(A) \setminus \mathcal{C}(A)$. So let $B \in \mathcal{I}(A) \setminus \mathcal{C}(A)$. Since $B \in \mathcal{I}(A)$, we have that $B \cap A \neq \emptyset$. On the other hand, since X is hereditarily indecomposable,

$B \in \mathcal{C}(X) \setminus \mathcal{C}(A)$, $B \cap (X \setminus A) \neq \emptyset$, we obtain that $A \subseteq B$. Thus $A \in \mathcal{C}(B)$. Since B is indecomposable, by Theorem 2, we have that $B \in \xi_B(A) \subseteq \xi_X(A)$.

Now suppose that for each $A \in \mathcal{C}(X)^*$, $\xi_X(A) = \mathcal{I}(A) \setminus \mathcal{C}(A)$. Note that this assumption implies that for all $A \in \mathcal{C}(X)^*$, $X \in \xi_X(A)$, so X is indecomposable by Theorem 2. Let $B \in \mathcal{C}(X)^*$, we want to show that B is indecomposable. Take $A \in \mathcal{C}(B)^*$. Since $\xi_X(A) = \mathcal{I}(A) \setminus \mathcal{C}(A)$, we have $B \in \xi_X(A)$ and then $B \in \xi_B(A)$, which implies that $Int_B(A) = \emptyset$, hence B is indecomposable (see [H–Y], Theorem 3–41). □

Let us observe the following:

(7) $\xi_X(A)$ is not necessarily connected. To see this, for each $n \in \mathbb{N}$ let $A_n = \left\{t\left(1, \frac{1}{n}\right) \in \mathbb{R}^2 \mid t \in [0,1]\right\}$ and $A_0 = [0,1] \times \{0\}$. Let $X = \bigcup_{n=0}^{\infty} A_n$, i. e. X is the harmonic fan. If $A = \left[\frac{1}{2}, \frac{3}{4}\right] \times \{0\}$, then the set $\{B \in \xi_X(A) \mid \left(\frac{3}{4}, 0\right) \in B\}$ is open and closed in $\xi_X(A)$, thus $\xi_X(A)$ is not connected. Observe that in this case $X \setminus A$ is connected.

A similar argument shows that there exists a subcontinuum A of the Knaster continuum (see [K] p. 204) such that $\xi_X(A)$ is not connected. Hence it is not necessarily true that $\xi_X(A)$ is connected even for indecomposable continua (compare with Corollary 10, below).

(8) If $B, D \in \xi_X(A)$ then it is not necessarily true that whenever $C \in \mathcal{C}(X)$ satisfying $B \subset C \subset D$ we have that $C \in \xi_X(A)$. To show this, let X be the harmonic fan and let A be as in (7). For each $n \in \mathbb{N}$, let $B_n = \left\{t\left(\frac{3}{4}, \frac{1}{n}\right) \in \mathbb{R}^2 \mid t \in [0,1]\right\}$, and let $B_0 = \left[0, \frac{3}{4}\right] \times \{0\}$. Let $B = \bigcup_{n=0}^{\infty} B_n$ and $C = B \cup ([0,1] \times \{0\})$. Then B and X both belong to $\xi_X(A)$ and $C \notin \xi_X(A)$. To see this note that for any subcontinuum Z of C such that $\mathcal{H}(Z, C) < \frac{1}{8}$, $A \subseteq Z$.

(9) If $X \in \xi_X(A)$ then X may be locally connected at every point of A. As we can see in the following space: Let $A_0 = \{(x,0) \in \mathbb{R}^2 \mid x \in [0,1]\}$, for each $n \in \mathbb{N}$, let $A_n = \{(x, \frac{1}{2^{n-1}}) \in \mathbb{R}^2 \mid x \in [0,1]\}$. For each $n \in \mathbb{N} \cup \{0\}$ and each $m \in \{0, \dots, 2^{n+1}\}$, let $B_{n,m} = \{(\frac{m}{2^{n+1}}, y) \in \mathbb{R}^2 \mid y \in [0, \frac{1}{2^n}]\}$. Let

$$X = \left(\bigcup_{n=0}^{\infty} A_n\right) \cup \left(\bigcup_{n=0}^{\infty}\left(\bigcup_{m=0}^{2^{n+1}} B_{n.m}\right)\right)$$

Then X is a locally connected continuum and for every subcontinuum A of A_0, $X \in \xi_X(A)$. To see this, let $Y_n = \{(x,y) \in X \mid y \geq \frac{1}{n}\}$, for each $n \in \mathbb{N}$. Then the sequence $\{Y_n\}_{n=1}^{\infty}$ of subcontinua of X converges to X and for each $n \in \mathbb{N}$, $Y_n \cap A_0 = \emptyset$.

We are going to state Lemma 7.5 of [I2] of A. Illanes:

Lemma 4. *If X is a generalized Warsaw circle with rolling map f, then $\mathcal{C}(X) = \{f([a,b]) \mid 0 \leq a \leq b\} \cup \{f([0,b]) \cup f([a,\infty)) \mid b \geq 1\}$.*

(10) There exists an a–triodic arcwise connected continuum X such that if $A \in \mathcal{C}(X)^*$ and $Int_X(A) = \emptyset$ then $X \in \xi_X(A)$. Let W be any generalize Warsaw circle with rolling map f (see Lemma 4). To prove this claim, let us observe that if $A = \{f(a)\} \in \mathcal{F}_1(W)$ with $a > 0$ then taking $A_n = f\left(\left[0, a - \frac{1}{n}\right]\right) \cup f\left(\left[a + \frac{1}{n}, \infty\right)\right)$, for each $n \in \mathbb{N}$, we see that $W \in \xi_W(A)$. If $a = 0$ then, it is enough to take $A_n = f([1,n])$, for each $n \in \mathbb{N}$. If $A \in \mathcal{C}(W)^* \setminus \mathcal{F}_1(X)$, and $Int_W(A) = \emptyset$ then $A \subseteq f([0,1])$. In this case taking $A_n = f([1,n])$, for each $n \in \mathbb{N}$, we have that $W \in \xi_W(A)$.

On the other hand, let us note that if $A \in \mathcal{C}(W)^*$ is of the form $A = f([0,b]) \cup f([a,\infty))$, with $b \geq 1$ (in this case, we have that $b < a$, since A is a proper subcontinuum of W). Let $\mathcal{A} = \{C \in \mathcal{C}(W) \mid C \cap A = \{f(b)\}\}$ and $\mathcal{B} = \{C \in \mathcal{C}(W) \mid C \cap A = \{f(a)\}\}$, then $Cl_{\mathcal{C}(W)}(\mathcal{A})$

and $Cl_{\mathcal{C}(W)}(\mathcal{B})$ are order arcs whose end points are $\{f(b)\}$ and $f([b,a])$, and $\{f(a)\}$ and $f([b,a])$, respectively. Note that $(Cl_{\mathcal{C}(W)}(\mathcal{A}) \setminus \{\{f(b)\}\}) \cap (Cl_{\mathcal{C}(W)}(\mathcal{B}) \setminus \{\{f(a)\}\}) = f([b,a])$, and that $\xi_W(A) = (Cl_{\mathcal{C}(W)}(\mathcal{A}) \setminus \{\{f(b)\}\}) \cup (Cl_{\mathcal{C}(W)}(\mathcal{B}) \setminus \{f(a)\}\})$. Hence $\xi_W(A)$ is an arc segment. Similar arguments show that for the other types of proper subcontinua A of W, $\xi_W(A)$ is either an arc segment or consists of at most two half open arcs.

(11) Observe that if X is a continuum such that for every $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is connected, then X is not necessarily indecomposable. To see this, let $\mathcal{S}^1$ be the unit circle. Let $A = \{e^{i\theta} \in \mathcal{S}^1 \mid \theta_b \leq \theta \leq \theta_e\} = [\![e^{i\theta_b}, e^{i\theta_e}]\!]$, where $\theta_b \neq \theta_e$. Without loss of generality, we may assume that $\theta_b, \theta_e \in [0, 2\pi]$. The other cases are similar. Hence A is nondegenerate subcontinuum of $\mathcal{S}^1$. Let $\mathcal{A} = \{C \in \mathcal{C}(\mathcal{S}^1)^* \mid C \cap A = \{e^{i\theta_b}\}\}$, then $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A})$ is an order arc whose end points are $\{e^{i\theta_b}\}$ and $B = \{e^{i\theta} \mid \theta \leq \theta_b \text{ or } \theta_e \leq \theta\}$, and $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A}) \setminus \{\{e^{i\theta_b}\}\} \subseteq \xi_X(A)$. Similary, if $\mathcal{B} = \{C \in \mathcal{C}(\mathcal{S}^1)^* \mid C \cap A = \{e^{i\theta_e}\}\}$, then $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B})$ is an order arc whose end points are $\{e^{i\theta_e}\}$ and B, and $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B}) \setminus \{\{e^{i\theta_e}\}\} \subseteq \xi_X(A)$. Observe that $\left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A}) \setminus \{\{e^{i\theta_b}\}\}\right) \cap \left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B}) \setminus \{\{e^{i\theta_e}\}\}\right) = \{B\}$, and $\xi_X(A) = \left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A}) \setminus \{\{e^{i\theta_b}\}\}\right) \cup \left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B}) \setminus \{\{e^{i\theta_e}\}\}\right)$. Therefore, $\xi_X(A)$ is an arc segment. In particular, $\xi_X(A)$ is connected.

Now, suppose that $A \in \mathcal{F}_1(\mathcal{S}^1)$. Assume that $A = \{p\}$. Let $\mathcal{A} = \{C \in \mathcal{C}(\mathcal{S}^1)^* \mid C = [\![e^{i\theta_b}, e^{i\theta_e}]\!], \text{and } e^{i\theta_b} = p\}$. Then $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A})$ is an order arc whose end points are A and $\mathcal{S}^1$, and $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A}) \setminus \{A\} \subseteq \xi_X(A)$. Similarly, if $\mathcal{B} = \{C \in \mathcal{C}(\mathcal{S}^1)^* \mid C = [\![e^{i\theta_b}, e^{i\theta_e}]\!], \text{and } e^{i\theta_e} = p\}$, then $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B})$ is an order arc whose end points are A and $\mathcal{S}^1$, and $Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B}) \setminus \{A\} \subseteq \xi_X(A)$. Observe that $\left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{A}) \setminus \{A\}\right) \cap \left(Cl_{\mathcal{C}(\mathcal{S}^1)}(\mathcal{B}) \setminus \{A\}\right) = \{\mathcal{S}^1\}$, and

$\xi_X(A) = \big(Cl_{\mathcal{C}(S^1)}(\mathcal{A})\setminus\{A\}\big)\cup\big(Cl_{\mathcal{C}(S^1)}(\mathcal{B})\setminus\{A\}\big)$. Therefore, $\xi_X(A)$ is an arc segment. In particular, $\xi_X(A)$ is connected

Let us note that if X is any solenoid then for each $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is an arc segment. Hence, we ask the following:

Question. *If X is a continuum such that for every $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is an arc segment, then must X be a solenoid?*

Theorem 5. *Let X be an irreducible unicoherent continuum, then X is indecomposable if and only if for every $A \in \mathcal{C}(X)^*$ with $Int_X(A) = \emptyset$, $X \in \xi_X(A)$.*

Proof: If X is an indecomposable continuum, then by Theorem 2, $X \in \xi_X(A)$ for each $A \in \mathcal{C}(X)^*$. Now suppose X is decomposable. Then $X = A \cup B$, where A and B are proper subcontinua of X. Since X is irreducible, $X \setminus A$ is connected (see [N3], 11.16). Let $E = Cl_X(X \setminus A)$ and $F = Cl_X(X \setminus E)$. Then E and F are subcontinua of X, $X = E \cup F$ and $E \cap F = \partial_X(E)$. Observe that since X is unicoherent, $\partial_X(E)$ is subcontinuum of X, and it is nowhere dense. On the other hand, $X \setminus \partial_X(E) = (E \setminus F) \cup (F \setminus E)$, where $E \setminus F$ and $F \setminus E$ are open and disjoint, hence $X \notin \xi_X(\partial_X(E))$. □

Corollary 6. *Let X be a chainable continuum, then X is indecomposable if and only if for every $A \in \mathcal{C}(X)^*$ with $Int_X(A) = \emptyset$, $X \in \xi_X(A)$.*

Proof: It is known that each chainable continuum is a–triodic (see [N3], 12.5) and unicoherent (see [C–V], Theorem (9.C.12)). Hence, by Sorgenfrey's Theorem (see [S], 3.2), chainable continua are irreducible. Then the result follows from Theorem 5. □

Next we will state Theorem 2 of [N–Q] of Nadler and Quinn:

Theorem 7. *A pathwise connected continuum X is a–triodic if and only if X is an arc, a simple closed curve or a generalized Warsaw circle.*

Theorem 8. *Suppose X is a pathwise connected continuum. Then, for each $A \in \mathcal{C}(X)^*$, either $\xi_X(A)$ is an arc segment or it consists of at most two half open arcs if and only if X is an arc, a simple closed curve, or a generalized Warsaw circle.*

Proof: Let X be a pathwise connected continuum satisfying the statement of the Theorem. We will show that X is a–triodic. Since X is pathwise connected, it is enough to see that X does not contain simple triods.

Suppose, to the contrary, that X contains a simple triod $T = \alpha_1 \cup \alpha_2 \cup \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are arcs and $\alpha_1 \cap \alpha_2 = \alpha_1 \cap \alpha_3 = \alpha_2 \cap \alpha_3 = \alpha_1 \cap \alpha_2 \cap \alpha_3 = \{p\}$. Suppose that the end points of α_j are p and q_j, for $j \in \{1,2,3\}$. Let $r_j \in \alpha_j \setminus \{p, q_j\}$ and let β_j the subarc of α_j from p to r_j, $j \in \{1,2,3\}$. Then $T_1 = \beta_1 \cup \beta_2 \cup \beta_3$ is a simple triod and $T_1 \in \mathcal{C}(T)^*$. Observe that $\xi_T(T_1)$ consists of three half open arcs Γ_1,Γ_2, and Γ_3. Since $\xi_T(T_1) \subseteq \xi_X(T_1)$ and either $\xi_X(T_1)$ is an arc segment or it consists of at most two half open arcs, we have that there exists $j \in \{1,2,3\}$ such that both end points of Γ_j belong to $\xi_X(T_1)$, but one of the end points of Γ_j is $\{r_j\}$ and $\{r_j\} \notin \xi_X(T_1)$, a contradiction. Hence X is a–triodic. Then the result follows from Theorem 7.

For the converse, let us observe that if X is a generalized Warsaw circle, then, by remark (10), for each $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is either an arc segment or it consists of at most two half open arcs. If X is a simple closed curve, then, using remark (11), it may be shown that for each $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is an arc segment. Finally, if X is an arc, it is easy to prove that for each $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ consists of one or two half open arcs. □

Theorem 9. *If X is a hereditarily indecomposable continuum then for each $A \in \mathcal{C}(X)^*$, we have that $\xi_X(A)$ is the order arc from A to X minus its end point A.*

Proof: It is known that a continuum Y is hereditarily indecomposable if and only if for each subcontinuum C of Y there is one and only one order arc from C to Y (see [N2], (1.11) and (1.59)).

Let $A \in \mathcal{C}(X)^*$. If B belongs to the order arc from A to X minus A, then $A \in \mathcal{C}(B)$ and, since B is indecomposable, $B \in \xi_B(A) \subseteq \xi_X(A)$. Now, if $B \in \xi_X(A)$, then, since X is hereditarily indecomposable, $A \in \mathcal{C}(B)$. Thus B belongs to the unique order arc from A to X minus its end point A. □

Corollary 10. *If X is a hereditarily indecomposable continuum then for each $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is arcwise connected.*

Theorem 11. *If X is a continuum such that for every $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is an order arc minus one end point then X is hereditarily indecomposable.*

Proof: Suppose X is decomposable, so $X = A \cup B$, where A and B are proper subcontinua of X. Let C be a component of $A \cap B$. Let a be a point of $A \setminus B$ in the composant of A containing C. Then there exists an order arc $\Lambda_1 : [0,1] \to \mathcal{C}(A)$ such that $\Lambda_1(0) = \{a\}$ and $\Lambda_1(1) = A$. Let $t_A \in [0,1]$ be such that $\Lambda_1(t_A) \cap C \neq \emptyset$ and such that for each $s < t_A$, $\Lambda_1(s) \cap C = \emptyset$. Thus $\Lambda_1(t_A) \in \xi_X(C)$. Similarly, we can find an order arc Λ_2: $[0,1] \to \mathcal{C}(B)$ and a point $t_B \in [0,1]$ such that $\Lambda_2(t_B) \in \xi_X(C)$. Observe that $\Lambda_1(t_A) \not\subseteq \Lambda_2(t_B)$ and $\Lambda_2(t_B) \not\subseteq \Lambda_1(t_A)$. This contradicts the fact that $\xi_X(C)$ is an order arc minus one end point. Thus X is indecomposable. A similar argument shows that every subcontinuum of X is indecomposable. □

Corollary 12. *The continuum X is hereditarily indecomposable if and only if for every $A \in \mathcal{C}(X)^*$, $\xi_X(A)$ is an order arc minus one end point.*

The author wishes to thank very much Professor Alejandro Illanes for all the useful conversations they had. The author also thanks the referee for his (her) suggestions which have improved the paper.

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