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SUBSETS AND SUBSPACES THAT DETERMINE GROUP TOPOLOGIES

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ABSTRACT. The concept of a free topological group over a pointed space was introduced by Graev in 1948 [3]. Later in Hewitt and Ross[4], this concept was expanded to consider the inclusion map $i : (X, t) \to G$ where G is any group. We will follow the terminology of [2] and call the finest group topology on G making i continuous the "associated Graev topology". In this paper we will develop conditions that insure that a given extension of a subspace topology is in fact the associated Graev topology. We will discover the associated Graev topology of various subspaces of locally compact groups. We will also discover subsets of groups for which every topology that can be extended to a group topology can be extended in only one way.

The concept of a free topological group over a pointed space was introduced by Graev in 1948 [3]. Later in Hewitt and Ross[4], this idea was expanded to consider a general inclusion map $i : (X, t) \to G$ where G is any group. We will follow the terminology of [2] and call the finest group topology on G making i continuous the "associated Graev topology". In this

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paper we will develop conditions that insure that a given extension of a subspace topology is in fact the associated Graev topology. We will discover the associated Graev topology of various subspaces of locally compact groups. We will also discover subsets of groups for which every topology that can be extended to a group topology has in fact a unique group topology extension.

We define the following notations, if G is a group with $S \subseteq G$ and t is some topology on S, then let $\prod_{i=1}^{n} S$ denote the Cartesian product of n copies of S with the product topology generated by placing t on each copy of S. Let $m_n : \prod_{i=1}^{n} S \to G$ denote the map defined by $m_n(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n$. If T is a topology on G and $S \subseteq G$, then $S|_T$ denotes the subset S endowed with the relative topology from (G, T). We will assume as additional hypothesis throughout the paper that if (S, t) is a subspace of G then there exists a group topology T on G such that $S|_T = (S, t)$. We note that if (G, t) is a topological group and S is a subset of G with nonempty interior in T then in fact T is the associated Graev topology for $S|_T$.

The following theorem is useful when deciding if a locally compact group topology is the associated Graev topology for some compact subspace.

Theorem 1. *If (G, T) is a locally compact Hausdorff group and S is a compact subspace of G that algebraically generates G, then T is the associated Graev topology for $S|_T$, In addition there is no other locally compact group topology on G which induces the relative topology $S|_T$.*

Proof: Since inversion and translations are homeomorphisms for any topological group we may without loss of generality assume that S is symmetric and contains the identity element of G. By the Baire category theorem we have that the interior of S^n is nonempty for some natural number n, let T^* denote the associated Graev topology for $S|_T$. Clearly T^* is Hausdorff.

Also since $\prod_{i=1}^{n} S$ is compact, $m_n : \prod_{i=1}^{n} S \to (G, T^*)$ is a closed map. Therefore T and T^* agree on $S^n \subseteq G$.

Corollary 2. *For n a natural number, the Euclidean topology on $\mathbf{R}^{n+1}$ is the unique locally compact Hausdorff group topology that extends the usual topology on a standard n-sphere.*

Corollary 3. *The Euclidean topology on the reals is the unique locally compact Hausdorff group topology that extends the usual topology on the standard Cantor set.*

Proof: If K is the standard Cantor set then $K + K = [0, 2]$.

Corollary 4. *Let (G, T) be a compact connected Lie group and Γ a maximal torus. Then $\Gamma|_T$ has a unique extension to a locally compact group topology.*

Proof: Let T' be any extension of $\Gamma|_T$ to a group topology on G. If T' fails to be Hausdorff then we can find $x \in G$ with $x \neq e$, the identity element of G, and $x \in cl_{T'}\{e\}$. But since the conjugates of Γ covers G[6], we can find a $y \in G$ such that $x \in y\Gamma y^{-1}$. Since $\Gamma|_T$ is Hausdorff we must conclude that T' is also Hausdorff. Now since $\Gamma|_T$ is compact and G can be written as a finite product of maximal tori, we have by the proof of Theorem 1 that $T = T'$.

The importance of using a compact subspace in the above arguments is made clear by T. Christine Stevens[5]. She demostrates the existence of a metric group topology on $\mathbf{R}^n$ for $n \geq 2$ that agrees with the usual topology on every line. Yet her topology has an unbounded sequence that converges to the origin. Therefore by letting S be the union of the x and y axis in $\mathbf{R}^2$, we can provide a counterexample to many attempts at weakening the hypothesis of Theorem 1.

Theorem 5. *If H is a subgroup of G of index greater than 2 and W is an open set in the topological group (G, T) such that $W - H \neq \varnothing$, then T is the associated Graev topology for $(W - H)|_T$.*

Proof: As before we may assume that $e \in W$ and that W is symmetric. If $x \in G - \overline{H}$ where $\overline{H}$ is the closure of H in T, then the topology on W will determine the topology on the open set $xW \cap (G - x\overline{H})$. Thus T is the associated Graev topology for $T|_{W-H}$.

Now suppose that H is dense in G and let T^* denote the associated Graev topology for $T|_{W-H}$. Let $y \in V$ where $V \in T^*$. Since the index of H in G is greater than 2, we can find elements s and t of G such that sH, tH, and yH are pairwise disjoint. Since H is dense we may assume that $s \in yW \cap sH$ and $t \in yW \cap tH$. Since T and T^* agree on $sW - sH$ and $tW - tH$, we can find sets u_s and u_t in T such that $u_s \cap (sW - sH) = V \cap (sW - sH)$ and $u_t \cap (tW - tH) = V \cap (tW - tH)$. Thus $y \in u_s \cap u_t \cap sW \cap tW$ and $u_s \cap u_t \cap sW \cap tW \in T$.

Suppose that $x \in u_s \cap u_t \cap sW \cap tW \cap (G - V)$. Then $x \in u_s \cap sW \cap (G - V)$ and hence since $u_s \cap (sW - sH) = V \cap (sW - sH)$ and $x \notin V$, we have that $x \notin u_s \cap (sW - sH)$. But this means that $x \notin sW - sH$. Also since $x \in sW$ we must have that $x \in sH$. In a similar fashion we can conclude $x \in tH$. This is a contradiction since sH and tH are distinct cosets. Therefore $u_s \cap sW \cap u_t \cap tW \subseteq V$ and hence $V \in T$.

Corollary 6. *If T is a group topology for the real numbers, then T is the unique topology which extends the relative topology for T on the irrationals.*

Corollary 7. *Let T be a group topology for G and let H be a subgroup of G that is of index greater than 2. Then T is the unique group topology that extends the relative topology for T on $G - H$.*

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