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A SPECIAL SUBSET OF THE REAL LINE AND REGULARITY OF WEAK TOPOLOGIES

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ABSTRACT. We introduce the notion of a σ'-set in the real line and use it to characterize regularity of spaces with a weak topology.

1. INTRODUCTION

The letter N denotes the set of natural numbers, I the closed unit interval $[0,1]$, and the symbol ω stands for the first infinite cardinal.

Let A be a non-empty subset of I. We set $X(A) = (A \times \{0\}) \cup (\cup_{n \in N} I \times \{1/n\})$. Let $\mathcal{E}(A)$ be the relative Euclidean topology of $X(A)$ in the plane. We introduce a new topology for $X(A)$. For every $x \in A$ let $S_x = \{(x,0)\} \cup \{(x,1/n) : n \in N\}$. Consider the cover $\mathcal{C}(A) = \{S_x : x \in A\} \cup \{A \times \{0\}\} \cup \{I \times \{1/n\} : n \in N\}$, where each element of $\mathcal{C}$ has the usual topology. Let

$$\mathcal{T}(A) = \{U : U \subset X(A), U \cap C \text{ is relatively open in } C \text{ for every } C \in \mathcal{C}(A)\}.$$

Obviously $\mathcal{T}(A)$ is a topology on $X(A)$. The space $(X(A), \mathcal{T}(A))$ is Hausdorff because of $\mathcal{E}(A) \subset \mathcal{T}(A)$.

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In general, a space X is said to be determined by a cover $\mathcal{P}$, if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$ [2]. Our space $(X(A), \mathcal{T}(A))$ is determined by the cover $\mathcal{C}(A)$. The space $(X(I), \mathcal{T}(I))$ was introduced in [10, Example 1.6.(2)] to show that there is a regular separable, Lindelöf space which is determined by a point-finite cover of compact metric spaces, but is not dominated by a cover of metric spaces. But, as we see later, the space $(X(I), \mathcal{T}(I))$ is *not* regular. The purpose of the paper is to give a characterization of regularity of $(X(A), \mathcal{T}(A))$.

When we try to characterize regularity of $(X(A), \mathcal{T}(A))$, We naturally reach a special subset of the real line. First we recall some special subsets of the real line, refer [4, §40] or [6] as a survey. Let A be a subset of the real line, then A is called a λ-set (or rarefied set in the sence of Kuratowski) if every countable subset of A is a G_δ-set in A, and A is called a λ'-set if for every countable subset F of the real line, F is a G_δ-set in $A \cup F$. Obviously every λ'-set is a λ-set. It is easy to see that A is a λ'-set iff for every countable subset F of the real line, $A \cup F$ is a λ-set, moreover iff for every countable subset F of the real line, there is an F_σ-set H of the real line such that $F \cap H = \emptyset$ and $A \subset F \cup H$. It is known that there is a λ'-set of cardinality ω_1 in ZFC, for example see [6, p. 215]. Rothberger showed that not every λ-set is a λ'-set [7].

When we replace "countable subset" in the definition of a λ-set by "F_σ-set", we obtain the notion of a σ-set. A set A is called a σ-set if every F_σ-set of A is a G_δ-set in A. It is known that it is consistent that there are no uncountable σ-sets [5, Theorem 22]. It is easy to see that A is a σ-set iff for every F_σ-set F of the real line, there is an F_σ-set H in the real line such that $F \cap H \cap A = \emptyset$ and $A \subset F \cup H$.

Now we introduce the notion of a σ'-set to characterize regularity of $(X(A), \mathcal{T}(A))$.

Definition 1.1. A set A in the real line is called a σ'-set if for every F_σ-set F of the real line there is an F_σ-set H in the real line such that $F \cap H = \emptyset$ and $A \subset F \cup H$.

Obviously every σ'-set is both a σ-set and a λ'-set. Not every σ-set is a σ'-set. In [6, Theorem 5.7] Miller constructed an uncountable σ-set X which is concentrated on a countable set under the continuum hypothesis, where a set A is said to be concentrated on a set D if for every open set G with $D \subset G$, $A - G$ is countable. Since every λ'-set concentrated on a countable set is countable, X is not a λ'-set, in particular not a σ'-set. Not every λ'-set is a σ'-set. Let Y be a λ'-set of cardinality ω_1 in ZFC, for example see [6, p. 215]. In a model guaranteeing that there are no uncountable σ-sets, see [5, Theorem 22], such a Y is a λ'-set which is not a σ'-set.

A subset A of the real line is called a Sierpiński set if it is uncountable and for every Lebesgue measure zero set F, the set $F \cap A$ is countable. We can construct a Sierpiński set by assuming the continuum hypothesis [8]. It is known that every Sierpiński set is a σ-set [9]. By using the same idea, we can see the following fact.

Fact 1.2. Every Sierpiński set is a σ'-set.

Proof: Let A be a Sierpiński set and F an F_σ-set of the real line R. Since every Lebesgue measurable set is the union of an F_σ-set and a measure zero set, we can set $R - F = H \cup Z$, where H is an F_σ-set and Z is a measure zero set. Since $Z \cap A$ is countable, the set $H \cup (Z \cap A)$ is a desired F_σ-set. $\square$

Let C be the Cantor set obtained by deleting the open middle third of I, and let f a homeomorphism of I onto I such that $f(C)$ has a positive Lebesgue measure, for example see [3, p. 83]. By the standard method to obtain a Sierpiński set we can construct a Sierpiński set X in $f(C)$ under the continuum hypothesis. Then $f^{-1}(X)$ is a σ'-set which is not a Sierpiński set.

2. Main results

For every $E \subset X(A)$, we set $E_0 = \{x \in A : (x,0) \in E\}$ and $E_n = \{x \in I : (x, 1/n) \in E\}$ for every $n \in N$. The following lemma is easy to check.

Lemma 2.1. *Let U be a subset of $X(A)$, then the following are equivalent.*

(1) *U is open in $(X(A), \mathcal{T}(A))$,*

(2) *U_0 is open in A, U_n is open in I for every $n \in N$ and $U_0 \subset \bigcup_{n\in N}(\bigcap_{m\geq n} U_m)$ holds.*

We denote by $X_q(A)$ the quotient space of $(X(A), \mathcal{T}(A))$ obtained by collapsing $A \times \{0\}$ to one point.

Theorem 2.2. *Let A be a non-empty subset of I, then the following are equivalent.*

(1) *A is a σ'-set,*

(2) *$(X(A), \mathcal{T}(A))$ is regular,*

(3) *$X_q(A)$ is regular.*

Proof: (1)$\to$(2). Fix any $a \in A$ and let F be a closed subset of $(X(A), \mathcal{T}(A))$ such that $(a,0) \notin F$. We show that $(a,0)$ and F can be separated by open sets. Since $F \cap (A \times \{0\})$ is closed in $(X(A), \mathcal{E}(A))$, there is a $V \in \mathcal{E}(A) (\subset \mathcal{T}(A))$ such that $(a,0) \in V$ and $\overline{V} \cap F \cap (A \times \{0\}) = \emptyset$. Thus we have only to see that $(a,0)$ and $\bigcup_{n\in N} F \cap (I \times \{1/n\})$ can be separated by open sets. For every $x \in A$ let ord(x)= $|\{n \in N : x \in F_n\}|$, where recall $F_n = \{x \in I : (x, 1/n) \in F\}$. Since $S_a \cap F$ is closed in S_a, ord(a) is finite. Hence, for simplicity, we may assume $a \notin \bigcup_{n\in N} F_n$. For every $n \in N$ let $H_n = \bigcup_{m\geq n} F_m$. Since A is a σ'-set, for every $n \in N$ there is an F_σ-set J_n in I such that $J_n \cap H_n = \emptyset$ and $A \subset J_n \cup H_n$. We set $J_n = \bigcup_{m\in\omega} J_{nm}$, where J_{nm} is closed in I, $J_{nm} \subset J_{nm+1}$ and $a \in J_{nm}$ for every $m \in \omega$. For every $k \in N$ $(\cup_{n+m=k} J_{nm}) \cap F_k = \emptyset$ holds. Therefore there is an open set U_k in I such that $\cup_{n+m=k} J_{nm} \subset U_k$ and $\overline{U}_k \cap F_k = \emptyset$. Let $B = \overline{\{x \in A : ord(x) = \omega\}}$, where the closure is taken in A. Obviously $a \notin B$. Now we claim $A - B \subset \cup_{n\in N}(\cap_{m\geq n} U_m)$. In fact, let $x \in A - B$, then ord(x) is finite. Hence there is an $n \in N$ with $x \notin H_n$. Moreover there is an $m \in \omega$ such that $x \in J_{nm}$. Let $k = n + m$, then for every $l \geq k$, $x \in J_{nm} \subset J_{n\,l-n} \subset \cup_{i+j=l} J_{ij} \subset U_l$. Thus $x \in \cap_{l\geq k} U_l$. We set $W = (A - B) \times \{0\} \cup (\cup_{n\in N} U_n \times \{1/n\})$. By Lemma 2.1 W is an open neighborhood of $(a, 0)$. Obviously $\overline{W} \subset A \times \{0\} \cup (\cup_{n\in N} \overline{U}_n \times \{1/n\})$, thus $\overline{W} \cap (\cup_{n\in N} F \cap (I \times \{1/n\})) = \emptyset$. The set $V \cap W$ is a desired open neighborhood of $(a, 0)$.

(2)$\rightarrow$(3). Since $(X(A), \mathcal{T}(A))$ is a regular Lindelöf space, it is normal [1, Theorem 3.8.2]. Hence $A \times \{0\}$ and a closed subset $F \subset X(A)$ with $(A \times \{0\}) \cap F = \emptyset$ can be separated by open sets. This means that $X_q(A)$ is regular.

(3)$\rightarrow$(1). Assume that A is not a σ'-set. Then there is an F_σ-set B in I such that for every F_σ-set B' with $B \cap B' = \emptyset$, $\{B, B'\}$ does not cover A. Suppose that every $x \in A$ has an open neighborhood U_x in A and an F_σ-set B_x in I such that $B_x \cap B = \emptyset$ and $U_x \subset B_x \cup B$. Since A is Lindelöf, $A = \cup_{i\in\omega} U_{x_i}$ for some $\{x_i\}_{i\in\omega} \subset A$. Then the set $C = \cup_{i\in\omega} B_{x_i}$ is an F_σ-set satisfying $C \cap B = \emptyset$ and $A \subset C \cup B$, which is a contradiction. Thus there is a point $a \in A$ such that for every open neighborhood U of a in A and every F_σ-set B' in I with $B \cap B' = \emptyset$, $\{B, B'\}$ does not cover U. We set $B = \cup_{n\in N} B_n$, where B_n is closed in I. Since each B_n is a zero-set in I, we may assume that $\{B_n\}_{n\in N}$ is point-finite in I. Let $E = \cup_{n\in N} B_n \times \{1/n\}$. By point-finite property of $\{B_n\}_{n\in N}$ the set E is closed in $(X(A), \mathcal{T}(A))$. Suppose that there is an open neighborhood U of $(a, 0)$ such that $\overline{U} \cap E = \emptyset$. We set $U_0 = \{x \in A : (x, 0) \in U\}$ and $U_n = \{x \in I : (x, 1/n) \in U\}$ for every $n \in N$. Obviously $\overline{U}_n \cap B_n = \emptyset$ for every $n \in N$, and by Lemma 2.1 $U_0 \subset \cup_{n\in N}(\cap_{m\geq n} U_m)$. Let $F_n = \cap_{m\geq n} \overline{U}_m$ for every $n \in N$. If $n \leq m$, then $F_n \cap B_m \subset \overline{U}_m \cap B_m = \emptyset$. Hence

$F_n \cap (\cup_{m \geq n} B_m) = \emptyset$ for every $n \in N$. This implies that every $F_n \cap B$ is closed in I. So we can set $F_n - B = \cup_{m \in N} F_{nm}$, where F_{nm} is closed in I. Let $F = \cup_{n,m \in N} F_{nm}$. The set F is an F_σ-set and $F \cap B = \emptyset$, moreover it is easy to see $U_0 - B \subset F$. This is a contradiction to the choice of $a \in A$. We conclude that for every open neighborhood U of $(a, 0)$, $\overline{U}$ intersects with E. In other words, for every open neighborhood V of E, $\overline{V}$ intersects with $A \times \{0\}$. This means that $X_q(A)$ is not regular. □

By the theorem above, it is consistent with ZFC that there is an uncountable subset A of the real line such that $(X(A), \mathcal{T}(A))$ is regular. Let A be a non-empty subset of I. We set $X'(A) = (A \times \{0\}) \cup (\cup_{n \in N} A \times \{1/n\})$. We consider the cover $\mathcal{C}'(A) = \{S_x : x \in A\} \cup \{A \times \{0\}\} \cup \{A \times \{1/n\} : n \in N\}$. Let $\mathcal{T}'(A)$ be the topology on $X'(A)$ determined by the cover $\mathcal{C}'(A)$. By the same idea as Theorem 2.2 we can see that the notion of a σ-set characterizes regularity of $(X'(A), \mathcal{T}'(A))$. We denote by $X'_q(A)$ the quotient space of $(X'(A), \mathcal{T}'(A))$ similar to $X_q(A)$.

Theorem 2.3. *Let A be a non-empty subset of I, then the following are equivalent.*

(1) *A is a σ-set,*
(2) *$(X'(A), \mathcal{T}'(A))$ is regular,*
(3) *$X'_q(A)$ is regular.*

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