Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 23, 1998

Research Announcement

THE STRUCTURE OF LOCALLY COMPACT T_5 SPACES UNDER STRONG AXIOMS

PETER NYIKOS

In the research announcement [EN1], some new topological independence results were detailed which were made possible by the discovery of new posets which do not add reals when they are forced with. Since these posets are proper, they can be used in proving certain statements to be consequences of (or at least compatible with) PFA or PFA$^+$, even though these axioms deny CH (in fact, they imply $\mathfrak{c} = \aleph_2$). A remarkable recent example is:

Theorem 1. [N1] *If it is consistent that there is a supercompact cardinal, it is consistent that every hereditarily normal (T_5), hereditarily cwH manifold of dimension greater than 1 is metrizable.*

[Here "cwH" stands for "collectionwise Hausdorff"—see Definition 1 directly below. All through this announcement, the Hausdorff separation axiom is assumed, but the cwH property is much stronger than this.]

Definition 1. An *expansion* of a subset D of a set X is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = d$ for all $d \in D$. A space X is [*strongly*] *collectionwise Hausdorff* (abbreviated [*s*]*cwH*) if every closed discrete subspace has an expansion to a disjoint [*resp.* discrete] collection of open sets.

Set-theoretic hypotheses sufficient for Theorem 1 are obtainable by revised countable support forcing with semi-proper posets beginning with a ground model that has a supercompact cardinal. The simplest and strongest set of sufficient hypotheses can be summed up as $SSA + PFA^+$, where PFA^+ is as in Baumgartner's article in *Handbook of Set-Theoretic Topology* and SSA is the axiom that there is a stationary subset S of ω_1 such that ideal of nonstationary subsets of S is $(\omega_2\,;\,\omega_2,\omega)$-saturated:

Definition 2. Let S be a stationary subset of ω_1. We say that *the ideal of nonstationary subsets of S is $(\kappa\,;\,\lambda,\mu)$-saturated* if for every collection $\mathcal{Z}$ of κ-many stationary subsets of S, there is a subcollection $\mathcal{W}$ of $\mathcal{Z}$ such that $|\mathcal{W}| = \lambda$ and such that every subcollection of $\mathcal{W}$ having μ or fewer members has stationary intersection.

Axiom SSA was shown consistent in [Sh, XIII, 4.3] and the forcing used can easily be modified to get PFA^+ at the same time. The use of PFA^+ in Theorem 1 can be broken down into the use of PFA and an axiom CC_{22} which is one of the strongest of the axioms in a schema introduced in [EN].

Definition 3. A subset S of a poset P is *downward closed* if whenever $s \in S$ and $p < s$, then $p \in S$ also. A collection of subsets of a set X is an *ideal* if it is downward closed with respect to $\subset$, and closed under finite union. An ideal $\mathcal{J}$ of countable subsets of a set X is *countable-covering* if $\mathcal{J} \restriction Q$ is countably generated for each countable $Q \subset X$.

Definition 4. *Axiom CC_{22}* is the axiom that if $\mathcal{J}$ is a countable-covering ideal on a stationary subset E of ω_1, then either:

(i) there exists a stationary subset A of E such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there exists a stationary subset B of E such that $B \cap J$ is finite for all $J \in \mathcal{J}$.

As shown in [N1], Axiom CC_{22} is a consequence of PFA^+ and it is also compatible with CH [EN2]. It plays a key role in all known proofs of Theorem 1 and of the following much more general result [N4]:

Theorem 2. $[SSA + CC_{22} + PFA]$ *If X is a locally compact, locally connected, T_5, hereditarily cwH space, then:*

(a) *Every component of X is ω_1-compact, AND*
(b) *Every non-Lindelöf component of X has uncountably many cut points, AND*
(c) *X is hereditarily collectionwise normal and hereditarily countably paracompact, AND*
(d) *X is the union of a rim-finite closed subspace R and a family of open Lindelöf subspaces with countable discrete boundaries, AND*
(e) *For every open cover $\mathcal{U}$ of X, there is a discrete family $\mathcal{K}$ of copies of regular cardinals such that $X \setminus \bigcup \mathcal{K}$ can be covered by a relatively locally finite collection of open sets, each of which is a subset of some $U \in \mathcal{U}$, AND*
(f) *X admits a perfect map onto a monotone normal, locally compact, locally connected, rim-finite space Z such that every point of Z has perfectly normal preimage.*

Since no manifold of dimension > 1 has any cut points, Theorem 1 is immediate from Theorem 2. In the set-theoretic axioms used in Theorem 2, SSA can be replaced by the following axiom [which it implies, see Theorem 3 (f) below]:

Axiom F. Given any family of functions $\{f_\alpha : \alpha < \omega_2\}$ from ω_1 to ω, there is a stationary set S and an infinite $Z \subset \omega_2$ and $n \in \omega$ such that $f_\zeta(\sigma) < n$ for all $\zeta \in Z$ and all $\sigma \in S$.

Axiom F clearly implies that there are no Kurepa trees, and hence it implies there is an inner model with an inaccessible cardinal. It would be good to know what its consistency strength actually is. It fits in nicely with the following concepts.

Definition 5. If $\mathcal{I}$ is an ideal of subsets of a set S, we say a collection $\mathcal{A}$ of subsets of S is *$\mathcal{I}$-disjoint* if the intersection of any pair of members of $\mathcal{A}$ is in $\mathcal{I}$. A collection $\mathcal{A}$ of sets in

$\mathcal{P}(S) \setminus \mathcal{I}$ is *maximal $\mathcal{I}$-disjoint* if it is $\mathcal{I}$-disjoint and for each $B \in \mathcal{P}(S) \setminus \mathcal{I}$ there exists $A \in \mathcal{A}$ such that $(B \cap A) \notin \mathcal{I}$.

Definition 6. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on a set S. We say *the pair $\langle \mathcal{I}, \mathcal{J} \rangle$ is $(\kappa, \iota\,;\, \lambda, \mu)$-para-saturated* [resp $(\kappa, <\iota\,;\, \lambda, \mu)$-*para-saturated*] if for each family $\mathbb{K}$ of κ maximal $\mathcal{I}$-disjoint collections of sets in $\mathcal{P}(S) \setminus \mathcal{I}$, each of which has $\leq \iota$ [*resp.* fewer than ι] members, it is possible to choose a subfamily $\mathbb{L}$ of λ collections and a member of each collection in the subfamily $\mathbb{L}$ so that the intersection of any set of μ chosen members is not in J. We say *$\mathcal{I}$ is $(\kappa, \iota\,;\, \lambda, \mu)$-para-saturated* if the pair $\langle \mathcal{I}, \mathcal{I} \rangle$ is $(\kappa, \iota\,;\, \lambda, \mu)$-para-saturated.

Of course, this concept only makes sense if $\kappa \geq \lambda \geq \mu$. On the other hand, there are no restrictions on ι vis-a-vis the other cardinals. If an ideal $\mathcal{I}$ is $(\kappa\,;\, \lambda, \mu)$-saturated it is clearly $(\kappa, \iota\,;\, \lambda, \mu)$-para-saturated for all ι. In general, the smaller κ is, and the bigger any of the remaining cardinals are, the more demanding the condition. Laver has observed that the pair $\langle \mathcal{I}, \mathcal{J} \rangle$ is not $(\kappa, 2\,;\, \omega, \omega)$-para-saturated in a model obtained by adding κ Cohen reals, where $\mathcal{I}$ and $\mathcal{J}$ are the nonstationary and countable ideals on ω_1, respectively. This is the weakest possible kind of nontrivial para-saturation property involving infinite μ. We also have a number of consistency results going the other way, some of them summarized in the following theorem along with two negative results.

Theorem 3. [N2] *Let $\mathcal{I}$ be the nonstationary ideal on a stationary subset E of ω_1. Then:*

(a) *If $\mathfrak{p} > \omega_1$, then $\mathcal{I}$ is $(\omega, n\,;\, \omega, \omega)$-para-saturated for all finite n*
(b) *$\mathcal{I}$ is not $(\omega, <\omega\,;\, \omega, \omega)$-para-saturated;*
(c) *If $\mathfrak{p} > \omega_1$, then $\mathcal{I}$ is $(\omega_1, <\omega\,;\, \omega, \omega)$-para-saturated;*
(d) *$\mathcal{I}$ is not $(\omega_1, \omega\,;\, \omega, \omega)$-para-saturated*
(e) *If it is consistent that there is a supercompact cardinal, it is consistent that there is an E such that $\mathcal{I}$ is $(\omega_2, 2^{\omega_1}\,;\, \omega_2, \omega)$-para-saturated;*

(f) *If $\mathcal{I}$ is $(\omega_2, \omega\,;\, \omega_1, \omega)$-para-saturated, then Axiom F holds, and hence there is an inner model with an inaccessible cardinal.*

(g) *If $\mathfrak{p} > \omega_1$ and Axiom F holds, and $E = \omega_1$, then $\mathcal{I}$ is $(\omega_2, \omega\,;\, \omega, \omega)$-para-saturated.*

A natural question raised by this theorem is:

Problem 1. Is it possible to show the consistency of the nonstationary ideal on ω_1 being $(\omega_2, \omega\,;\, \omega, \omega)$-para-saturated without using large cardinal axioms?

In our context, however, the most relevant unsolved problem about para-satura-tion at this time is:

Problem 2. Let S be a stationary subset of ω_1. If $\mathcal{I}$ is the nonstationary ideal on S and $\mathcal{J}$ is the countable one, is it possible to show the consistency of $\langle \mathcal{I}, \mathcal{J} \rangle$ being $(\omega_2, 2^{\omega_1}\,;\, \omega, \omega)$-para-saturated without using large cardinal axioms?

The existence of S for which this para-saturation property holds is called Axiom S in [N1] and is used there in place of Axiom F to prove Theorem 1. Otherwise the same axioms are used as were used in proving Theorem 2, and PFA was actually only used in Theorems 1 and 2 for two things: (1) every locally compact Hausdorff space of countable spread is hereditarily Lindelöf and (2) every first countable, countably compact Hausdorff space is either compact or contains a copy of ω_1. No large cardinals are needed for this much: (1) follows simply from $MA(\omega_1)$ and (2) has been obtained in a forcing model of $MA(\omega_1)$ by Alan Dow without any necessity for large cardinals.

The proof of Theorem 1 in [N1] also uses the following 'intuitively obvious' fact about domains (*i.e.*, open connected sets) in $\mathbb{S}^n$:

Theorem A *Suppose that D is a domain in $\mathbb{S}^n$. Then for every component C of the complement of D, the frontier ∂C of C is connected.*

This is trivial for the case $n = 1$. I am indebted to David Gauld for the proof for $n > 1$. As might be expected, it uses algebraic topology.

The following concept played a role in the proofs of Theorems 1 and 2, and also in the proofs of some theorems that do not require local connectedness:

Definition 7. An *alignment* of a space X is a family $\langle X_\alpha : \alpha < \theta \rangle$ of open subspaces of X whose union is X, such that $\overline{X_\alpha}$ is a proper subset of X_β whenever $\alpha < \beta$. The ordinal θ is called the *length* of the alignment, while the *width at* α of the alignment equals the Lindelöf degree of $X_\alpha \setminus \bigcup_{\xi<\alpha} X_\xi$, and the *width* of the alignment is the supremum of the widths at all ordinals $< \theta$. An alignment is *continuous* if $X_\alpha = \bigcup_{\xi<\alpha} X_\xi$ whenever α is a limit ordinal.

Recall that the *Lindelöf degree* of a space X denoted $L(X)$, is the least infinite cardinal number κ such that every open cover of X has a subcover of cardinality $\leq \kappa$. Spaces of Lindelöf degree $\leq \aleph_1$ will be called $\aleph_1$-*Lindelöf* below. The following two theorems are shown in [N2] and are also used in proving Theorem 2.

Theorem 4. [MA(ω_1)] *Let X be a locally compact, hereditarily scwH, ω_1-compact space. Then X has a continuous alignment $\{X_\alpha : \alpha < \theta\}$ of countable width such that each X_α is ω_1-compact and each $\overline{X_\alpha} \setminus X_\alpha$ is hereditarily Lindelöf. Moreover, if η is a limit ordinal of uncountable cofinality, then $\bigcup\{X_\alpha : \alpha < \eta\}$ has (countable, closed) discrete boundary.*

Theorem 5. [PFA + CC_{22} + Axiom F] *Every locally compact, T_5, hereditarily cwH space X satisfying $L(X) = \aleph_1$ has a continuous alignment $\{X_\alpha : \alpha < \omega_1\}$ of countable width such that the boundary of each X_α is countable.*

An example is given in [N5] to show that one cannot make the boundary discrete in Theorem 5. However, if one adds ω_1-compactness (or local connectedness: see Theorem 2 (a)) to the hypotheses, one can make it closed discrete for all α. This has a consequence similar to that in Theorem 2 (b):

Theorem 6. [PFA + CC_{22} + Axiom F] *Every locally compact, T_5, hereditarily cwH, ω_1-compact space X satisfying $L(X) = \aleph_1$ is collectionwise normal and countably paracompact.*

The hypotheses may seem almost ridiculously strong, but this and Theorem 2 (b) are among the very few consistency results that say such-and-such a kind of Dowker space does not exist. [Recall that a space is called *Dowker* if it is normal but not countably paracompact.] All others are, if anything, even more specialized.

We close with a pair of theorems [N3] used in the proof of Theorem 2 that used no special set-theoretic hypotheses at all, and the topological hypotheses involved a concept so weak that it is implied by each of the following individually: normality (even pseudonormality), countable paracompactness, the scwH property, and realcompactess:

Definition 7. A space X satisfies *Property wD* if every infinite closed discrete subspace D has an infinite subspace D' that has an expansion to a discrete collection of open sets.

Of course, Property wD need only be verified for countable sets.

Theorem 7. *Let X be a locally compact space satisfying wD hereditarily, and let $\langle X_\alpha : \alpha < \theta \rangle$ be a continuous alignment of X. For each limit ordinal γ of uncountable cofinality, the boundary of X_γ in X is a closed discrete subspace.*

Theorem 8. *If X is a locally compact, locally connected space satisfying Property wD hereditarily, and every $\aleph_1$-Lindelöf subspace of X has $\aleph_1$-Lindelöf closure, then X has a continuous alignment of width $\leq \omega_1$.*

Acknowledgements:

I am indebted to Todd Eisworth for his timely help in looking up and relaying the information in [Sh] which made Theorem 1 possible. I announced this theorem at the international topology conference in Gyula, Hungary, on August 12, 1998, one day after Eisworth informed me of the existence of what is here called SSA. I also wish to thank the organizers of that conference, sponsored by the Bolyai János Mathematical Society, for inviting me to give a 50-minute talk and arranging for Internet access which was utilized in the communication between Eisworth and me. He was in Jerusalem at the time.

References

[EN1] T. Eisworth and P. Nyikos, *Recent applications of totally proper forcing,* research announcement submitted to Topology Proceedings

[EN2] ________, *Applications of some PFA-like axioms mostly compatible with CH,* in preparation.

[N1] P. Nyikos, *Complete normality and metrization theory of manifolds,* submitted to Top. Appl.

[N2] ________, *Locally compact T_5 spaces and para-saturation of ideals,* submitted to Fund. Math.

[N3] ________, *Alignments and the theory of locally compact T_5 spaces,* in preparation

[N4] ________, *The fine structure of locally compact, locally connected T_5 spaces,* in preparation.

[N5] ________, *Some test spaces in the theory of locally compact T_5 spaces,* in preparation.

[Sh] S. Shelah, *Proper and Improper Forcing,* Springer-Verlag, 1998.

University of South Carolina, Columbia SC 29208
E-mail address: nyikosmath.sc.edu