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ON SOME PROPERTIES OF LINEARLY LINDELÖF SPACES

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Abstract

Improving an earlier result in [4], we show that the cardinality of a sequential linearly Lindelöf Tychonoff space X does not exceed 2^ω if and only if the pseudocharacter of X does not exceed 2^ω. We also establish that every locally metrizable linearly Lindelöf regular space is separable and metrizable.

1. Introduction

It is well known that the following condition:

(CAP) *every uncountable subset A of X of regular cardinality has a point of complete accumulation in X*,

does not characterize Lindelöf spaces, though it is not difficult to show that all Lindelöf spaces satisfy (CAP). The first example of a non-Lindelöf space of this kind was constructed by A.S. Mischenko [7]. The spaces satisfying (CAP) are nowadays called *linearly Lindelöf* spaces. Notice, that the condition (CAP) is equivalent to the following condition: *every open covering γ of X which is a chain* (that is, for any two elements of γ, one is a subset of the other one) *contains a countable subcovering of X*.

It is natural to consider, which additional assumptions turn a linearly Lindelöf space into a Lindelöf one, and which theorems on Lindelöf spaces can be extended to linearly Lindelöf spaces.

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It was proved in [4], that the cardinality of every first countable linearly Lindelöf Tychonoff space does not exceed 2^ω (Theorem 3.1.), which is a generalization of a theorem in [1]. In this note, we improve this result. We make an extensive use of some technical results from [4]; we provide their precise formulations, omitting the proofs, which the reader can find in [4]. All spaces considered are assumed to be T_1 and Tychonoff. If τ is a cardinal number, then τ^+ stands for the smallest cardinal number greater than τ. We put $c = 2^\omega$. Other terminology and notation are such as in [5].

2. Some Notions, the Main Result, and Its Reduction

In what follows, τ is an infinite cardinal number, which we also interpret as the first ordinal number of cardinality τ. We call a subset A of a space X a *G_τ-set* in X, or a subset of *the type* G_τ, if there is a family γ of open subsets of X such that the cardinality of γ is not greater than τ and A is the intersection of γ. We denote by $Ch(X)$ the smallest infinite cardinal number τ such that X is of the type G_τ in $\beta(X)$ (or, which is equivalent, X is of the type G_τ in some other compactification of X). Following the tradition, we call G_ω-sets G_δ-sets. A point x of a space X is said to be *G_τ-separated from a subset Y of X*, if there is a closed G_τ-set P in X such that $x \in P$ and the sets Y and P are disjoint.

We recall also the notion of a free sequence introduced in [1]. Let κ be an ordinal number. A *κ-long free sequence in a space X* is a transfinite sequence $S = \{x_\alpha : \alpha < \kappa\}$ of elements of X such that for every $\alpha < \kappa$ the closures in X of the sets $L_S(\alpha) = \{x_\beta : \beta < \alpha\}$ and $R_S(\alpha) = \{x_\beta : \alpha \leq \beta < \kappa\}$ are disjoint. If every free sequence in X is countable, we write $F(X) \leq \omega$.

A space X *ω_1-Lindelöf*, if every open covering γ of X such that $|\gamma| \leq \omega_1$ contains a countable subcovering. Every linearly Lindelöf space is ω_1-Lindelöf. We denote by $l(X)$ *the Lindelöf degree of* X, that is, the smallest infinite cardinal number τ such that from every open covering of X one can choose a subcovering, the cardinality of which does not exceed τ. Here is our main result:

Theorem 2.1. *Let X be a linearly Lindelöf sequential space. Then the cardinality of X does not exceed 2^ω if and only if $\psi(X) \le 2^\omega$.*

We will derive Theorem 2.1 from the next three assertions.

Proposition 2.2. *Suppose X is a sequential ω_1-Lindelöf space and every closed subset A of X such that $|A| \le 2^\omega$ is a G_{2^ω}-set in X. Then $|X| \le 2^\omega$.*

This assertion is proved by a standard saturation argument (see [1], [2], [6]). Besides, it differs from Lemma 2.11 in [4] only in one respect: we no longer require that X should be first countable; instead, we assume that X is sequential, since, in fact, sequentiality is all we need from first countability in the proof of Lemma 2.11 in [4]. We present the proof of Proposition 2.2, for the sake of completeness.

Proof of Proposition 2.2. Let $\mathcal{P}$ be the family of all closed subsets A of X such that $|A| \le 2^\omega$. For each $A \in \mathcal{P}$ we fix a family $\eta(A)$ of open sets in X such that $|\eta(A)| \le 2^\omega$ and $\cap\eta(A) = A$. For every countable family γ of open sets in X such that $X \setminus \cup\gamma$ is not empty, we fix a point c_γ in $X \setminus \cup\gamma$. Now we are going to define by transfinite recursion an ω_1-long sequence $\{A_\alpha : \alpha < \omega_1\}$ of elements of $\mathcal{P}$.

Put $A_0 = \emptyset$. Let us assume that $A_\beta \in \mathcal{P}$ is already defined for every $\beta < \alpha$, for some ordinal $\alpha < \omega_1$. We will now define the set $A_\alpha \in \mathcal{P}$. Put $\mathcal{U}_\alpha = \cup\{\eta(A_\beta) : \beta < \alpha\}$, and let $\mathcal{W}_\alpha$ be the family of all countable subfamilies γ of the family $\mathcal{U}_\alpha$ such that $X \setminus \cup\gamma \ne \emptyset$. Put $C_\alpha = \{c_\gamma : \gamma \in \mathcal{W}_\alpha\}$ and $H_\alpha = (\cup\{A_\beta : \beta < \alpha\}) \cup C_\alpha$. Clearly, $|\mathcal{U}_\alpha| \le 2^\omega$, $|\mathcal{W}_\alpha| \le 2^\omega$, and $|H_\alpha| \le 2^\omega$. Let A_α be the closure of the set H_α in X. Then $|A_\alpha| \le 2^\omega$, since X is sequential and Hausdorff. Hence $A_\alpha \in \mathcal{P}$. The definition of the sets A_α for each $\alpha < \omega_1$ is complete.

It is clear from the construction that if $\beta < \alpha$ then $A_\beta \subset A_\alpha$. Since the tightness of X is countable and each A_α is closed in

X, it follows that the set $L = \cup\{A_\alpha : \alpha < \omega_1\}$ is closed in X. Therefore, the subspace L of X is ω_1-Lindelöf. Also $|L| \leq 2^\omega$.

Let us show that $L = X$. Assume the contrary, and fix $b \in X \setminus L$. Then, for each $\alpha < \omega_1$, we can choose $U_\alpha \in \eta(A_\alpha)$ such that b is not in U_α. Also $A_\alpha \subset U_\alpha$. Therefore, the family $\{U_\alpha : \alpha < \omega_1\}$ is an open covering of L. This covering contains a countable subfamily γ such that $L \subset \cup\gamma$, since L is ω_1-Lindelöf. Then b is not in $\cup\gamma$; therefore, a point c_γ is defined such that $c_\gamma \in X \setminus \cup\gamma \subset X \setminus L$. On the other hand, since ω_1 is not countably cofinal and γ is countable, there is $\alpha^* < \omega_1$ such that $\gamma \subset \mathcal{U}_{\alpha^*}$ and $\gamma \in \mathcal{W}_{\alpha^*}$. Then $c_\gamma \in C_{\alpha^*} \subset H_{\alpha^*} \subset A_{\alpha^*} \subset L$, a contradiction. □

Lemma 2.3. *Let X be a linearly Lindelöf space, and Y a closed subspace of X such that $\psi(y, X) \leq 2^\omega$, for each $y \in Y$, $|Y| \leq 2^\omega$, $F(Y) \leq \omega$, and $l(X) \leq c^+$. Then Y is a G_{2^ω}-set in X.*

Lemma 2.3 is, indeed, the key new step in the argument. We will prove it a little later, after showing how to derive Theorem 2.1 from Lemma 2.3 and Proposition 2.2. For this reduction, we need the next easy to prove result from [4]:

Lemma 2.4. *Let X be a sequential space, the cardinality of which is greater than 2^ω. Then there is a closed subspace Y of X such that $|Y| = c^+$.*

Proof of Theorem 2.1 from Proposition 2.2 and Lemmas 2.3 and 2.4. It is obvious, that the condition is necessary. Let us prove sufficiency.

Assume the contrary. Then, by Lemma 2.4, there exists a linearly Lindelöf sequential space X such that $|X| = c^+$. Then $l(X) \leq c^+$, and, by Lemma 2.3, every closed subset A of X such that $|A| \leq 2^\omega$ is a G_{2^ω}-set in X. Since X is ω_1-Lindelöf, Proposition 2.2 implies that $|X| \leq 2^\omega < c^+$, a contradiction. □

3. A Proof of Lemma 2.3

In this section, we present a sequence of technical results, the last of which is Lemma 2.3. The following assertion is known (see [2]):

Lemma 3.1. *Let Y be a subspace of X and $x \in X$. Assume also that $F(Y) \leq \omega$. Then either a) There is a countable subset A of Y such that x is in the closure of A, or b) There is a closed G_ω-set P in X such that $x \in P$ and $P \cap Y = \emptyset$.*

The next two results from [4] remain the key steps in our argument:

Lemma 3.2. *Let X be a linearly Lindelöf space and $\mathcal{P}$ a family of closed subsets of X, satisfying the next two conditions: a) $\cap\mathcal{P} = \emptyset$, and b) for every closed subset P of X there is $F \in \mathcal{P}$ such that either $F \subset P$ or $F \cap P = \emptyset$. Then there is a countable subfamily ξ of $\mathcal{P}$ such that $\cap\xi = \emptyset$.*

Lemma 3.3. [4] *Let X be a space, Y a linearly Lindelöf subspace of X such that $F(Y) \leq \omega$, and $x \in X \setminus Y$. Let also $\mathcal{P}$ be a family of closed subsets of Y such that every closed separable subspace F of the space Y, such that x is in the closure of F in the space X, belongs to $\mathcal{P}$, and for every closed G_δ-subset B of the space X such that $x \in B$, the set $Y \cap B$ belongs to $\mathcal{P}$. Then there is a countable subfamily ξ of $\mathcal{P}$ such that $\cap\xi = \emptyset$.*

The next assertion is slightly more general than Proposition 2.6 in [4].

Proposition 3.4. [4] *Let X be a linearly Lindelöf space such that $|X| \leq 2^\omega$, $w(X) \leq 2^\omega$, and $F(X) \leq \omega$. Then $Ch(X) \leq 2^\omega$.*

Proof. Let $Z = b(X)$ be a Hausdorff compactification of X such that $w(Z) \leq 2^\omega$, and $\mathcal{S}$ the family of all closed subsets B of Z such that $B \cap X$ is separable and dense in B. Since $|X| \leq 2^\omega$, the cardinality of $\mathcal{S}$ is not greater than 2^ω. Let $\mathcal{G}$ be the family of all closed G_ω-subsets of Z. Since Z is compact and $w(Z) \leq 2^\omega$, we have $|\mathcal{G}| \leq 2^\omega$. Therefore, the cardinality of the family $\mathcal{E}$ of all sets $A \subset Z$ such that $A = \cap\lambda$ for some countable subfamily λ of the family $\mathcal{S} \cup \mathcal{G}$ also does not exceed 2^ω.

Let us show that for every $z \in (Z \setminus X)$ there exists $K \in \mathcal{E}$ such that $z \in K \subset (Z \setminus X)$. Let $\mathcal{S}_z$ be the family of all $B \in \mathcal{S}$ such that z is in the closure of $B \cap X$, and $\mathcal{G}_z$ the family of all $B \in \mathcal{G}$ such that $z \in B$. Consider the family

$$\mathcal{P}_z = \{X \cap B : B \in (\mathcal{S}_z \cup \mathcal{G}_z)\}.$$

The family $\mathcal{P}_z$ satisfies conditions imposed on $\mathcal{P}$ in Lemma 3.3. Therefore, there exists a countable subfamily η of $\mathcal{S}_z \cup \mathcal{G}_z$ such that $\cap\eta \subset (Z \setminus X)$. Clearly, $z \in \cap\eta \in \mathcal{E}$. It follows that there exists a subfamily $\mathcal{K}$ of $\mathcal{E}$ such that $Z \setminus X = \cup\mathcal{K}$. Since $|\mathcal{K}| \leq |\mathcal{E}| \leq 2^\omega$ and all elements of $\mathcal{K}$ are closed in Z, we conclude that X is a G_{2^ω}-set. □

The next assertion is well known (and easily proved, see [5]).

Proposition 3.5. *Let Z be a space, H a subspace of Z, and $\tau = l(H)$. Then for every closed in Z subset F, contained in $Z \setminus H$, there exists a G_τ-set P in Z such that $F \subset P \subset Z \setminus H$.*

The next assertion is a new step in the argument, essential for obtaining the announced generalization.

Proposition 3.6. *Let X be a linearly Lindelöf space such that $|X| \leq 2^\omega$ and $F(X) \leq \omega$, and let $b(X)$ be a compactification of X. Then there exist families $\mathcal{P}$ and $\mathcal{K}$ of compact subsets of $b(X)$ such that the following conditions are satisfied:*

1) $b(X) = (\cup\mathcal{P}) \cup (\cup\mathcal{K})$;

2) $|F \cap X| = 1$, for each $F \in \mathcal{P}$;

3) $F \cap X = \emptyset$, for each $F \in \mathcal{K}$;

4) $|\mathcal{P}| \leq 2^\omega$ and $|\mathcal{K}| \leq 2^\omega$.

Proof. Since $|X| \leq 2^\omega$ and X is Tychonoff, there exists a one-to-one continuous mapping f of X onto a Tychonoff space Y such that the weight of Y is not greater than 2^ω (see [4]). Then, clearly, $F(Y) \leq F(X) \leq \omega$, and $|Y| \leq |X| \leq 2^\omega$.

Therefore, by Proposition 3.4, $Ch(Y) \leq 2^\omega$. Take the continuous extension g of f to the Stone-Čech compactifications βX

and βY, and fix a family γ of compact subsets of βY such that $\cup\gamma = \beta Y \setminus Y$ and $|\gamma| \leq 2^\omega$. Put

$$\mathcal{P}_1 = \{g^{-1}(y) : y \in Y\}, \mathcal{K}_1 = \{g^{-1}(K) : K \in \gamma\}.$$

Now, let us fix a continuous mapping h of βX onto bX which is identity on X (see [5]), and put

$$\mathcal{P} = \{h(F) : F \in \mathcal{P}_1\}, \mathcal{K} = \{h(K) : K \in \mathcal{K}_1\}.$$

Obviously, the families $\mathcal{P}$ and $\mathcal{K}$ satisfy conditions 1)–4). □

Now everything is ready for the proof of Lemma 2.3.

Proof of Lemma 2.3. Since c^+ is not countably cofinal, and X is linearly Lindelöf, it follows that $l(X) \leq 2^\omega$. Indeed, from any open covering of X we can choose a subcovering γ such that $|\gamma| \leq c^+$, by the assumption. Now, since X is linearly Lindelöf, we can choose a subcovering γ_1 of γ such that either γ_1 is finite or the cardinality of γ_1 is cofinal to ω. Since $|\gamma_1| \leq |\gamma| \leq c^+$, it follows that $|\gamma_1| \leq 2^\omega$. Therefore, $l(X) \leq 2^\omega$.

Let B be a compact space, containing X as a subspace, and let $b(Y)$ be the closure of Y in B. By Proposition 3.6, there exists a family $\mathcal{K}$ of compact subspaces of $b(Y)$, satisfying the following conditions: 1) $|\mathcal{K}| \leq 2^\omega$; 2) $\cup\mathcal{K} = b(Y)$; and 3) $|K \cap X| \leq 1$, for each $K \in \mathcal{K}$.

Consider the subspace $Z = b(Y) \cup X$ of the space B. Take any $K \in \mathcal{K}$. If $K \cap X = \emptyset$, put $H_K = X$. If $K \cap X$ is not empty, then, by the conditions 2) and 3), $K \cap X = \{y_K\}$, for some $y_K \in Y$; then put $H_K = X \setminus \{y_K\}$. Since $l(X) \leq 2^\omega$, and $\psi(y_K, X) \leq 2^\omega$, we have $l(H_K) \leq 2^\omega$, for each $K \in \mathcal{K}$. Also, it is clear that $K \subset Z$, K is closed in Z, and $K \cap H_K = \emptyset$. By Proposition 3.5, there exists a family γ_K of open sets in Z such that $|\gamma_K| \leq 2^\omega$ and $K \subset \cap\gamma_K \subset Z \setminus H_K$. Then, clearly,

$$(\cap\gamma_K) \cap (X \setminus b(Y)) = \emptyset.$$

Put $\mathcal{B} = \cup\{\gamma_K : K \in \mathcal{K}\}$. Obviously, $|\mathcal{B}| \leq 2^\omega$. Since $\cup\mathcal{K} = b(Y)$, and $K \subset \cap\gamma_K \subset b(Y)$, for each $K \in \mathcal{K}$, it follows that *for each* $z \in b(Y)$, *and each* $x \in X \setminus b(Y)$, *there exists* $U \in \mathcal{B}$ *such that* $z \in U$ *and* $x \notin U$.

Let μ be the family of all open subsets W of Z such that W is the union of a finite collection of elements of $\mathcal{B}$ and $b(Y) \subset W$. Then from the properties of the family $\mathcal{B}$ we just established it follows that $|\mu| \leq 2^\omega$ and $\cap\mu = b(Y)$. Therefore, the set $b(Y)$ is of the type G_{2^ω} in Z. Since $b(Y) \cap X = Y$, it follows that the set Y is of the type G_{2^ω} in the space X. □

Notice, that Lemma 2.3 is obvious in the case of Lindelöf spaces. It is amazing that it took us much effort to prove it for linearly Lindelöf spaces. Lemma 2.3 is a strengthening of Lemma 2.10 in [4].

4. The Case of Linearly Lindelöf Locally Metrizable Spaces, and Some Open Questions

Question 1. *Is it true in ZFC that every first countable linearly Lindelöf space is Lindelöf?*

Notice, that consistently the answer is “yes”, since under CH every linearly Lindelöf space has cardinality not greater than ω_1 (this follows from Theorem 3.1, but can be easily proved directly).

Question 2. *Is every locally compact linearly Lindelöf space Lindelöf?*

We do not even know, if the answer to the last question is consistently “yes”.

Locally metrizable linearly Lindelöf spaces constitute an important special case of first countable linearly Lindelöf spaces. Here we have a definitive result.

Theorem 4.1. *Every locally metrizable linearly Lindelöf space* X *is a separable metrizable space (hence, Lindelöf).*

Proof. Obviously, it is enough to show that X is hereditarily Lindelöf.

Assume the contrary. Then X contains a subspace Y such that $|Y| = \omega_1$ and each uncountable subspace Z of Y is not

Lindelöf (let Y be a standard right separated subspace of X such that $|Y| = \omega_1$, see [6]).

Since X is linearly Lindelöf, there is a point $a \in X$ of complete accumulation of Y. Take any open neighbourhood U of a such that $\bar{U}$ is metrizable. Since $\bar{U}$ is also linearly Lindelöf, it follows that $\bar{U}$ is a separable metrizable space. On the other hand, $Z = U \cap Y$ is uncountable; therefore, by the choice of Y, Z is not Lindelöf. This is impossible, since Z is a subspace of a separable metrizable space U. □

Remark. Clearly, in Theorem 4.1 we can replace the assumption that X is linearly Lindelöf by the weaker assumption that X is ω_1-Lindelöf.

Corollary 4.2. *On the real line R there is no locally compact linearly Lindelöf non-Lindelöf topology which is stronger than the usual topology of the real line.*

Proof. Indeed, such a topology on R would turn R into a locally metrizable space (since every one-to-one continuous mapping on a compact space is a homeomorphism). □

Question 3. Is it true in ZFC that every locally compact first countable linearly Lindelöf space is Lindelöf?

There is another open question of this kind closely related to Theorem 4.1. Recall that a base $\mathcal{B}$ of a space X is called *a base of countable order* if for each $x \in X$, and every strictly decreasing sequence $\eta = \{U_n : n \in \omega\}$ of elements of $\mathcal{B}$ containing x, η is a base of X at x. Bases of countable order were introduced in [3], where it was shown that every paracompact space with a base of countable order is metrizable. In [9] spaces with a base of countable order were studied in depth. In particular, H.H. Wicke and J. Worrell proved in [9] that if a space X can be covered by open subspaces with a base of countable order, then X also has a base of countable order. For example, it follows from this result, that the space of countable ordinals, with the usual order topology, has a base of countable order. Therefore, no property of paracompactness type is inherent in the presence of a base of countable order. We also see from Wicke and Worrell result that every locally metrizable space has a base of countable order. Hence, a positive answer to the next question would provide a reasonable generalization of Theorem 4.1:

Question 4. Is there a non-metrizable linearly Lindelöf space with a base of countable order?

Notice, that by the metrization theorem in [3], mentioned above, this question is equivalent to the following one: *is every linearly Lindelöf space with a base of countable order Lindelöf?*

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