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SURJECTIVITY OF ISOMETRIES

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Abstract

Isometries $X \to X$ certainly do not have to be surjective, generally. Yet, they do sometimes; if X is finite, for example, an isometry $X \to X$ must be surjective. Isometries of $\mathbf{R}$ and $\mathbf{C}$ (combinations of reflection, translation and conjugation) must be surjective. More generally, any norm-preserving additive map of a finite-dimensional normed space must be surjective. When isometries are surjective there can be potent algebraic consequences. For example, a surjective isometry between real Banach spaces must be an affine map. There is a strong connection between *linear* isometries H of spaces of functions and the property $fg = 0 \Rightarrow HfHg = 0$. We call a map H with this property *separating.* [Other aliases include *Lamperti operators*, *separation-preserving operators, disjoint operators, disjointness-preserving operators* and *d-homomorphisms.*] For Banach spaces $C(X)$ and $C(Y)$ of real-valued continuous functions on the compact spaces X and Y, a surjective linear isometry $H : C(X) \to C(Y)$ must be separating. And even though the metric structure is quite different, a linear isometry of $L_p[0,1]$ (or ℓ_p), $1 \leq p \leq \infty$, $p \neq 2$, onto itself must also be separating ([4], pp. 170-175). In this article we consider additive–rather than linear–separating isometries $H : C(X) \to C(Y)$ where X and Y are compact Hausdorff spaces and the functions in $C(X)$

and $C(Y)$ take values in $F = \mathbf{R}$ or $\mathbf{C}$. We characterize surjectivity of additive separating isometries $H : C(X) \to C(Y)$ in Theorems 7 and 8. They are surjective iff they satisfy the following functional separation condition: (Def. 1) For any two distinct points $y, y' \in Y$ there exist $f, g \in C(X)$ such that $\operatorname{cl}\operatorname{coz} f \cap \operatorname{cl}\operatorname{coz} g = \emptyset$ while $Hf(y)Hg(y') \neq 0$. We also specify what form such isometries must have.

1. Background

By *separating map* we mean a map H defined on some space Λ of functions, such that for all $f, g \in \Lambda$, $fg = 0 \Rightarrow HfHg = 0$. Any multiplicative map such as any ring homomorphism between rings of functions is obviously separating. We give some other examples in Example 2 below.

As mentioned in our introductory remarks many isometries between function spaces must be separating. Suppose that X and Y are compact Hausdorff spaces and that $C(X)$ and $C(Y)$ denote the sup-normed Banach spaces of $\mathbf{R}-$ or $\mathbf{C}-$valued continuous functions on X and Y, respectively. Then any additive separating map $H : C(X) \to C(Y)$ induces a continuous map $h : Y \to X$. The mechanism is as follows. For each $y \in Y$ and $f \in C(X)$, let $y\hat{} \circ H(f) = Hf(y)$. The set

$$\bigcap_{y\hat{}\circ Hf \neq 0} \operatorname{cl}\operatorname{coz} f = \operatorname{supp} y\hat{} \circ H$$

is a singleton (see Theorem 3(a) below) called the SUPPORT of $y\hat{} \circ H$. The map $h : Y \to X$, $y \mapsto \operatorname{supp} y\hat{} \circ H$, is called the SUPPORT MAP of H and is continuous; we list some basic properties of the support map in Theorem 3.

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There is a certain duality here: For compact X and Y, H 1-1 implies h onto. For $\mathbf{R}$-valued functions and H a *linear* continuous separating map, H is surjective if and only if h injective. The support map h of H is injective if and only if H satisfies the following functional separation condition (see Theorem 3(d)).

Definition 1. *The separating map* $H : C(X) \to C(Y)$ *is* DETACHING *if for any two distinct points* $y, y' \in Y$ *there exist* $f, g \in C(X)$ *such that* $\operatorname{cl}\operatorname{coz} f \cap \operatorname{cl}\operatorname{coz} g = \emptyset$ *while* $Hf(y)Hg(y') \neq 0$.

Example 2. EXAMPLES OF SEPARATING MAPS

(a) COMPOSITION Let K be any ring. Let X and Y be sets and let Λ and Γ be any spaces of K-valued functions closed under multiplication on X and Y, respectively, with multiplication defined pointwise in Λ and Γ. Let $h : Y \to X$ be any function. The composition map $f \mapsto f \circ h$ is separating.
(b) WEIGHTED COMPOSITION Let K be any topological ring. Let $C(X)$ and $C(Y)$ denote the spaces of K-valued continuous functions on the Tihonov spaces X and Y. Let $h : Y \to X$ be continuous and let $w \in C(Y)$. The *weighted composition map* $H : C(X) \to C(Y)$, $f \mapsto w \cdot (f \circ h)$ is separating; we call w the *weight function.* If $C(X)$ and $C(Y)$ are endowed with their compact-open topologies, weighted compositions are continuous.
(c) ISOMETRIES Let $(K, |\,|)$ be any valued field and let $X = \{x_0\}$ be any point. Then $C(\{x_0\})$ "=" K (identify $a \in K$ with the map $\mathbf{a}(x_0) = a$). Any field isometry (i.e., additive and multiplicative) $u : K \to K$ is separating.
(d) DIFFERENTIATION Let $D(\mathbf{R}) \subset C(\mathbf{R})$ be the subspace of continuously differentiable functions with compact-open topology. The differentiation operator is separating and discontinuous.
(e) INTEGRATION IS NOT SEPARATING The map $Hf(t) = \int_a^t f(x)\,dx$, on any space of integrable functions (closed under multiplication) is not separating: it maps triangles (i.e., hat functions) into functions that are eventually constant; thus, even

though f and g may be disjoint 'triangles,' there will points x such that $Hf\,(x)\,Hg\,(x) \neq 0$.

Let $C(X)$ and $C(Y)$ denote the sup-normed Banach spaces of real- or complex-valued continuous functions on the compact Hausdorff spaces X and Y. If $H : C\,(X) \to C\,(Y)$ is a surjective linear isometry then not only must H be separating, it must be the following weighted composition map:

$$Hf(y) = H\mathbf{1}(y) \cdot f(h(y)) \text{ for any } f \in C\,(X) \text{ and } y \in Y \quad (1)$$

where $h : Y \to X$ is a surjective homeomorphism, $\mathbf{1} \in C\,(X)$ maps everything in X into 1 and $|H\mathbf{1}\,(y)| \equiv 1$. The continuous function $H\mathbf{1}$ is the weight function in this case. Thus, the Banach space structure of $C\,(X)$ is enough to characterize a compact Hausdorff space X (i.e., $C\,(X)$ norm-isomorphic to $C\,(Y) \Rightarrow X \cong Y$; moreover, for example, $C\,[0,1]$ is not norm-isomorphic to $C\left([0,1]^2\right)$ because $[0,1] \ncong [0,1]^2$). Now drop compactness and assume only that X and Y are realcompact; instead of assuming that H is a linear isometry, assume only that H is a linear *biseparating* map (H a bijection with H and H^{-1} separating). Then H still must be a weighted composition (Eq. (1)) with a homeomorphism h from Y onto X ([2], Prop. 3). This latter result generalizes Hewitt's well-known result that the ring structure of $C\,(X)$ characterizes the realcompactification υX of X: if $C\,(X)$ is ring-isomorphic to $C\,(Y)$ then the realcompactification υX of X is homeomorphic to the realcompactification υY of Y.

Results like these are what motivated our interest in separating maps.

2. Notation

- $F = \mathbf{R}$ or $\mathbf{C}$.
- X and Y are compact Hausdorff spaces.

- $C(X)$ and $C(Y)$ denote the sup-normed Banach spaces of F-valued continuous functions on X and Y.
- $\mathbf{1}$ denotes the function that maps every $x \in X$ into $1 \in F$.
- $H : C(X) \to C(Y)$ is an additive separating map such that $H\mathbf{1}$ never vanishes, i.e., $H\mathbf{1}(y) \neq 0$ for any $y \in Y$.
- For $y \in Y$, the function $y\hat{} \circ H : C(X) \to F$, denotes the map $f \mapsto Hf(y)$, i.e., $y\hat{}$ is the evaluation of Hf at y.
- For any function f, $\operatorname{coz} f$ denotes the cozero set of f.
- $\operatorname{cl} U$ and $\mathbf{C}U$ denote, respectively, the topological closure and the complement of the set U.

3. How Separating Maps Work

We list some elementary properties of additive separating maps in Theorem 3. Proofs can be found in [1] (Theorem 2.4) as well as [9] and [10].

Theorem 3. *Let $H : C(X) \to C(Y)$ be an additive separating map.*
(a) *For each $y \in Y$, $\operatorname{supp} y\hat{}\circ H = \cap_{y\hat{}\circ Hf\neq 0} \operatorname{cl}\operatorname{coz} f$ is a singleton. Also, the* SUPPORT MAP OF H, *the map $h : Y \to X$, $y \mapsto \operatorname{supp}(y\hat{} \circ H)$ is continuous.* ***We reserve the letter h to denote the support map in the sequel.***
(b) *If $f = 0$ on an open subset U of X, then $Hf = 0$ on $h^{-1}(U)$; equivalently, if $f = g$ on U, then $Hf = Hg$ on $h^{-1}(U)$.*
(c) *For any $f \in C(X)$, $h(\operatorname{coz} Hf) \subset \operatorname{cl}\operatorname{coz} f$, where $\operatorname{cl}\operatorname{coz} f$ denotes the topological closure of $\operatorname{coz} f$. It follows from this that if H is injective then $h(Y)$ is dense in X.*
(d) *The support map h is 1-1 if and only if H is* DETACHING *in the sense of Def. 1 that for any two distinct points $y, y' \in Y$ there exist $f, g \in C(X)$ such that $\operatorname{cl}\operatorname{coz} f$ and $\operatorname{cl}\operatorname{coz} g$ are disjoint while $Hf(y)Hg(y') \neq 0$.*

(e) ([9], Theorem 2.2) *For any $y \in Y$, $y\hat{} \circ H$ is continuous if and only if $[y\hat{} \circ H](f) = Hf(y) = H[f(h(y)\mathbf{1}](y)$ for all $f \in C(X)$.*

Corollary 4. *If H is a continuous linear separating map then H is a weighted composition map.*

Proof. If H is continuous, then $y\hat{} \circ H$ is continuous for all $y \in Y$. By Theorem 3(e) it follows that $[y\hat{} \circ H](f) = Hf(y) = H[f(h(y)\mathbf{1}](\mathbf{y})$ for all $f \in C(X)$ and $y \in Y$. Since H is linear, $Hf(y) = H[f(h(y)\mathbf{1}](y) = f(h(y)H\mathbf{1}(y)$. □

As the referee has pointed out, to prove Cor. 4 we could use some representation theorems and argue as follows: Every continuous linear map $H : C(X) \to C(Y)$ has the form $Hf(y) = \int f\,\mu_y$ for some continuous map $y \mapsto \mu_y$ from Y to the space $M(X) = C(X)^*$ of measures on X, endowed with the weak* topology. If H is separating, the support of every measure μ_y has at most one point. If we assume $H1(y) \neq 0$, then $\mu_y \neq 0$. Therefore μ_y is an atomic measure; it has the form $\mu_y = w(y)h(y)$ for some $h(y) \in X$. Thus $Hf(y) = \int f\,\mu_y = w(y)f(h(y))$. In the complex case, every continuous additive separating map $H : C(X) \to C(Y)$ has the following form. Let $P_1(X)$ be the space of measures on X whose support has at most one point. The space $P_1(X)$ can be identified with the quotient $\mathbf{C} \times X/\{0\} \times X$. For $f \in C(X)$, let f_L be the natural (linear) extension of f over $P_1(X)$. Then $Hf(y) = f_L(h_1(y)) + \bar{f}_L(h_2(y))$, where $h_1, h_2 : Y \to P_1(X)$, and there is no $y \in Y$ such that the measures $h_1(y)$ and $h_2(y)$ have distinct non-empty supports.

4. Detaching Implies Surjective

There is very little variety in the additive isometries of $\mathbf{R}$ and $\mathbf{C}$. They must be as in Theorem 5.

Theorem 5. *Let $A : F \to F$ be an additive isometry.*

(a) *If $F = \mathbf{R}$, then $Ax = \pm x$ for all $x \in \mathbf{R}$.*

(b) *If $F = \mathbf{C}$, then $Az = e^{i\theta} z$ or $e^{i\theta}\overline{z}$ for all $x \in \mathbf{C}$ (θ arbitrary).*

Consequently, any additive isometry of $\boldsymbol{R}$ or $\boldsymbol{C}$ is surjective.

Compare these forms to those of the maps in Theorems 7 and 8.

Our ultimate goal is to show that an additive separating isometry H is surjective if and only if it is detaching, i.e., if and only if its support map h is injective. We do half of it in this section: We show in Theorem 7 that if H is detaching then H is surjective. First we need Theorem 6 which shows that constant functions $a\mathbf{1}$, $a \in F$, must map into functions of constant magnitude $|a|$.

Theorem 6. *Let $H : C(X) \to C(Y)$ be an additive separating isometry such that $H\mathbf{1}$ never vanishes. If H is detaching then, for any $a \in F$ and $y \in Y$,*
(a) $|H(a\mathbf{1})(y)| = |a|$ *so the map* $g_y : F \to F$, $a \mapsto H(a\mathbf{1})(y)$ *is a surjective isometry;*
(b) *If* $F = \mathbf{R}$, $H(a\mathbf{1})(y) = aH\mathbf{1}(y) = \pm a$;
(c) *If* $F = \mathbf{C}$, $H(a\mathbf{1})(y) = aH\mathbf{1}(y) = ae^{i\theta}$ *or* $\overline{a}e^{i\theta}$ *where* $H\mathbf{1}(y) = e^{i\theta}$ *where* θ *depends on* y.

Proof. (a) By Theorem 3(c), $h(Y)$ is dense in X. As X is compact, $h(Y) = X$. Since H is detaching, h is 1-1 (Theorem 3(d)), so h is a surjective homeomorphism.

Let $a \in F$. Since H is an isometry, $|H(a\mathbf{1})(y)| \leq |a|$ for every $y \in Y$. If, for some $y_0 \in Y$, $|H(a\mathbf{1})(y_0)| < |a|$, then, for some $\epsilon > 0$ and neighborhood U of y_0, $|H(a\mathbf{1})(y)| \leq |a| - \epsilon$ for all $y \in U$. Since h is a surjective homeomorphism, there exists a closed neighborhood V of $h(y_0)$ in X such that $V \subset h(U)$. By Urysohn's lemma, there exists $f \in C(X)$, $0 \leq f \leq 1$, such that $f \equiv 1$ on V, and $f \equiv 0$ on $\mathbf{C}[h(U)]$. Since H additive and continuous, it must be $\mathbf{R}$-linear; hence, for all $y \in Y$ (since f is real-valued),

$$H[af](y) = H[af(h(y))\mathbf{1}](y) = f(h(y))H(a\mathbf{1})(y)$$

(Theorem 3(e)). Since $|H(a\mathbf{1})(y)| \leq |a| - \epsilon$ for all $y \in U$ and $f(h(y)) = 0$ for all $y \in \mathbf{C}U$, $\|H(af)\| < |a|$. Since $\|af\| = |a|$ and H is an isometry, this is a contradiction.

(b,c) This follows from (a) and the form that additive isometries of **R** or **C** must have (Theorem 5). □

Theorem 7 shows that additive detaching separating isometries are surjective as well as what form such maps must have. Note that (Theorem 7(a)), H must be linear in the real case. In any event we get a result concerning the form of the isometry very much like that of the Stone-Banach theorem (namely $Hf(y) = (f \circ h)(y) H\mathbf{1}(y)$) on the form of surjective linear isometries between spaces of continuous functions on compact spaces.

Theorem 7. *Let $H : C(X) \to C(Y)$ be an additive separating isometry such that $H\mathbf{1}$ never vanishes. If H is detaching then H is surjective, and H must have one of the following forms:*
(a) *If $F = \mathbf{R}$, then $Hf(y) = f(h(y)) H\mathbf{1}(y)$ and there exists a clopen set $U \subset Y$ such that, for all $f \in C(X)$,*

$$Hf(y) = \begin{cases} f(h(y)), & y \in U \\ -f(h(y)), & y \in \mathbf{C}U \end{cases}$$

Thus, if Y is connected, $Hf(y) = f(h(y))$ or $-f(h(y))$ for all $y \in Y$ and $f \in C(X)$.
(b) *If $F = \mathbf{C}$, then there exists a clopen set $U \subset Y$ such that, for all $f \in C(X)$,*

$$Hf(y) = \begin{cases} f(h(y)) H\mathbf{1}(y), & y \in U \\ \\ \overline{f(h(y))} H\mathbf{1}(y), & y \notin U \end{cases}$$

Proof. (a) Since H is an additive continuous map, it is **R**-linear by the density of the rationals in **R**. Since the continuity of H implies the pointwise continuity of H it follows from by Theorem 3(e) that

$$Hf(y) = H[f(h(y))\mathbf{1}](y) = f(h(y)) H\mathbf{1}(y)$$

for all $f \in C(X)$ and $y \in Y$. Since $|H\mathbf{1}(y)| \equiv 1$ by Theorem 6, $H\mathbf{1}(y) = \pm 1$. By Theorem 3(c), $h(Y)$ is dense in X. As Y is compact, $h(Y) = X$. Since H is detaching, h is 1-1 (Theorem 3(d)), so h is a surjective homeomorphism. Given $g \in C(Y)$, let

$$f = \left(\frac{1}{H\mathbf{1} \circ h^{-1}}\right)\left(g \circ h^{-1}\right) \in C(X).$$

Then $Hf = g$ and H is seen to be surjective. Let $U = \{ y \in Y : Hf(y) = +f(h(y)) \text{ for all } f \in C(X)\}$. We show that U is clopen by showing that U and $\mathbf{C}U$ are open. For any $y_0 \in U$, we claim that there must be a neighborhood V of y_0 contained in U. If not, then every neighborhood of y_0 meets $\mathbf{C}U$. This means that there must be a net (y_s) of points of $\mathbf{C}U$ such that $y_s \to y_0$. Thus, by continuity, we have the contradictory result that

$$H\mathbf{1}(y_s) = -\mathbf{1}(h(y_s)) = -1 \to H\mathbf{1}(y_0) = \mathbf{1}(h(y_0)) = 1.$$

Hence U is open. An identical argument shows that $\mathbf{C}U$ is open and therefore that U is clopen.

(b) It follows from Theorem 3(e) that

$$Hf(y) = H[f(h(y))\mathbf{1}](y)$$

for each $f \in C(X)$ and $y \in Y$. Since H is an isometry, by Theorem 6(c), for any $f \in C(X)$ and $y \in Y$,

$$H[f(h(y))\mathbf{1}](y) = \begin{cases} f(h(y))\, H\mathbf{1}(y) \\ \text{or} \\ \overline{f(h(y))} H\mathbf{1}(y) \end{cases} \tag{2}$$

where $|H\mathbf{1}(y)| \equiv 1$ by Theorem 6(a). It follows that

$$U = \{y \in Y : Hf(y) = f(h(y))\, H\mathbf{1}(y) \text{ for all } f \in C(X)\}$$

is clopen by the argument used to show that the U of (a) was clopen; in this case we use $i\mathbf{1}$ instead of $\mathbf{1}$.

We now show that H is surjective. Since $H\mathbf{1}(y) \neq 0$ for all $y \in Y$, it follows that $\overline{1/(H\mathbf{1})} \in C(Y)$. Consider the map $\overline{H} : C(X) \to C(Y)$ where $\overline{H}f(y) = \overline{H\mathbf{1}(y)}Hf(y)$ for all $y \in Y$. Hence $\overline{H}\mathbf{1}(y) = |H\mathbf{1}(y)|^2 = 1$ for all $y \in Y$, i.e., $\overline{H}\mathbf{1} = \mathbf{1}$. It is easy to verify that $\overline{H}$ is a separating isometry whose support map is also h. Let $g \in C(Y)$, and consider

$$f(x) = \begin{cases} g\left(h^{-1}(x)\right), & x \in h^{-1}(U) \\ \\ \overline{g\left(h^{-1}(x)\right)}, & x \in h^{-1}(\mathbf{C}U). \end{cases}$$

By Eq. (2), with $\overline{H}$ instead of H, since $\overline{H}\mathbf{1} = \mathbf{1}$, it follows that $\overline{H}f = g$ and $\overline{H}$ is seen to be surjective. Thus, for each $g \in C(Y)$, there exists $w \in C(X)$ such that $\overline{H}w = \left(\overline{H\mathbf{1}}\right)w = g$. Thus if $\overline{H}w' = \overline{H\mathbf{1}}Hw' = \overline{H\mathbf{1}}g$, then $Hw' = g$, and H is seen to be surjective. ∎

5. Surjective Implies Detaching

We have shown in Th. 7 that if the additive separating isometry H is detaching, then H is surjective. We prove the converse in Th. 8, thereby showing that surjectivity is equivalent to detaching for additive separating isometries. This equivalence fails if the functions take values in valued fields (such as the p-adic numbers Q_p, for example) other than $\mathbf{R}$ or $\mathbf{C}$ ([3]).

Theorem 8. *Let $H : C(X) \to C(Y)$ be an additive separating isometry such that $H\mathbf{1}$ never vanishes. Then if H is surjective, H is detaching.*

Proof. First, let $F = \mathbf{R}$. As an additive, continuous map, H is linear. Hence, by Theorem 3(e), for all $f \in C(X)$ and $y \in Y$,

$$Hf(y) = H[f(h(y)\mathbf{1}](y) = f(h(y))H\mathbf{1}(y) \tag{3}$$

Thus $Hf(y) = 0$ if and only if $f(h(y)) = 0$. If h is not injective, there exist distinct points $y_1, y_2 \in Y$ such that $h(y_1) = h(y_2)$.

For any $f \in C(X)$, by Eq. (3),

$$\begin{aligned} Hf(y_1) &= f(h(y_1))H\mathbf{1}(y_1) \\ &= f(h(y_2))H\mathbf{1}(y_1) \end{aligned}$$

so $f(h(y_2)) = Hf(y_1)/H\mathbf{1}(y_1)$. Thus, by Eq. (3),

$$Hf(y_2) = \left(\frac{H\mathbf{1}(y_2)}{H\mathbf{1}(y_1)}\right) Hf(y_1).$$

Since there must exist $g \in C(Y)$ such that $g(y_1) = 0$ and $g(y_2) \neq 0$, it follows H is not surjective.

Now assume that $F = \mathbf{C}$. By Theorems 3(e) and the $\mathbf{R}$-linearity of additive continuous maps, for all $f \in C(X)$ and $y \in Y$,

$$\begin{aligned} Hf(y) &= H[f(h(y)\mathbf{1}](y) \\ &= \operatorname{Re} f(h(y))H\mathbf{1}(y) + \operatorname{Im} f(h(y))H(i\mathbf{1})(y). \end{aligned} \tag{4}$$

Suppose that for distinct points $y_1, y_2 \in Y$, $h(y_1) = h(y_2) = x \in X$. Choose open neighborhoods V_1 and U_1 of y_1 and V_2 and U_2 of y_2, respectively, such that

$$V_1 \subset \operatorname{cl} V_1 \subset U_1 \text{ and } V_2 \subset \operatorname{cl} V_2 \subset U_2$$

with $U_1 \cap U_2 = \emptyset$. Choose $k_1, k_2 \in C(Y)$, such that $0 \leq k_1, k_2 \leq 1$, $k_i(\mathbf{C}U_i) \equiv 0$, and $k_i(V_i) \equiv 1$ for $i = 1, 2$, respectively. Then $k_1 k_2 = 0$. Let W be a neighborhood of x such that $|f(w) - f(x)| < \varepsilon$ for all $w \in W$. Choose a neighborhood O of x such that $O \subset \operatorname{cl} O \subset W$. Choose $k_{O,W} \in C(X)$ such that $0 \leq k_{O,W} \leq 1$, $k_{O,W}(O) \equiv 1$ and $k_{O,W}(\mathbf{C}W) \equiv 0$. As H is surjective, there exist $f, g \in C(X)$ such that $Hf = k_1$ and $Hg = k_2$. As H is an isometry and $k_{O,W}$ is real-valued,

$$H(k_{O,W} f)(y) = k_{O,W}(h(y))\, Hf(y) = k_{O,W}(h(y))\, k_1(y) \tag{5}$$

and

$$H(k_{O,W} g)(y) = k_{O,W}(h(y))\, Hg(y) = k_{O,W}(h(y))\, k_2(y) \tag{6}$$

for all $y \in Y$. Hence, since $k_1(y_1) = 1$ and $h(y_1) = x$,

$$H(k_{O,W} f)(y_1) = k_{O,W}(h(y_1))\, k_1(y_1) = k_{O,W}(x) = 1.$$

Since H is an isometry, $\|f\| = \|Hf\| = \|k_1\| = 1$. Therefore $|f(x)| \leq 1$. Clearly $\|(k_{O,W} \circ h)\, k_1\| \leq 1$. Since $(k_{O,W} \circ h)(y_1)\, k_1(y_1) = k_{O,W}(x) = 1$, it follows that

$$\|(k_{O,W} \circ h)\, k_1\| = \|H(k_{O,W} f)\| = \|k_{O,W} f\| = 1. \tag{7}$$

For all $w \in W$,

$$\begin{aligned} |k_{O,W}(w)\, f(w)| &= |k_{O,W}(w)|\, |f(w)| \\ &\leq |k_{O,W}(w)|\, (|f(x)| + \varepsilon) \\ &\leq |f(x)| + \varepsilon. \end{aligned} \tag{8}$$

Since $k_{O,W}(\mathbf{C}W) \equiv 0$, $\|k_{O,W} f\| = 1 \leq |f(x)| + \varepsilon$. Hence $|f(x)| \geq 1$ and we conclude that $|f(x)| = 1$. Similarly, by Eq. (6), $|g(x)| = 1$. Now choose a neighborhood G of x such that $|(f+g)(w) - (f+g)(x)| < \varepsilon$ for all $w \in G$. Choose a neighborhood O of x such that $O \subset \operatorname{cl} O \subset G$. Choose $k_{O,G} \in C(X)$, $0 \leq k_{O,G} \leq 1$, $k_{O,G}(O) \equiv 1$ and $k_{O,G}(\mathbf{C}G) \equiv 0$. By Eqs. (5) and (6), for all $y \in Y$,

$$H[k_{O,G}(f+g)](y) = k_{O,G}(h(y))\, (k_1(y) + k_2(y)).$$

Since $k_1 k_2 = 0$ and H is an isometry, it follows as in Eq. (7) that

$$\begin{aligned} \|k_{O,G}(f+g)\| &= \|(k_{O,G} \circ h)(k_1 + k_2)\| \\ &= \max(\|(k_{O,G} \circ h)\, k_1\|, \|(k_{O,G} \circ h)\, k_2\|) = 1. \end{aligned}$$

Therefore $|f(x) + g(x)| \leq 1$. As in Eq. (8) above, however, it follows that $|f(x) + g(x)| \geq 1$; therefore $|f(x) + g(x)| = 1$. Similarly, $|f(x) - g(x)| = 1$. As there cannot be two complex numbers $f(x)$ and $g(x)$ of magnitude 1 whose sum and difference have magnitude 1, this is a contradiction. Consequently h is injective. Therefore H is detaching by Th. 3(d). □

We now know that H is surjective if and only if H is detaching in which case H takes the forms of Theorem 7; this generalizes the results about the character of additive isometries of $\mathbf{R}$ and $\mathbf{C}$ of Theorem 5.

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