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CHARACTERIZATIONS OF HYPERSPACES OF BITOPOLOGICAL SPACES

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Abstract

We find an internal characterization of those bitopological spaces which are homeomorphic to a certain kind of bitopological hyperspace. We remark that such a characterization represents an alternative to certain characterization results for traditional hyperspaces. Some categorical characterizations of various bitopological hyperspace functors are also developed.

This paper continues the investigation in [B2] and [B3] into the properties of a certain asymmetric construction of a hyperspace for bitopological spaces. The original motivation for this construction was to have a context in which to separately consider the upper and lower Vietoris topologies on a hyperspace and explore the interactions between them. In [B3] we defined the property "quasisober" for bitopological spaces, and explored many of its consequences. This property plays an important role in Section 1 here.

A *bitopological space* is a triple $(X, \mathcal{T}, \mathcal{T}^*)$ where $\mathcal{T}$ and $\mathcal{T}^*$ are topologies on X. Our terminology is largely influenced by [Ko]. The $\mathcal{T}$-closure operator in X is designated by c and the $\mathcal{T}^*$-closure operator is c^*. A map $f : X \to Y$ is a *continuous (open)* map $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ of bitopological

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spaces if it is continuous (open) with respect to first topologies $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ and with respect to second topologies $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U}^*)$. When we use a result about spaces with only one topology we will refer to this as a result from *traditional* topology.

Given a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ let 2^X be the set of non-empty subsets of X which are closed relative to $\mathcal{T}$. If $\{A_1, A_2, \dots, A_n\}$ is a family of subsets of X, let $\langle A_1, A_2, \dots, A_n\rangle = \{B \in 2^X \mid B \subseteq \cup_{i=1}^n A_i$ and for each $i = 1, \dots, n,\ A_i \cap B \neq \phi\}$. Let $L(\mathcal{T})$ be the topology on 2^X generated by the subbasis consisting of sets of the form $\langle O, X\rangle$ where $O \in \mathcal{T}$ and let $U(\mathcal{T}^*)$ be the topology on 2^X generated by the basis consisting of sets of the form $\langle O\rangle$ where $O \in \mathcal{T}^*$. Then $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$ is the *hyperspace* of $(X, \mathcal{T}, \mathcal{T}^*)$.

For a traditional space $(X, \mathcal{T})$ the topologies $L(\mathcal{T})$ and $U(\mathcal{T})$ are respectively the *lower Vietoris* and *upper Vietoris* topologies on 2^X, and their supremum is the *Vietoris* topology [Vi].

A space $(X, \mathcal{T}, \mathcal{T}^*)$ is an R_0 space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then $c^*x \subseteq O$. We will designate by R_0^* the property that the dual space $(X, \mathcal{T}^*, \mathcal{T})$ is R_0. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a T_0 space if for any two distinct points in X there is an element of $\mathcal{T} \cup \mathcal{T}^*$ which contains one point but not the other.

A space $(X, \mathcal{T}, \mathcal{T}^*)$ has two *specialization* orders: $x \leq y$ iff $x \in cy$ and $x \leq^* y$ iff $x \in c^*y$. For any $A \subseteq X$ define $\operatorname{sat} A = \cap\{O \in \mathcal{T} \mid A \subseteq O\}$. sat^* similarly refers to saturation with respect to $\mathcal{T}^*$. We see that X is R_0 iff for each $x, y \in X$, if $x \leq^* y$ then $y \leq x$. We also have that X is R_0 iff for any $A \subseteq X$, $\operatorname{sat}^* A \subseteq cA$.

For a partial order $(X, \leq)$ a point $p \in X$ is *prime*[1] if whenever z is a supremum of $\{x, y\}$, and $p \leq z$, then either $p \leq x$ or $p \leq y$.

[1] This is called *coprime* in [LM]. We hope that this inconsistency is justified by the fact that it is probably the *reverse* inclusion on the hyperspace that of greater interest for applications in computer science. Therefore, "prime" here can be thought of as "prime with respect to $\supseteq$."

For $A \subseteq X$ let $\downarrow A$ be the set $\{x \in X | \; x \leq a$ for some $a \in A\}$.

A non-empty set $A \subseteq X$ is $\mathcal{T}$-*irreducible* if it is $\mathcal{T}$-closed and for any $\mathcal{T}$-closed sets B and C, if $A = B \cup C$ then either $A = B$ or $A = C$. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is *quasisober* if for any $\mathcal{T}$-irreducible $A \subseteq X$, there is a point $x \in A$ such that for any $O \in \mathcal{T}^*$ if $x \in O$ then $A \subseteq O$.

1. Internal Characterization of Hyperspaces

We wish to characterize those spaces which are homeomorphic to a hyperspace of some bitopological space. Our approach will be that, given a space $(X, \mathcal{T}, \mathcal{T}^*)$ we will try to define a homeomorpism from X to the hyperspace of one of its subspaces. This approach fails for some examples of hyperspaces of non-quasisober spaces, so the proposition below will give a characterization of those spaces which are hyperspaces of quasisober spaces.

The fact that some kind of sobriety property is involved here is not surprising in view of the fact that the property "sober" was developed in order to describe spaces where one can reconstruct the points from the partial order on the open sets [AGV]. Part of what we do in the following proposition is to reconstruct the points of the base space by looking at the bitopological structure on the closed sets.

Characterization theorems for traditional (uniform) hyperspaces can be found in [Ha] and [B1]. In both cases the characterization came via an assertion that the space admitted a partial order with certain properties. In the bitopological version of this technique the partial order to be used is implicit in the topological structure—it is the specialization order of the first topology.

There is some overlap here with work that has already been done in the study of continuous partial orders. This becomes apparent if one compares Proposition 1.1 here (and its proof) with the overviews of Spectral Theory in [La] and [LM]. Here we emphasize topological properties over order theoretic properties as much as possible; what may be the most significant difference, however, is that here we do not assume any local compactness property.

Proposition 1.1. *Let* $(X, \mathcal{T}, \mathcal{T}^*)$ *be a space and let* P *be the set of primes of* X *in the specialization order* $\leq$. *For* $(X, \mathcal{T}, \mathcal{T}^*)$ *to be homeomorphic to the hyperspace of some quasisober space it is necessary and sufficient that* $(X, \mathcal{T}, \mathcal{T}^*)$ *be a* T_0, R_0^* *space satisfying the following conditions:*

(1) *Every non-empty subset of* X *has a supremum with respect to* $\leq$.

(2) *For each* $x, y \in X$ *if* $\downarrow \{x\} \cap P \subseteq \downarrow \{y\}$ *then* $x \leq y$.

(3) *For each* $x \in X$ *and* $O \in \mathcal{T}$ *such that* $x \in O$, *there are* $x_1, x_2, \ldots, x_n \in X$ *such that* $x \in X - \downarrow \{x_1, x_2, \ldots, x_n\} \subseteq O$, *i.e.,* $\mathcal{T}$ *has for a subbasis the complements of closures of points.*

(4) *For any* $\mathcal{T}|_P$*-closed set* $A \subseteq P$ *and* $O \in \mathcal{T}^*$ *if* $A \subseteq O$ *and* $x \in X$ *is a supremum for* A *then there exists an* $O' \in \mathcal{T}^*$ *such that* $x \in O'$ *and* $O' \cap P \subseteq O$.

(5) *For each* $x \in X$ *and* $O \in \mathcal{T}^*$ *if* $x \in O$ *then there is an* $O' \in \mathcal{T}^*$ *such that* $x \in O'$ *and for any* $\mathcal{T}|_P$*-closed set* $A \subseteq O' \cap P$ *there is a supremum of* A *in* O.

Proof. (Sufficiency) Since X is T_0 and R_0^* we have the following property of $\leq$: if $x \leq y$ and $y \leq x$ then $x = y$. Therefore the suprema guaranteed by Property 1 are always unique.

Consider the subspace $(P, \mathcal{T}|_P, \mathcal{T}^*|_P)$ of $(X, \mathcal{T}, \mathcal{T}^*)$. We show that P is quasisober. Suppose $A \subseteq P$ is $\mathcal{T}|_P$-irreducible. Let $x = \sup A$. Then $x \in P$ since if $x \leq \sup\{a, b\}$ then $A \subseteq \downarrow \{a\} \cup \downarrow \{b\}$ which implies that either $A \subseteq \downarrow \{a\}$ or $A \subseteq \downarrow \{b\}$, hence $x \leq a$ or $x \leq b$. We show that $x \in A$. Suppose not. Then by Property 3 there are $x_1, x_2, \ldots, x_n \in X$ such that

$x \in X - \downarrow \{x_1, x_2, \ldots, x_n\} \subseteq X - A$. But then $A \subseteq \downarrow \{x_i\}$ for some i and so $x \leq x_i$, a contradiction. Finally we note that for any $O \in \mathcal{T}^*$, if $x \in O$ then by the R_0^* property $A \subseteq O$.

Define $f : X \to 2^P$ by $f(x) = \downarrow \{x\} \cap P$. This is well defined since, on the one hand, $f(x)$ is closed in P since $\downarrow \{x\}$ is the closure of $\{x\}$, and on the other hand, if $f(x) = \phi$ then by Property 2 we would have $x \leq y$ for all $y \in X$, which would make x a prime, and that would imply that $x \in f(x)$.

Define $g : 2^P \to X$ by $g(A) = \sup A$. We now show that f and g are inverses. Let $y = g(f(x))$. Then $y \leq x$ by definition and $x \leq y$ by Property 2. So $x = y$.

Now let $B = f(g(A))$. Clearly $A \subseteq B$. Suppose $x \in B$. Given $O \in \mathcal{T}$ with $x \in O$ we have by Property 3 some $x_1, x_2, \ldots, x_n \in X$ such that $x \in X - \downarrow \{x_1, x_2, \ldots, x_n\} \subseteq O$. Let $y = \sup\{x_1, x_2, \ldots, x_n\}$. Since x is prime we have $x \not\leq y$ so $g(A) \not\leq y$. Therefore y is not an upper bound for A so choose some $a \in A$ with $a \not\leq y$. Then $a \in O$. This shows that $x \in cA \cap P = A$. This, in turn, shows that $B \subseteq A$, and so we have $A = B$.

We now show that $f : (X, \mathcal{T}, \mathcal{T}^*) \to (2^P, L(\mathcal{T}|_P), U(\mathcal{T}^*|_P))$ and $g : (2^P, L(\mathcal{T}|_P), U(\mathcal{T}^*|_P)) \to (X, \mathcal{T}, \mathcal{T}^*)$ are continuous.

There is by Property 3 a subbase for $L(\mathcal{T}|_P)$ consisting of the sets $< P, O \cap P >$ where $O = X - \downarrow \{x_1, x_2, \ldots, x_n\}$ for some $x_1, x_2, \ldots, x_n$. We have $O \cap P = P - \downarrow \{x_1, x_2, \ldots, x_n\} = P - \downarrow \{\sup\{x_1, x_2, \ldots, x_n\}\}$. Then by Property 2,

$$f^{-1}[< P, O \cap P >] = X - \downarrow \{\sup\{x_1, x_2, \ldots, x_n\}\} \in \mathcal{T}.$$

Similarly, if $O = X - \downarrow \{x\}$ is a subasis element for $\mathcal{T}$ then $g^{-1}[O] = < P, O \cap P > \in L(\mathcal{T}|_P)$.

Given an $x \in X$ and a basis element $< O \cap P >$ for $U(\mathcal{T}^*|_P)$ such that $f(x) \in < O \cap P >$ there is by Property 4 an $O' \in \mathcal{T}^*$ such that $x \in O'$ and $O' \cap P \subseteq O$. If $y \in O'$ then by the R_0^* property we have $f(y) \subseteq O' \cap P \subseteq O \cap P$ and so $f(y) \in < O \cap P >$.

Given a $\mathcal{T}|_P$-closed set $A \subseteq P$ and $O \in \mathcal{T}^*$ with $g(A) =$

$\sup A \in O$ there is by Property 5 an $O' \in \mathcal{T}^*$ such that $g(A) \in O'$ and for any $\mathcal{T}|_P$-closed $B \subseteq O' \cap P$ we have $g(B) \in O$. This means that $A \in\, < O' \cap P >$ and for any $B \in\, < O' \cap P >$ we have $g(B) \in O$.

(Necessity) Let $(Y, \mathcal{S}, \mathcal{S}^*)$ be a quasisober space and identify $(X, \mathcal{T}, \mathcal{T}^*)$ with $(2^Y, L(\mathcal{S}), U(\mathcal{S}^*))$. We have shown in [B2] that X is T_0 and R_0^*, and that the $\mathcal{T}$ specialization order $\leq$ on X is just set inclusion in 2^Y.

If $\mathcal{C} \subseteq 2^Y$ is non-empty then $\sup \mathcal{C} = c(\cup \mathcal{C})$, which gives us Property 1. The primes of X are the $\mathcal{S}$-irreducible subsets of Y. Since the closures of points are irreducible we have Property 2. $L(\mathcal{S})$ has for a subbasis the collection of sets of the form $< Y, O >$ for $O \in \mathcal{S}$. But $< Y, O > = 2^Y - < Y - O > = 2^Y - c\{Y - O\}$, where the closure is relative to $L(\mathcal{S})$. This gives us Property 3.

Let $\mathcal{P}$ be the set of $\mathcal{S}$-irreducible subsets of Y. Given $\mathcal{A}$ relatively closed in $\mathcal{P}$ we note that $\cup \mathcal{A}$ is $\mathcal{S}$-closed. Therefore $\sup \mathcal{A} = \cup \mathcal{A}$. Suppose $\mathcal{A} \subseteq \cup_{\alpha \in L} < O_\alpha >$ where each $O_\alpha \in \mathcal{S}^*$. Let $O = \cup_\alpha O_\alpha \in \mathcal{S}^*$. Then $\sup \mathcal{A} \in\, < O >$. Furthermore if $B \in\, < O > \cap\, \mathcal{P}$ then B is an $\mathcal{S}$-irreducible set which is covered by the collection $\{O_\alpha |\ \alpha \in L\} \subseteq \mathcal{S}^*$. Since $(Y, \mathcal{S}, \mathcal{S}^*)$ is quasisober, we must have $B \subseteq O_\alpha$ for some α, and so $B \in \cup_{\alpha \in L} < O_\alpha >$. This shows that Property 4 is satisfied.

Given $B \in \cup_{\alpha \in L} < O_\alpha >$ with each $O_\alpha \in \mathcal{S}^*$ we must have $B \in\, < O_\alpha >$ for some $\alpha \in L$. If $\mathcal{A}$ is relatively closed in $\mathcal{P}$ then $\sup \mathcal{A} = \cup \mathcal{A}$ as in the last paragraph. So if $\mathcal{A} \subseteq\, < O_\alpha >$ then $\sup \mathcal{A} \in\, < O_\alpha > \subseteq \cup_{\alpha \in L} < O_\alpha >$. Thus we have Property 5. □

Example 1.1. Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\} = \mathcal{T}^*$. Consider the hyperspace $(2^X, \mathcal{L}, \mathcal{U})$ of $(X, \mathcal{T}, \mathcal{T}^*)$. We leave it to the reader to verify that no subspace of 2^X has a hyperspace homeomorphic to 2^X. This shows that the method of proof in the last proposition cannot be applied to characterize hyperspaces in general—it is possible for a given space to be homeomorphic to a hyperspace (of a non-quasisober bispace) but not to the hyperspace of any of its own subspaces. If a more general characterization theorem is possible it will require a different approach.

The set of conditions in Proposition 1.1 may have many equivalent formulations. For example, property (2) is equivalent to the statement that P *generates* X, i.e., every $x \in X$ is a supremum of a subset of P. This property is mentioned briefly on page 141 in [La] in such a way as to further illustrate the connection between this work here and existing characterization theorems in the theory of continuous lattices.

2. A Categorical Look at Hyperspaces and Multifunctions

In contrast to the last section, here we explore an approach to characterization of the hyperspace which establishes the hyperspace construction as a functor with certain properties. In the last section the internal characterization had to be limited to a class of hyperspaces, the hyperspaces of quasisober spaces. Here we find that the approach we take meets with greater or lesser difficulties depending on how we define the points of the hyperspace; and the choice of the $\mathcal{T}$-closed sets may be the worst of the reasonable choices.

In contrast to [Ha], where uniform hyperspaces were characterized as the free **H**-algebras[2], i.e., the uniform hyperspace functor is a left adjoint of a forgetful functor, here we will explore the idea of characterizing the hyperspace as a right adjoint to a certain inclusion functor, one that embeds a category of objects and functions into a category of objects and relations. This reflects one of the traditional applications of hyperspaces: the study of multifunctions.

Definition 2.1. We define $\mathbf{T}$ to be the category of bitopological spaces and continuous maps and $\mathbf{T^m}$ to be the category

[2] See [Pa] section 2.3 as a convenient source for the terminology used in the paper [Ha].

whose objects are the same as $\mathbf{T}$ but whose morphisms are relations (i.e., multifunctions) satisfying the following: $f \subseteq X \times Y$ is a morphism in $\mathbf{T^m}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$ if

(1) the domain of f is X,

(2) for every $O \in \mathcal{U}$ we have $f^{-1}[O] \in \mathcal{T}$,

(3) and for every $\mathcal{U}^*$-closed set C we have $f^{-1}[C]$ is $\mathcal{T}^*$-closed.

Composition of morphisms is the usual composition of relations. Let $\iota : \mathbf{T} \to \mathbf{T^m}$ be the inclusion functor which maps each object to itself, and each function to itself considered as a relation.

Definition 2.2. For a set X let $\mathcal{P}_0(X)$ be the set of non-empty subsets of X. If $\mathcal{T}$ and $\mathcal{T}^*$ are topologies on X, let us abuse notation by letting $L(\mathcal{T})$ and $U(\mathcal{T}^*)$ be topologies on $\mathcal{P}_0(X)$ with otherwise the same definitions we used above on 2^X.

Proposition 2.1 *There is a right adjoint* $\eta : \mathbf{T^m} \to \mathbf{T}$ *to the functor* $\iota : \mathbf{T} \to \mathbf{T^m}$*, where for objects* $\eta(X, \mathcal{T}, \mathcal{T}^*)$ *is the power set hyperspace* $(\mathcal{P}_0(X), L(\mathcal{T}), U(\mathcal{T}^*))$*, and for morphisms* $\eta(f)$ *is defined by* $\eta(f)(A) = f[A]$.

The result in Proposition 2.1 is the ideal we wish to aim for. The hyperspace construction is realized as a right adjoint of an inclusion functor, and therefore as a coreflection. The hyperspace that arises, however, is not T_0. For the rest of this paper we shall look (in order of increasing complexity) at three different ways to choose the points of the hyperspace in order to make it T_0, and we shall try to make the analog of Proposition 2.1 work for each one.

Definition 2.3. Take $\mathbf{S}$ to be the category of R_0 bitopological spaces. We define $\mathbf{S^m_s}$ to be the category whose objects are R_0 bitopological spaces and whose morphisms are relations satisfying the following: $f \subseteq X \times Y$ is a morphism in $\mathbf{S^m_s}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$ if

(1) the domain of f is X,

(2) for every $O \in \mathcal{U}$ we have $f^{-1}[O] \in \mathcal{T}$,

(3) for every $\mathcal{U}^*$-closed set C we have $f^{-1}[C]$ is $\mathcal{T}^*$-closed,

(4) and for every $x \in X$ we have $f[x] = \operatorname{sat}^* f[x]$.

Composition of morphisms is the usual composition of relations. For each object X the identity relation i_X is $\cup_{x\in X}(\{x\} \times \operatorname{sat}^* x)$. Let $\iota_s : \mathbf{S} \to \mathbf{S_s^m}$ be the inclusion functor which maps each object to itself, and each function $f : X \to Y$ to the relation $\cup_{x\in X}(\{x\} \times \operatorname{sat}^* f(x))$.

Lemma 2.1. *Each i_X and $\iota_s(f)$ is an $\mathbf{S_s^m}$-morphism.*

Proof. If $\operatorname{sat}^* x \cap O \neq \phi$ then there is a $y \in O$ with $x \in c^*y$. In an R_0 bispace this means that for $O \in \mathcal{T}$ we have $\operatorname{sat}^* x \cap O \neq \phi$ if and only if $x \in O$. So it is always true that $i_X^{-1}[O] = O$ and $\iota_s(f)^{-1}[O] = f^{-1}[O]$. This shows that property (2) in Definition 2.3 is satisfied, and the other properties are relatively easier. ☐

Lemma 2.2. *Each i_X is a two sided identity: $i_Y \circ f = f$ and $g \circ i_X = g$.*

Proof. $i_Y \circ f = f$ follows from the fact that sat^* respects the subset relation.

Suppose g is a morphism in $\mathbf{S_s^m}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$. Suppose $y \notin g[x]$. Then $g[x] \cap c^*y = \phi$. $g^{-1}[c^*y]$ is $\mathcal{T}^*$-closed and doesn't contain x so $\operatorname{sat}^* x \cap g^{-1}[y] = \phi$. Hence $y \notin g[\operatorname{sat}^* x] = g[i_X[x]]$. This shows that $g \circ i_X = g$. ☐

Lemma 2.3. *The morphisms in $\mathbf{S_s^m}$ are closed under composition.*

Proof. Properties (1) to (3) above are obviously satisfied by compositions. Property (4) is satisfied by $f \circ g$ as long as it is satisfied by f, since the saturation of a set is the union of the saturations of its points. (This means that property (4) in Definition 2.3 is equivalent to the statement that $f[A]$ is $\mathcal{T}^*$-saturated for any set A.) ☐

Lemma 2.4. *ι_s is a functor.*

Proof. ι_s clearly preserves identity morphisms. Given functions $g : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ and $f : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$ we show $\iota_s(f \circ g) = \iota_s(f) \circ \iota_s(g)$. Suppose that $z \notin \iota_s(f \circ g)[x]$. Then $f(g(x)) \notin c^* z$. So $g(x) \notin f^{-1}[c^* z]$, and, since $f^{-1}[c^* z]$ is $\mathcal{U}^*$-closed, $\iota_s(g)[x] \cap f^{-1}[c^* z] = \phi$. Then $f[\iota_s(g)[x]] \cap c^* z = \phi$ and so $z \notin \iota_s(f)[\iota_s(g)[x]]$. □

Definition 2.4. For a bispace $(X, \mathcal{T}, \mathcal{T}^*)$ let II^X be the set of non-empty $\mathcal{T}^*$-saturated subsets of X. Let us abuse notation by letting $L(\mathcal{T})$ and $U(\mathcal{T}^*)$ be topologies on II^X with otherwise the same definitions we used above on 2^X. This hyperspace has been discussed in the last section of [B2].

Proposition 2.2. *There is a right adjoint $\eta_s : \mathbf{S_s^m} \to \mathbf{S}$ to the functor $\iota_s : \mathbf{S} \to \mathbf{S_s^m}$, where for objects $\eta_s(X, \mathcal{T}, \mathcal{T}^*)$ is the saturated set hyperspace $(\mathrm{II}^X, L(\mathcal{T}), U(\mathcal{T}^*))$, and for morphisms $\eta_s(f) = \mathrm{II}^f$ is defined by $\mathrm{II}^f(A) = f[A]$.*

Proof. We showed in [B2] that the II^X hyperspace is always R_0. We note that II^f is well-defined since we have observed above that $f[A]$ is always $\mathcal{T}^*$-saturated. The rest of the proof should be routine. □

The power set space $\mathcal{P}_0(X)$ has a T_0-identification $\overline{\mathcal{P}_0}(X)$, and this can be regarded as a subspace of $\mathcal{P}_0$ in the following way.

Definition 2.5. For a bispace $(X, \mathcal{T}, \mathcal{T}^*)$ let $\overline{\mathcal{P}_0}(X)$ be the set of non-empty subsets A of X with the property that $A = cA \cap \mathrm{sat}^* A$. Let the operators L and U again be adjusted to give topologies on this set $\overline{\mathcal{P}_0}(X)$. (This is a T_0-identification for $\mathcal{P}_0(X)$ in the sense that A, $B \in \mathcal{P}_0(X)$ have the same neighborhoods in both $L(\mathcal{T})$ and $U(\mathcal{T}^*)$ if and only if $cA \cap \mathrm{sat}^* A = cB \cap \mathrm{sat}^* B$, and that $cA \cap \mathrm{sat}^* A$ has the same neighborhoods as A.)

To make this new hyperspace arise as an adjoint we will again have to make several modifications to the category $\mathbf{T^m}$. Firstly, the relations clearly have to satisfy the equation $f[x] = c(f[x]) \cap \operatorname{sat}^* f[x]$ for all x in the domain of f. This entails that, in addition to the sort of changes we have seen above, the definition of composition will have to be modified.

Definition 2.6. Taking $\mathbf{T}$ as before, we define $\mathbf{T^m_t}$ to be the category whose objects are bitopological spaces and whose morphisms are relations satisfying the following: $f \subseteq X \times Y$ is a morphism in $\mathbf{T^m_t}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$ if

(1) the domain of f is X,

(2) for every $O \in \mathcal{U}$ we have $f^{-1}[O] \in \mathcal{T}$,

(3)for every $\mathcal{U}^*$-closed set C we have $f^{-1}[C]$ is $\mathcal{T}^*$-closed,

(4) and for every $x \in X$ we have $f[x] = c(f[x]) \cap \operatorname{sat}^* f[x]$.

Composition of morphisms is defined as follows: if $f : X \to Y$ and $g : Y \to Z$ then $(x, z) \in g \circ f$ iff $z \in c(g[f[x]]) \cap \operatorname{sat}^* g[f[x]]$. For each object X the identity relation i_X is $\cup_{x \in X}(\{x\} \times (cx \cap \operatorname{sat}^* x))$. Let $\iota_t : \mathbf{T} \to \mathbf{T^m_t}$ be the inclusion functor which maps each object to itself, and each function $f : X \to Y$ to the relation $\cup_{x \in X}(\{x\} \times (c(f(x)) \cap \operatorname{sat}^* f(x)))$.

We owe thanks to Frank Oles for the idea of modifying composition in the above definition. The work of Oles in knowledge representation employs a continuity property for relations that is exactly what we require in $\mathbf{T^m_t}$ (with $\mathcal{T} = \mathcal{T}^*$). What's more, Oles has provided a counterexample that shows that $\mathbf{T^m_t}$ is not closed under the usual composition.

Lemma 2.5. *Each i_X and $\iota_t(f)$ is a $\mathbf{T^m_s}$-morphism.*

Proof. We don't need any separation assumption this time—both of the properties (2) and (3) in Definition 2.6 are easily satisfied. ∎

Lemma 2.6. *Each i_X is a two sided identity: $i_X \circ f = f$ and $g \circ i_X = g$.*

Proof. $i_Y \circ f = f$ follows from the fact that $A \to cA \cap \operatorname{sat}^* A$ respects the subset relation.

Suppose g is a morphism in $\mathbf{T_t^m}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$. Suppose $y \notin g[x]$. Then either there is an $O \in \mathcal{U}$ with $y \in O$ and $O \cap g[x] = \phi$, or $g[x] \cap c^* y = \phi$. In the former case, $g^{-1}[O] \in \mathcal{T}$ doesn't contain x and so $y \notin g[cx]$. In the latter case, $g^{-1}[c^* y]$ is $\mathcal{T}^*$-closed and doesn't contain x so $y \notin g[\operatorname{sat}^* x]$. So in either case $y \notin g[cx \cap \operatorname{sat}^* x] = g[i_X[x]]$. This shows that $g[I_X[x]] = g[x]$. But then $c(g[I_X[x]]) \cap \operatorname{sat}^* g[I_X[x]] = c(g[x]) \cap \operatorname{sat}^* g[x] = g[x]$. So $g \circ i_X = g$. ☐

Lemma 2.7. *The morphisms in $\mathbf{T_t^m}$ are closed under composition.*

Proof. Given $\mathbf{T_t^m}$-morphisms $g : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ and $f : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$ we note that if A is either $\mathcal{V}$-open or $\mathcal{V}^*$-closed then $(f \circ g)^{-1}[A] = g^{-1}[f^{-1}[A]]$ and thus properties (2) and (3) of Definition 2.6 are satisfied by $f \circ g$. On the other hand, the definition of composition has been formulated so that property (4) is automatically satisfied. ☐

Lemma 2.8. *Composition in $\mathbf{T_t^m}$ is associative.*

Proof. Given $\mathbf{T_t^m}$-morphisms $h : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$, $g : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$ and $f : (Z, \mathcal{V}, \mathcal{V}^*) \to (W, \mathcal{W}, \mathcal{W}^*)$ we will show $(f \circ g) \circ h[x] = c(f[g[h[x]]]) \cap \operatorname{sat}^* f[g[h[x]]] = f \circ (g \circ h)[x]$. We observe that $c(f[g[h[x]]]) \cap \operatorname{sat}^* f[g[h[x]]]$ is clearly a subset of the other two.

Suppose $w \notin c(f[g[h[x]]]) \cap \operatorname{sat}^* f[g[h[x]]]$. If $w \notin c(f[g[h[x]]])$ then there is $O \in \mathcal{W}$ with $w \in O$ and $O \cap f[g[h[x]]] = \phi$. $O \cap c(f[g[y]]) = \phi$ for any $y \in h[x]$, and so $O \cap (f \circ g)[h[x]] = \phi$. Hence $w \notin (f \circ g) \circ h[x]$. What's more, $f^{-1}[O] \cap g[h[x]] = \phi$ and

$f^{-1}[O] \in \mathcal{V}$. So $f^{-1}[O] \cap (g \circ h)[x] = \phi$, hence $O \cap f[(g \circ h)[x]] = \phi$, and therefore $w \notin f \circ (g \circ h)[x]$.

On the other hand, suppose $w \notin \text{sat}^* f[g[h[x]]]$. Then $c^* w \cap f[g[h[x]]] = \phi$, hence $c^* w \cap \text{sat}^* f[g[y]] = \phi$ for any $y \in h[x]$, and so $c^* w \cap (f \circ g)[h[x]] = \phi$. Hence $w \notin (f \circ g) \circ h[x]$. What's more, $f^{-1}[c^* w] \cap g[h[x]] = \phi$ and $f^{-1}[c^* w]$ is $\mathcal{V}$-closed. So $f^{-1}[c^* w] \cap (g \circ h)[x] = \phi$, hence $c^* w \cap f[(g \circ h)[x]] = \phi$, and therefore $w \notin f \circ (g \circ h)[x]]$. □

Lemma 2.9. ι_t *is a functor.*

Proof. ι_t clearly preserves identity morphisms.

Given functions $g : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ and $f : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$ we show $\iota_t(f \circ g) = \iota_t(f) \circ \iota_t(g)$. Suppose that $z \notin \iota_t(f \circ g)[x] = c(f(g(x))) \cap \text{sat}^* f(g(x))$. Then either $z \notin c(f(g(x)))$ or $f(g(x)) \notin c^* z$. In the first case we have an $O \in \mathcal{V}$ with $z \in O$ and $g(x) \notin f^{-1}[O]$. So $f^{-1}[O] \cap c(g(x)) = \phi$, hence $O \cap f[c(g(x))] = \phi$, and so $z \notin \iota_t(f) \circ \iota_t(g)[x]$. On the other hand, if $f(g(x)) \notin c^* z$ then $g(x) \notin f^{-1}[c^* z]$, and, since $f^{-1}[c^* z]$ is $\mathcal{U}^*$-closed, $\iota_s(g)[x] \cap f^{-1}[c^* z] = \phi$. Then $f[\iota_s(g)[x]] \cap c^* z = \phi$ and so $z \notin \iota_t(f) \circ \iota_t(g)[x]$. □

Proposition 2.3. *There is a right adjoint* $\eta_t : \mathbf{T_t^m} \to \mathbf{T}$ *to the functor* $\iota_t : \mathbf{T} \to \mathbf{T_t^m}$*, where for objects* $\eta_t(X, \mathcal{T}, \mathcal{T}^*)$ *is the hyperspace* $(\overline{\mathcal{P}_0}(X), L(\mathcal{T}), U(\mathcal{T}^*)$*, and for morphisms* $\eta_t(f)$ *is defined by* $\eta_t(f)(A) = c(f[A]) \cap \text{sat}^* f[A]$*.*

Proof. Suppose $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is a $\mathbf{T_t^m}$-morphism. We show that $\eta_t(f)$ is continuous by noting that for $O \in \mathcal{U}$ and $O^* \in \mathcal{U}^*$, we have $\eta_t(f)^{-1}[\langle Y, O\rangle] = \langle X, f^{-1}[O]\rangle$ and $\eta_t(f)^{-1}[\langle O\rangle] = \langle f^{-1}[O]\rangle$.

For $A \in \mathcal{P}_0(X)$ consider the relation $h : (W, \mathcal{W}, \mathcal{W}^*) \to (X, \mathcal{T}, \mathcal{T}^*)$ where $W = \{w\}$ and $h = \{w\} \times A$. Then $\eta_t(f)(A) = f \circ h[w]$ for any relation f defined on X. Given $\mathbf{T_t^m}$-morphisms $g : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ and $f : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$

we see that $\eta_t(f \circ g)(A) = (f \circ g) \circ h[w] = f \circ (g \circ h)[w] = \eta_t(f) \circ \eta_t(g)(A)$ by Lemma 2.8.

The rest of the proof should be routine. ◻

Definition 2.7. For any relation $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ let the function $2^f : (2^X, L(\mathcal{T}), U(\mathcal{T}^*)) \to (2^Y, L(\mathcal{U}), U(\mathcal{U}^*))$ be defined by $2^f(A) = c(f[A])$.

We note that 2^f need not be continuous even when f is a continuous function.

Example 2.1. Let $X = \{a, b\}$ and let $Y = \{a, b, c\}$. Let $\mathcal{T}$ be the discrete topology on X and let $\mathcal{T}^*$ be the indiscrete topology on X. Let $\mathcal{U} = \{\phi, \{b, c\}, Y\}$ and let $\mathcal{U}^* = \{\phi, \{a, b\}, Y\}$. Let $f(a) = a$ and $f(b) = b$. Then $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ is continuous but $2^f : (2^X, L(\mathcal{T}), U(\mathcal{T}^*)) \to (2^Y, L(\mathcal{U}), U(\mathcal{U}^*))$ is not—$(2^f)^{-1}[\langle\{a, b\}\rangle] = \{\{a\}\}$ which is not $U(\mathcal{T}^*)$-open.

Definition 2.8. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *normal* space if for any $\mathcal{T}^*$-closed $A \subseteq X$ and any $O \in \mathcal{T}$ if $A \subseteq O$ then there is an $O' \in \mathcal{T}$ such that $A \subseteq O'$ and $c^*O' \subseteq O$.

Lemma 2.10. *If $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ satisfies the continuity properties (2) and (3) of Definition 2.1 (or 2.3 or 2.6), and $(Y, \mathcal{U}, \mathcal{U}^*)$ is normal, then $2^f : (2^X, L(\mathcal{T}), U(\mathcal{T}^*)) \to (2^Y, L(\mathcal{U}), U(\mathcal{U}^*))$ is continuous.*

Proof. Continuity with respect to the L topologies works out without any separation assumptions. The more problematic question is the continuity with respect to the U topologies.

Suppose $A \in 2^X$ and $2^f(A) \in \langle O\rangle$ where $O \in \mathcal{U}^*$. Take $O' \in \mathcal{U}^*$ with $c(f[A]) \subseteq O'$ and $cO' \subseteq O$. Then $A \in \langle X - f^{-1}[Y - O']\rangle \subseteq (2^f)^{-1}[\langle O\rangle]$. ◻

Definition 2.9. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *regular* space if for any $x \in X$ and any $O \in \mathcal{T}$ if $x \in O$ then there is an $O' \in \mathcal{T}$ with $x \in O'$ and $c^*O' \subseteq O$.

Definition 2.10. A space $(X, \mathcal{T}, \mathcal{T}^*)$ is a *compact* if every $\mathcal{T}^*$-closed set is $\mathcal{T}$-compact. (This definition was introduced in [B3].)

Definition 2.11. Let **CR** be the category of compact regular R_0^* bitopological spaces and continuous maps. Define $\mathbf{CR_c^m}$ be the category with the same objects as **CR** but whose morphisms are relations satisfying the following: $f \subseteq X \times Y$ is a morphism in $\mathbf{CR_c^m}$ from $(X, \mathcal{T}, \mathcal{T}^*)$ to $(Y, \mathcal{U}, \mathcal{U}^*)$ if

(1) the domain of f is X,

(2) for every $O \in \mathcal{U}$ we have $f^{-1}[O] \in \mathcal{T}$,

(3) for every $\mathcal{U}^*$-closed set C we have $f^{-1}[C]$ is $\mathcal{T}^*$-closed,

(4) and for every $x \in X$ we have $f[x] = c(f[x])$.

Composition of morphisms is defined as follows: if $f : X \to Y$ and $g : Y \to Z$ then $(x, z) \in g \circ f$ iff $z \in c(g[f[x]])$. For each object X the identity relation i_X is $\cup_{x \in X}(\{x\} \times cx)$. Let $\iota_c : \mathbf{CR} \to \mathbf{CR_c^m}$ be the inclusion functor which maps each object to itself, and each function $f : X \to Y$ to the relation $\cup_{x \in X}(\{x\} \times c(f(x)))$.

Lemma 2.11. *Each i_X and $\iota_c(f)$ is an $\mathbf{CR_c^m}$-morphism.*

Proof. This will be similar to Lemma 2.1 and will use the R_0^*-property. □

Lemma 2.12. *Each i_X is a two sided identity: $i_X \circ f = f$ and $g \circ i_X = g$.*

Proof. Similar to Lemma 2.6. □

Lemma 2.13. *The morphisms in $\mathbf{CR_c^m}$ are closed under composition.*

Proof. As with Lemma 2.10, the problem case is showing that property (3) in Definition 2.11 holds for $f \circ g$. Given $\mathbf{CR_c^m}$-morphisms $g : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ and $f : (Y, \mathcal{U}, \mathcal{U}^*) \to (Z, \mathcal{V}, \mathcal{V}^*)$ suppose $x \notin (f \circ g)^{-1}[A]$ where A is $\mathcal{V}^*$-closed. Then

$(f \circ g)[x] = c(f[g[x]])$ is disjoint from A. Take $O \in \mathcal{V}$ and $O^* \in \mathcal{V}^*$ with $A \subseteq O$ and $c(f[g[x]]) \subseteq O*$ and $O \cap O^* = \phi$. Then $x \notin g^{-1}[f^{-1}[Z - O^*]]$ which is $\mathcal{T}^*$-closed. What's more, $(f \circ g)^{-1}[A] \subseteq g^{-1}[f^{-1}[Z - O^*]]$ since if $x' \notin g^{-1}[f^{-1}[Z - O^*]]$ then $f[g[x']] \subseteq O^*$ and therefore $(f \circ g)[x']$ is disjoint from O. Thus $(f \circ g)^{-1}[A]$ is $\mathcal{T}^*$-closed. □

Lemma 2.14. *Composition in* $\mathbf{CR_c^m}$ *is associative.*

Proof. See proof of lemma 2.8. □

Lemma 2.15. ι_c *is a functor.*

Proof. See proof of lemma 2.9. □

Proposition 2.4. *There is a right adjoint* $\eta_c : \mathbf{CR_c^m} \to \mathbf{CR}$ *to the functor* $\iota_c : \mathbf{CR} \to \mathbf{CR_c^m}$*, where for objects* $\eta_c(X, \mathcal{T}, \mathcal{T}^*)$ *is the hyperspace* $(2^X, L(\mathcal{T}), U(\mathcal{T}^*)$*, and for morphisms* $\eta_c(f)$ *is defined by* $\eta_c(f) = 2^f$*.*

Proof. We showed in [B2] that if $(X, \mathcal{T}, \mathcal{T}^*)$ is regular then so is $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$. In [B3] we showed that if $(X, \mathcal{T}, \mathcal{T}^*)$ is compact then so is $(2^X, L(\mathcal{T}), U(\mathcal{T}^*))$, and we noted that it was known that all compact regular bispaces are normal. In addition, we showed in [B2] that all hyperspaces were R_0^*. The rest of the proof should work out more or less as above. □

To have to state Proposition 2.4 for such a restricted category is somewhat disappointing. Whether it can be generalized is a question that will require further investigation.

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