Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 23, Summer 1998, 75–92

COMPANION THEOREMS FOR MONOTONE NORMALITY AND THE HAHN-MAZURKIEWICZ THEOREM

Dale Daniel

Abstract

Nikiel has asked whether every monotonically normal compactum is the continuous image of a compact ordered space. Partial results have been obtained by Nikiel, Treybig, and Tuncali and by Rudin. It is natural to ask whether certain crucial properties hold in both classes of spaces, monotonically normal compacta and those spaces which are the continuous image of a compact ordered space. We study this question for locally connected continua.

A Hausdorff space X is said to be an *IOK* if there exists a continuous map $f : K \to X$ where K is a compact ordered space. If K is connected then X is said to be an *IOC*. We will often refer to a non-degenerate linearly ordered continuum as an *arc* or a *generalized arc* .

A Hausdorff space X is said to be *monotonically normal* provided that there exists a function G which assigns, to each point $x \in X$ and each open set U of X containing x, an open set $G(x, U)$ such that

(1) $x \in G(x, U) \subset U$,

Mathematics Subject Classification: Primary: 54F15; Secondary: 54C05, 54F05, 54F50

Key words: ordered compactum, ordered continuum, continuous image, locally connected continuum, monotone normality

(2) if U' is open and $U \subset U'$, then $G(x, U) \subset G(x, U')$,

(3) if x and y are distinct points of X, then $G(x, X - \{y\}) \cap G(y, X - \{x\}) = \emptyset$.

Such a function G is called a *monotone normality operator* on X.
Let K be a compact ordered space with order $<$. Then a *gap* of K is a subset $\{x, y : x < y\} \subset K$ such that the set $\{p \in K : x < p < y\}$ is empty. We will often denote such a gap by (x, y).
Let X be a space and $Y \subset X$. Then Y° will denote the interior of Y with respect to $X, \overline{Y}$ will denote the closure with respect to S, and ∂Y will denote its boundary with respect to X.

In [11], Nikiel asked if every monotonically normal compactum is an IOK. Nikiel, Treybig, and Tuncali [15] have shown that if X is a monotonically normal continuum and for each pair a, b of distinct points of X there exists a continuous onto map $f : X \to [c, d]$ where $[c, d]$ is a non-metrizable arc, and $f(a) = c$ and $f(b) = d$, then X is an IOC. Rudin ([17] and [18]) has shown that every separable monotonically normal compactum is an IOK. It is natural to ask if certain crucial properties hold in both classes of spaces, monotonically normal compacta and IOK's. We examine this question for locally connected continua. We refer to such results as companion theorems since the reader's choice of "X is monotonically normal" and "X is an IOK" may be used.

We begin with the remark that each compactum which is locally monotonically normal (resp. locally an IOK) at each point is monotonically normal (resp. an IOK). The proofs are straightforward and are left to the reader.

We next exhibit sufficient conditions under which monotonically normal spaces may be 'pasted' and the monotone normality maintained.

Lemma 1. *Suppose X is a locally connected continuum and Y is a closed metric G_δ subset of X such that $Y^\circ = \emptyset$, and X*

is locally monotonically normal at each point of $X - Y$*. Then* $X - Y$ *is monotonically normal.*

Proof. Suppose $Y = \cap U_n = \cap \overline{U_n}$ and $X = \overline{U_1} = U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset U_3 \supset \cdots$ For each $n \geq 1$, define $A_n = \overline{U_n} - U_{n+1}$ and $C_n = A_n \cap A_{n+1}$. Each A_n is compact Hausdorff and locally monotonically normal at each point and is therefore monotonically normal; let T_n denote a monotone normality operator for A_n for each n. Let $p \in U$ with U open in $X - Y$. Define an operator T on $X - Y$ by

$$T(p,U) = \begin{cases} T_n(p, U \cap A_n)^\circ, & p \in A_n^\circ \text{ for some } n \\ [T_n(p, U \cap A_n) \cup T_{n+1}(p, U \cap A_{n+1})]^\circ, & p \in C_n \text{ for some } n. \end{cases}$$

We leave it to the reader to show that T is a monotone normality operator for $X - Y$. □

Theorem 2. *Suppose* X *is a locally connected continuum and* Y *is a closed subset of* X *such that* Y *is metric* G_δ *and* X *is locally monotonically normal at each point of* $X - Y$*. Then* X *is monotonically normal.*

Proof. Suppose $Y = \cap U_n = \cap \overline{U_n}$ and $X = \overline{U_1} = U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset U_3 \supset \cdots$ By [22], there is an upper semi-continuous decomposition H of X into continua so that X/H is compact metric, $\{y\} \in H$ if $y \in Y$, and no element h of H in $X - Y$ intersects more than one ∂U_i. Let $\phi : X \to X/H$ denote the natural map; X/H has a monotone normality operator M by Theorem 2.6 of [4]. Let T be a monotone normality operator for $X - Y$. Let $p \in U$ with U open in X. Define an operator K on X by

$$K(p,U) = \begin{cases} \cup(M(p, (\phi(U))^\circ)), & p \in Y \\ T(p, U \cap (X - Y)) \cap (\cup M(\phi(p), \phi(X - Y))), & p \in X - Y. \end{cases}$$

We leave the verification that K is indeed a monotone normality operator on X to the reader. □

Corollary 3. *Suppose* X *is a locally connected continuum containing a closed subset* Y *such that* Y *is* G_δ*,* Y *is monotonically* *normal, ∂Y is compact metric, and X is locally monotonically normal at each point of $X - Y$. Then X is monotonically normal.*

Proof. $X - Y$ is monotonically normal by Lemma 1, and Y is monotonically normal by hypothesis. Then X is the union of a finite number of closed monotonically normal spaces and is therefore monotonically normal. $\square$

We now show the result analogous to Corollary 3 for IOK's. First, we need the following lemma.

Lemma 4. *Suppose X is a locally connected continuum and Y is a closed metric G_δ subset of X such that $Y^\circ = \emptyset$. Then there is a sequence (U_n) of open sets and a sequence (H_n) of finite covers of Y by connected open sets such that*

i) $Y = \cap U_n = \cap \overline{U_n}$ and $X \supset \overline{U_1} \supset U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset U_3 \supset \dots,$

ii) H_{n+1} is a star-refinement of H_n for each $n \geq 1$,

iii) $\delta(h \cap Y) \leq (1/3)^n$ for each $h \in H_n$,

iv) if $p \in Y$, $\{p\} = \cap Star(p, H_n)$,

v) if $h \in H_n$, h does not intersect both U_{n+1} and $X - U_k$ for any $k \in \{1, 2, \dots, n\}$, and

vi) for each n, there exists a finite set P_n such that

(1) for each $g \in H_n$, there is an element x_g of ∂g in P_n,
(2) $P_n = \{x_g : g \in H_n\}$, and
(3) $P_n \subset X - \cup\{g \in H_n : g \cap Y \neq \emptyset\}$.

Proof. Suppose $V_1 \supset \overline{V_2} \supset V_2 \supset \overline{V_3} \supset V_v \supset \dots$ and $Y = \cap V_n = \cap \overline{V_n}$.

Select least i_1 such that $V_{i_1} \neq X$ and define $U_1 = V_{i_1}$. For $p \in Y$, select an open connected neighborhood $g_1(p)$ of p so that

$\delta(g_1(p) \cap Y) \leq (1/3)$ and $\overline{g_1(p)} \subset U_1$. Select a finite subcover G_1 of Y by elements of $\{g_1(p) : p \in Y\}$. Now, for each $g \in G_1$, select a point x_g in $g - Y$, and let Q_1 denote the set of all components of sets of the form $h - \cup_{g \in G_1}\{x_g\}$, $h \in G_1$, and select an irreducible finite cover Q_1' of Y by elements of Q_1. Also, for each set C of Q_1', let x_C denote a point of $\cup_{g \in G_1}\{x_g\}$ in ∂C, and let P_1 denote the collection of points x_C so selected. Define Z_1 to be Q_1'. Select least i_2 so that $\overline{V_{i_2}} \subset \cup Z_1$, and $P_1 \subset X - V_{i_2}$; define $U_2 = V_{i_2}$. For each $x \in X - \cup Z_1$, select an open connected neighborhood $N_1(x)$ of x so that $N_1(x) \cap \overline{U_2} = \phi$. Select an irreducible finite subcover H_1 of X by elements of $Z_1 \cup \{N_1(x) : x \in X - \cup Z_1\}$. For each $x \in X$, also define $I_1(x) = \cap\{h \in H_1 : x \in h\}$.
Now, for each $p \in Y$, select an open connected neighborhood $g_2(p)$ of p so that $g_2(p) \subset U_2 \cap I_1(p)$ and $\delta(g_2(p) \cap Y) \leq (1/3)^2$. Select a finite subcover G_2 of Y by elements of $\{g_2(p) : p \in Y\}$. Now, for each $g \in G_2$, select a point x_g in $g - Y$, and let Q_2 denote the set of all components of sets of the form $h - \cup_{g \in G_2}\{x_g\}$, $h \in G_2$, and select an irreducible finite cover Q_2' of Y by elements of Q_2. Also, for each set D of Q_2', let x_D denote a point of $\cup_{g \in G_1}\{x_g\}$ in ∂D, and let P_2 denote the collection of points x_D so selected. Define Z_2 to be Q_2'. Select the least i_3 so that $\overline{V_{i_3}} \subset \cup Z_2$, and $P_2 \subset X - V_{i_3}$; define $U_3 = V_{i_3}$. For each $x \in X - \cup Z_2$, select an open connected neighborhood $N_2(x)$ of x so that $N_2 \cap \overline{U_3} = \phi$, and $\overline{N_2(x)} \subset I_1(x)$. By [26], there is a star-refinement S_2 of $Z_2 \cup \{N_2(x) : x \in X - \cup Z_2\}$ by connected open sets. Select a finite subcover H_2 of X by elements of S_2. Also define $I_2(x) = \cap\{h \in H_2 : x \in h\}$ for each $x \in X$.

We proceed by induction. Assume $H_{k-1}, I_{k-1}, U_{k-1}, P_{k-1}, \ldots$ have been defined for all $k - 1 \leq n, n \geq 2$.

Select the least i_k so that $\overline{V_{i_k}} \subset \cup G_{k-1}$ and $P_{k-1} \subset X - V_{i_k}$; define $U_k = V_{i_k}$. For each $p \in Y$, select an open connected neighborhood $g_k(p)$ of p so that $g_k(p) \subset U_k \cap I_{k-1}(p)$ and $\delta(g_k(p) \cap Y) \leq (1/3)^k$. Select a finite subcover G_k of Y by elements of $\{g_k(p) : p \in Y\}$. Now, for each $g \in G_k$, select a point x_g in $g - Y$, and let Q_k denote the set of all components of sets of the form $h - \cup_{g \in G_k}\{x_g\}$, $h \in G_k$, and select an irreducible finite cover Q'_k of Y by elements of Q_k. Also, for each set E of Q'_k, let x_E denote a point of $\cup_{g \in G_k}\{x_g\}$ in ∂E, and let P_k denote the collection of points x_E so selected. Define Z_k to be Q'_k. Select the least i_{k+1} so that $\overline{V_{i_{k+1}}} \subset \cup G_k$, and define $U_{k+1} = V_{i_{k+1}}$. For each $x \in X - \cup Z_k$, select an open connected neighborhood $N_k(x)$ of x so that $N_k \cap \overline{U_{k+1}} = \phi$ and $\overline{N_k(x)} \subset I_{k-1}(x)$. By [26], there is a star-refinement S_k of $Z_k \cup \{N_k(x) : x \in X - \cup Z_k\}$ by connected open sets. Select a finite subcover H_k of X by elements of S_k. □

Theorem 5. *Suppose X is a locally connected continuum and X contains a closed subset Y such that Y is G_δ, metric, and $Y^\circ = \emptyset$. If X is locally an IOK at each point of $X - Y$ then X is an IOK.*

Proof. Let (U_n), (Y_n), and (P_n) be sequences of open sets of X, finite covers of Y by connected open sets, and sets of boundary points of elements of Y_n, respectively, as constructed in the previous lemma. For each $g \in Y_n$, let $p^n_g \in P_n$ denote the the boundary point of g as selected in the lemma. Recall that, by construction, $P_n \subset X - \cup\{g \in Y_n : g \cap Y \neq \emptyset\}$ for each n.
Since $X - Y$ is locally an IOK, $X - \cup Y_1$ is an IOK. Applying the previous lemma, there exists a continuous onto map f_1 such that $f_1 : K_1 \to X - \cup Y_1$, K_1 is a compact ordered space, and for each $g \in Y_1$, there exists a gap (a^1_g, b^1_g) in K_1 such that $f_1(a^1_g) = f_1(b^1_g) = p^1_g$ and $(a^1_g, b^1_g) \cap (a^1_d, b^1_d) = \emptyset$ if and only if $g \neq d$.

For each gap (a^1_g, b^1_g) in K_1, define a continuous onto map $f^g_2 : K^g_2 \to \overline{g} - \cup Y_2$ so that if $h \in Y_2$ and $h \subset g \in Y_1$, then there exists a gap (a^2_h, b^2_h) in K^g_2 so that $f^g_2(a^2_h) = f^g_2(b^2_h) = p^2_h$, and $(a^2_h, b^2_h) \cap (a^2_k, b^2_k) = \emptyset$ if and only if $h \neq k$.
Define an order $<$ on $K_2 = K_1 \cup (\cup_{g \in Y_1} K^g_2)$ so that if $x, y \in K_2$ then $x < y$ provided that

$x, y \in K_1$ and $x < y$ in K_1, or
$x, y \in K^g_2$ for some K^g_2 and $x < y$ in K^g_2, or

$x \in K_2^g$ and $b_2^g < y$ in K_1 or $y \in K_2^g$ and $x < a_2^g$ in K_1, or
$x \in K_2^{g_1}$, $y \in K_2^{g_2}$, and $b_{g_1}^2 < a_{g_2}^2$.

Note that there is a natural continuous map $f_2 : K_2 \to X - \cup Y_2$ defined by

$$f_2(x) = \begin{cases} f_1(x), & \text{if } x \in K_1, \\ f_2^g(x), & \text{if } x \in K_2^g. \end{cases}$$

Similarly, for each gap (a_h^2, b_h^2) in K_2, there exists a continuous onto map $f_3^h : K_3^h \to \overline{h} - \cup Y_3$ so that if $k \in Y_3$ and $k \subset h \in Y_2$, then there exists a gap (a_k^3, b_k^3) in K_3^h so that $f_3^h(a_k^k) = f_k^h(b_k^3) = p_k$, and $(a_l^3, b_l^3) \cap (a_k^3, b_k^3) = \emptyset$ if and only if $l \neq k$.

We define an order $<$ on $K_3 = K_2 \cup (\cup_{h \in Y_2} K_3^h)$ so that if $x, y \in K_3$ then $x < y$ provided that

$x, y \in K_2$ and $x < y$ in K_2, or
$x, y \in K_3^h$ for some K_3^h and $x < y$ in K_3^h, or
$x \in K_3^h$ and $b_3^h < y$ in K_2 or $y \in K_3^h$ and $x < a_3^h$ in K_2, or
$x \in K_3^{h_1}$, $y \in K_3^{h_2}$, and $b_{h_1}^3 < a_{h_2}^3$.

There is a natural continuous onto map $f_3 : K_3 \to X - \cup Y_3$ defined by

$$f_3(x) = \begin{cases} f_2(x), & \text{if } x \in K_2, \\ f_3^h(x), & \text{if } x \in K_3^h. \end{cases}$$

We proceed by induction, obtaining a sequence, (K_i), of compact ordered spaces and a sequence, (f_i), of continuous onto maps such that $f_i : K_i \to (X - \cup Y_i)$, $i = 1, 2, 3, \dots$. We now define a compact ordered space K and a continuous onto map $f : K \to X$. An element of K is either a point of $\cup_1^\infty K_i$ or is a point denoted by $p = (a_{g_1}^1, a_{g_2}^2, \dots)$, where $g_1 \supset g_2 \supset g_3 \supset \cdots$ and $g_i \in Y_i$. f is defined by

$$f(x) = \begin{cases} f_i(x), & \text{if } x \in K_i, \\ \cap_1^\infty \overline{g_i}, & \text{if } x \text{ is of the form } (a_{g_1}^1, a_{g_2}^2, \dots). \end{cases}$$

Recall that, by construction, each $\cap_1^\infty \overline{g_i}$ is a unique point in Y. We define an order $<$ on K so that $x < y$ provided that

$x, y \in K_i$ for some i and $x < y$ in K_i, or
$x = (a^1_{g_1}, a^2_{g_2}, \dots)$, $y \in K_i$, and $a^i_{g_i} < y$ in K_i, or
$x = (a^1_{g_1}, a^2_{g_2}, \dots)$, $y = (a^1_{h_1}, a^2_{h_2}, \dots)$, and if i is the first coordinate where x and y differ, then $b^i_{g_i} < a^i_{h_i}$.

We now show that f is continuous. Let $x \in K = \cup K_i$. Then f is continuous at x since each f_i is. Suppose $x = (a^1_{g_1}, a^2_{g_2}, \dots)$ as above and $f(x) = y \in U$ with U open in X. Then there exists an n such that some $g \in Y_n$ has the property that $y \in g \subset (U \cap Y_n)$, $x \in (a^n_g, b^n_g)$, and $f((a^n_g, b^n_g)) \subset U$. □

Corollary 6. *Suppose X is a locally connected continuum containing a closed connected subset Y such that Y is an IOK and Y is G_δ in X. Suppose also that X is locally an IOK at each point of $X - Y$ and $X - Y^\circ$ is a locally connected continuum. Then X is an IOK if and only if ∂Y is metric.*

Proof. ($\Rightarrow$) We apply Theorem 1 of [8].
($\Leftarrow$) Suppose f, g are continuous onto maps and K, N are disjoint compact ordered spaces such that $f : K \to Y$ and $g : N \to X - Y^\circ$; note that such a map g exists by the preceding theorem. Define $M = K \cup N$ and if $a, b \in M$ we say that $a < b$ if and only if
$a \in K$ and $b \in N$, or
$a, b \in K$ and $a < b$ in K, or
$a, b \in N$ and $a < b$ in N.
Define $h : M \to X$ by

$$h(p) = \begin{cases} f(p), & p \in K, \\ g(p), & p \in N. \end{cases}$$

Then M is clearly a compact ordered space and h is continuous since each of f and g is. □

We refer the reader to [22] for a proof of the following lemma.

Lemma 7. *Let X be a locally connected continuum. Let Y be a closed G_δ subset of X. Then there is an upper semi-continuous* *decomposition G of X into continua such that $\phi(Y)$ is compact metric in X/G and if $x \in X - Y$ then $\{x\} \in G$.*

Theorem 8. *Let X be a first countable locally connected continuum which is not an IOK (monotonically normal). Then there is an upper semi-continuous decomposition G of X into continua so that $N = \{g \in G : X/G$ is not locally an IOK (monotonically normal) at $g\}$ has uncountably many components.*

Proof. Let $P = \{x \in X : X$ is not locally an IOK (monotonically normal) at $x\}$. We consider two cases.

Case 1. Suppose P has uncountably many components. Then the result is immediate by defining G to be the points of X so that $N = P$.

Case 2. Suppose P has only countably many components. Then there is a component E of P such that E is a non-degenerate subcontinuum of X. Select distinct points a, b of E. We apply Urysohn's Lemma to obtain a continuous map $f : X \to [-1, 2]$ such that $f(a) = -1$ and $f(b) = 2$. We now define certain subsets of $[0, 1]$.

Define $I_1 = (1/3, 2/3)$, $I_2 = (1/9, 2/9) \cup (7/9, 8/9)$, $I_3 = (1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20, 27) \cup (25/27, 26/27)$, and in general, define

$$I_n = (\cup_{k=0}^{\infty}((1 + 3k)/3^n, (2 + 3k)/3^n) \cap ([0, 1] - I_{n-1})), n \geq 2.$$

Define $L_1 = (3/9, 4/9) \cup (5/9, 6/9)$, $L_2 = (3/27, 4/27) \cup (5/27, 6/27) \cup (21/27, 22/27) \cup (23/27, 24/27)$, and, in general, define

$$L_n = \cup_{k=1}^{\infty}((2k + 3)/3^{n+1}, (2k + 4)/3^{n+1}) \cap I_n, n \geq 1.$$

Define $R = \overline{\cup L_n}$ and $Y = f^{-1}(R)$. Note that R is G_δ in $[-1, 2]$ by Exercise 3H in [26]. Suppose $R = \cap U_n$ with each U_n open in X. Then $Y = f^{-1}(R) = f^{-1}(\cap U_n) = \cap f^{-1}(U_n)$ with each $f^{-1}(U_n)$ open in X. Then we have that Y is a closed G_δ subset of X. We now apply the previous lemma to obtain an upper semi-continuous decomposition G of X into continua such that $\{x\} \in G$ if $x \in X - Y$ and $\phi(Y)$ is compact metric in X/G.

The set W of components of $[-1,2] - R$ is a countable set consisting of two half-open intervals, $[-1,0)$ and $(1,2]$, and a countably infinite number of open intervals. Each $w \in W$ has the property that $f^{-1}(w)$ contains points of E, and moreover $\overline{\cup W}$ has uncountably many components since its collection of limit points which are components is a subset of the Cantor set. Therefore, each component C of $\overline{\cup W}$ has the property that $f^{-1}(C)$ contains an element of G at which X/G is not locally an IOK (monotonically normal). Furthermore, for each pair C, C' of distinct components of $\overline{\cup W}$, $f^{-1}(C)$ and $f^{-1}(C')$ are separated in X by $f^{-1}(w)$ for some $w \in W$. Thus, $N = \{g \in G : X/G$ is not locally an IOK (monotonically normal) at $g\}$ has uncountably many components. □

In the final pair of theorems, we give sufficient conditions for a locally connected continuum X to be an IOK and monotonically normal, respectively, based on the existence of maps of arcs onto decomposition spaces of X.

Theorem 9. *Let X be a locally connected continuum and let G be an upper semi-continuous decomposition of X into continua such that each $g \in G$ is an IOK, ∂g is totally disconnected for each $g \in G$, and there exists a continuous onto map $f : [a,b] \to X/G$ so that $[a,b]$ is a first countable generalized arc and for each $g \in G$ either $f^{-1}(g)$ is finite or g is a point. Then X is an IOK.*

Proof. Suppose that each $g \in G$ is an IOK. For $x \in [a,b]$, let $g_x = f(x) \in G$. If $f^{-1}(g_x)$ is finite, let (u_x, v_x) ($[a, v)$ or $(u, b]$) denote an open (half-open) interval containing x but which intersects $f^{-1}(f(x))$ only in x. Let (u_n) and (v_n) denote sequences so that $u_x < u_1 < u_2 < \cdots < x < \cdots < v_2 < v_1 < v_x$, and $u_n \to x$ and $v_n \to x$.

We first show that there is a point $h_x \in g_x$ (resp. $k_x \in g_x$) so that if U (resp. V) is an open set containing h_x (resp. k_x)

then for each natural i (resp. j) we have $[\cup f((u_i, x))] \cap U \neq \emptyset$ (resp. $[\cup f((x, v_j))] \cap V \neq \emptyset$). Assume that for every $h \in g_x$ there exists U open in X such that $h \in U$ and a natural i so that $\cup f((u_i, x)) \cap U = \emptyset$. By the compactness of g_x, there exists O open in X and a natural m so that $g_x \subset O$ and $\cup f((u_m, x)) \cap O = \emptyset$. Define $R = \{r : r \in G, r \subset O\}$. Then R is open in X/G and $\cup R$ is open in X. Then $\cup f((u_m, x)) \cap (\cup R) = \emptyset$ and therefore $\cup(\overline{f((u_m, x))}) \cap g_x = \emptyset$. Then since $g_x \in f([u_m, x])$, we have that $f(\overline{(u_m, x)}) = f([u_m, x]) \not\subset \overline{f((u_m, x))}$ which contradicts the continuity of f. The case for k_x is similar.

We now show that h_x (resp. k_x) is unique. Suppose $h'_x \in g_x$, $h_x \neq h'_x$, and h'_x has the property above. Let O and O' be disjoint open so that $\overline{O} \cap \overline{O'} = \emptyset$ and without loss of generality $h_x \in O$ and $h'_x \in O'$. Select a subsequence (u'_n) of (u_n) so that $[u'_1, u'_2] \cap [u'_3, u'_4] \cap \cdots = \emptyset$, $u'_n \to x$, and for each $n = 1, 3, 5, \dots$, $\cup f([u'_n u'_{n+1}]) = C_n$ is a continuum in X intersecting both O and O'. The limiting set C of (C_n) is a continuum in X. Then $C \cap \overline{O} \cap g_x \neq \emptyset$ and $C \cap \overline{O'} \cap g_x \neq \emptyset$. By the continuity of f, $C \subset \partial g_x$ which contradicts the connectedness of C. The case for k_x is similar.

For each $x \in [a, b]$, let $f_x : K_x \to g_x$ denote a continuous map from a compact ordered space $K_x = [c_x, d_x]$ onto g_x so that $f_x(c_x) = h_x$ and $f_x(d_x) = k_x$, and $K_x \cap K_y = \emptyset$ if and only if $x \neq y$. Define $K = \cup K_x$ and define $<$ on K such that if $r, s \in K$, then $r < s$ if and only if

$r, s \in K_x$ for some $x \in [a, b]$ and $r < s$ in K_x, or

$r \in K_x$, $s \in K_y$, and $x < y$ in $[a, b]$.

Then sets of the form $(\bullet, c) = \{x \in K : x < c\}$ and $(d, \bullet) = \{x \in K : d < x\}$ with $c, d \in K$ form a sub-basis S for the interval topology T on K.

We now show that K is compact. It is clearly sufficient, by the Alexander Subbase theorem (Exercise 17S of [26]), to show that if some subset D of S covers K then some finite subset of D covers K. We consider two cases.

Case 1. Suppose that there are elements $(\bullet, c)$ and $(d, \bullet)$ of D and an $x \in X$ such that $c, d \in K_x$. If $d < c$ then we are done. Otherwise, $D' = \{q \cap K_x : q \in D\}$ is a cover of $[c, d] \subset K_x$. Then some finite collection F' of D' covers $[c, d]$ and therefore $F = \{(\bullet, c), (d, \bullet)\} \cup \{q \in D : (q \cap K_x) \in F'\}$ is a finite cover of K.

Case 2. Suppose that there are no elements $(\bullet, c)$, $(d, \bullet)$ of D so that $c, d \in K_x$ for some $x \in [a, b]$. If $d < c$ for some $(\bullet, c)$ and $(d, \bullet)$ in D, we are done. Otherwise, define $L = \{x :$ there exists $(\bullet, c) \in D$ and $c \in K_x\}$ and $R = \{x :$ there exists $(d, \bullet) \in D$ and $d \in K_x\}$. Let $l = l.u.b.L$ and $r = g.l.b.R$. Then $l \leq r$. Now assume that $l < r$. Select y such that $l < y < r$; such a y exists since $l \geq r$ if not. But then some element of K_y fails to be covered by any element of D.

We then have that $l = r$. We also show that $l \in L$ and $r \in R$. Assume $l \notin L$. Then there exists $(d, \bullet) \in D$, $d \in K_x$, such that $[c_l, d_l] \cap (d, \bullet) \supset \{d_l\}$ but $[c_l, d_l] \cap (d, \bullet) \not\supset \{c_l\}$, where $[c_l, d_l] = K_l$. Suppose $[c_r, d_r] = K_r$. Then $c_l = c_r < d$ since $l = r$, and c_l fails to be covered by any element of D. The case for $r \in R$ is similar, and this completes the argument that X is compact.

Define $Z : K \to X$ by $Z(y) = f_x(y)$ if $y \in K_x$. We now show that Z is continuous. Suppose $p \in K$ and $Z(p) \in V$ with V open in X. We consider two cases.

Case 1. Suppose $Z(p) \in g_x \in G$ and $V \subset g_x$. Then by the continuity of f_x, there is an open set O in K_x so that $p \in O$ and $f_x(O) = Z(O) \subset V$.

Case 2. Suppose $Z(p) \in g_x$ and $V \not\subset g_x$. Note that $V_0 = \cup\{h \in G : h \subset V\}$ is open in X. Then $Z^{-1}(V_0) = \cup\{f_y^{-1}(h) : h = g_y \in V_0\} = \cup\{K_y : h = g_y \in V_0\}$ is open in K. By the continuity of f_x, there exists O, open in K_x, such that $Z(O) = f_x(O) \subset (V \cap g_x)$. Then $O \cup Z^{-1}(V_0)$ is open and $Z(O \cup Z^{-1}(V_0)) = f_x(O) \cup V_0 \subset V$. □

Theorem 10. *Let X be a locally connected continuum and let G be an upper semi-continuous decomposition of X into continua such that each $g \in G$ is monotonically normal, ∂g is totally disconnected for each $g \in G$, X is first countable at each point of ∂g for each non-degenerate $g \in G$, and there exists a continuous onto map $f : [a, b] \to X/G$ so that $[a, b]$ is a generalized arc and for each $g \in G$ either $f^{-1}(g)$ is finite or g is a point. Then X is monotonically normal.*

Proof. Assume that each $g \in G$ is monotonically normal.

We first show that ∂g is finite for each $g \in G$. If g is degenerate, then $\partial g = g$ and we are done. So, suppose that g is non-degenerate. Let $f^{-1}(g) = \{x_1, \ldots, x_{n_g}\}$. For each $i = 1, \ldots, n_g$, there are unique points l_i and r_i of g such that $\lim_{x \to x_i^-} f(x) = l_i$ and $\lim_{x \to x_i^+} f(x) = r_i$, respectively. Assume $b \in \partial g - \{l_1, \ldots, l_{n_g}, r_1, \ldots, r_{n_g}\}$. Let (U_n) be a countable basis at b such that $U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset \cdots$ and $\overline{U_1} \cap \{l_1, \ldots, l_{n_g}, r_1, \ldots, r_{n_g}\} = \emptyset$. For each n, let $t_n \in g_n \in G$ so that $t_n \in U_n$. Select $s_1 \in f^{-1}(t_1)$, $s_2 \in f^{-1}(t_2)$, and so on. If $s_m \to x_i$ for some $i = 1, \ldots, n_g$, there exists a subsequence (s_{m_k}) of (s_m) such that $f(s_{m_k}) \to l_i$ or $f(s_{m_k}) \to r_i$. But this is impossible by construction of (U_n). Then $s_n \to t \in [a, b]$ and $t \notin \{x_1, \ldots, x_{n_g}\}$. Suppose $f(t) \neq g$. Then there are disjoint open sets U and V such that $f(t) \subset U$ and $g \subset V$. Then U contains at most a finite number of elements of (t_n) and $f^{-1}(U)$ contains a finite subset of (s_n), a contradiction.

We now show that each $g \in G$ is locally connected. Suppose $g \in G$ is non-degenerate and $\partial g = \{b_1, \ldots, b_{n_g}\}$. Let $p \in g$ and $p \in V$ open in X. We consider two cases.

Case 1. Suppose $p \in g^\circ$. By local connectivity of X, there is an open connected O so that $p \in O \subset g^\circ \cap V$.

Case 2. Suppose $p \in \partial g$ and without loss of generality $p = b_1$. Select a neighborhood U of p such that U is connected and open, $\overline{U} \cap \partial g = \{p\}$, and if $h \in G$ and $h \cap U \neq \emptyset$ then $h = g$ or $h \subset V$. Let O be the component of $g \cap U$ which contains p. Then O is open in g and $\overline{O} \cap \partial g = \{p\}$, and g is locally connected at p.

For each non-degenerate $g \in G$ and $p \in \partial g$, let $\{U_n(p)\}$ be a countable local basis at p. For each $g \in G$, select a sequence (T_n^g) of finite covers of g by connected open sets so that T_{n+1}^g star-refines T_n^g for each n and if $s \in T_n^g$ and $p \in s$ then $s \subset U_n(p)$. Define $T_g = \{t \subset g : t = \cap St(x, T_n^g \text{ for some } x \in g\}$. By [22], T_g is an upper semi-continuous decomposition of g into continua and g/T_g is a locally connected metric continuum. Note also that if $x \in \partial g$ then $\{x\} \in T_g$.

We now show that $H = \cup_{g \in G} T_g$ is an upper semi-continuous decomposition of X into continua. Let $h \in H$ and $h \subset U$ with U open in X. We consider two cases.

Case 1. Suppose $h \subset g^\circ$. We apply the upper semi-continuity of T_g to obtain V, open in g°, with the property that if $h' \in H$ and $h' \cap V \neq \emptyset$ then $h' \subset g^\circ \subset U$.

Case 2. Suppose $h = \{x\} \in \partial g$.

Subcase 2a. Suppose $g = h \in G$. Then there is V open such that if $h' \in G$ and $h' \cap V \neq \emptyset$ the $h' \subset U$. But $h' \in H$ and $h' \cap V \neq \emptyset$ implies that $h' \subset g' \in G$ and $g' \cap V \neq \emptyset$ $\Rightarrow h' \subset g' \subset U$.

Subcase 2b. Suppose g is non-degenerate. Since $(\partial g) - \{x\}$ is finite, there exists an open set W so that $x \in W \subset \overline{W} \subset U$ and $(\partial(g) \cap \overline{W}) - \{x\} = \emptyset$. Select a countable basis (U_n) at x such that $x \in \cdots U_{n+1} \subset \overline{U_{n+1}} \subset U_n \subset \overline{U_n} \subset \cdots \subset U_1 \subset \overline{U_1} \subset W$. Assume that for each n there is an $h_n \in H$ such that $h_n \cap U_n \neq \emptyset$, $h_n \cap (X - W) \neq \emptyset$, and $h_n \cap g = \emptyset$. Suppose without loss of generality that $x_n \in (h_n \cap U_n)$ and $y_n \in (h_n \cap (X - W))$ so that $x_n \to x$ and $y_n \to y$. Suppose also that for each n we have $h_n \subset g_n \in G$. Select O open such that $g \cup (x_n) \subset O$ and $\overline{O} \cap ((y_n) \cup \{y\}) = \emptyset$. Then each open $V \subset O$ containing g meets g_k for some k but $g_k \not\subset O$. This contradicts the upper semi-continuity of G.

Now, let x be an arbitrary but fixed element of $[a, b]$. Suppose $f(x) = g \in G$. Since g/T_g is an IOC there is an arc $K'_x = [l', r']$

and a continuous onto map $g'_x : k'_x \to g/T_g$. Suppose m is the element of T_g so that $m = \lim_{t\to x^-} f(t)$ and q is the element of T_g so that $q = \lim_{t\to x^+} f(t)$. We now show that there exists a compact ordered space K_x, with least element l and greatest element r, and a continuous onto map $g_x : K_x \to g/T_g$ so that $g_x(l) = m$ and $g_x(r) = q$. Select $l \in (g'_x)^{-1}(m)$ and $r \in (g'_x)^{-1}(q)$. We consider cases.

Case 1. Suppose $l' = l$ and $r' = r$. Then set $g_x = g'_x$ and $K_x = K'_x$

Case 2. Suppose $l' \le l \le r \le r'$. Let $A_1 = [l_1, r_1]$ and $A_2 = [l_2, r_2]$ be disjoint copies of $[l, r]$ via homeomorphisms F_1 and F_2, respectively, and such that each is also disjoint from K'_x. Set $K_x = A_1 \cup K'_x \cup A_2$.

Case 3. Suppose $l' \le r \le l \le r'$. Let $A_1 = [l_1, r_1]$ and $A_2 = [l_2, r_2]$ be disjoint copies of $[l, r']$ and $[l', r]$, respectively, via homeomorphisms F_1 and F_2, respectively, and such that each is also disjoint from K'_x. Set $K_x = A_1 \cup K'_x \cup A_2$.

In the latter two cases, the order in K_x is indicated by the order in which the union is written, and g_x is the composition of g'_x with the appropriate homeomorphisms and is therefore continuous.

We now form a compact ordered space $K = \cup K_x$ with the following order. Suppose $y, z \in K$; then $y < z$ if and only if

$y, z \in K_x$ and $y < z$ in K_x, or

$y \in K_x$, $z \in K_w$, and $x < w$ in $[a, b]$.

We define a map $F : K \to X/H$ by $F(x) = \phi_g(x)$ where $x \in g \in G$ and $\phi_g : g \to g/T_g$ is the natural map. We now show that F is continuous. Let $t \in X/H$, $F(x) = t$, $t \subset g \in G$, and $t \in V$ open. We consider two cases.

Case 1. Suppose $t \subset g^\circ$. Then there is a U open such that $x \in U$ and $f(U) = \phi_g(U) \subset V$.

Case 2. Suppose $t \in \partial g$. Set $V_0 = \cup\{h \in H : h \subset V\}$. Then V_0 is open in X. Then $F^{-1}(V_0) = \cup\phi_g^{-1}(V_0)$ which is the union of open sets in K. Then $F(F^{-1}(V_0)) \subset V_0 \subset V$. Therefore, X/H is an IOK and is then monotonically normal by [4]. Let L denote a monotone normality operator on X/H and L_g a monotone normality operator on g for each $g \in G$. We then define a monotone normality operator M on X as follows:

$M(x,U) = (L_g(x, U \cap g))^\circ \cap (\cup L(h_x, \{t \in H : t \subset g^\circ\}))$, if $x \in g^\circ$ for some $g \in H$, and

$M(x,U) = \cup L(x, (\cup\{g \in H : g \subset A_x \text{ or } g \subset U - \{x\}\})^\circ)$ where $A_x = \cup\{L_{g'}(x, U \cap g') : x \in g'\}$, if $x \in \partial g$ for some $g \in H$.

We leave the verification that M is indeed a monotone normality operator on X to the reader. □

The author wishes to acknowledge the referee for many helpful comments and suggestions.

References

[1] C. J. R. Borges, *On Stratifiable Spaces*, Pacific J. Math, **17** (1966), 1-16.

[2] W. Bula and M. Turzanski, *A Characterization of Continuous Images of Compact Ordered Spaces and the Hahn-Mazurkiewicz Theorem*, Top. and Its Appl., **22** (1986), 7-17.

[3] P. Collins, *Monotone Normality*, Discrete Math., **164** (1997), 87–147.

[4] R. W. Heath, D. J. Lutzer, and P. L. Zenor, *Monotonically Normal Spaces*, Trans. Amer. Math. Soc., **178** (1973), 481-493.

[5] S. Mardešić, *On the Hahn-Mazurkiewicz Theorem in Non-Metric Spaces*, Proc. Amer. Math. Soc., **11** (1960), 929-937.

[6] S. Mardešić, *Mapping Ordered Continua onto Product Spaces*, Glasnik, **15** (1960), 85-89.

[7] S. Mardešić, *Continuous Images of Ordered Compacta and a New Dimension Which Neglects Metric Subcontinua*, Trans. Amer. Math. Soc., **121** (1966), 85-89.

[8] S. Mardešić, *Images of Ordered Compacta are Locally Peripherally Metric*, Pacific J. of Math., **23** (1967), 557-568.

[9] S. Mardešić and P. Papić, *Continuous Images of Ordered Continua*, Glasnik, **15** (1960), 171-178.

[10] J. Nikiel, *Images of Arcs - A Non-Separable Version of The Hahn-Mazurkiewicz Theorem*, Fund. Math., **129** (1988), 91-120.

[11] J. Nikiel, *Some Problems on Continuous Images of Compact Ordered Spaces*, Questions and Answers, Gen. Top., **4** (1988), 117-128.

[12] J. Nikiel, *The Hahn-Mazurkiewicz Theorem for Hereditarily Locally Connected Continua*, Top. and Its Appl., **32** (1989), 307-323.

[13] J. Nikiel, *A Surprising Method in General Topology and Continuous Images of Ordered Compacta*, Bull. Pol. Acad. Sci., **38** (1990), 1-6.

[14] J. Nikiel, *Images of Arcs - A Non-Separable Version of The Hahn-Mazurkiewicz Theorem*, Fund. Math., **129** (1988), 91-120.

[15] J. Nikiel, L. B. Treybig, and H. M. Tuncali, *Local Connectivity and Maps onto Non-Metrizable Arcs*, Internat. J. Math. & Math. Sci., **20 (4)** (1997), 681-688.

[16] A. J. Ostaszewski, *Monotone Normality and G_δ Diagonals in The Class of Inductively Generated Spaces*, Topology, **23** (1978), 905-930.

[17] M. E. Rudin, *Compact, Separable, Linearly Ordered Spaces*, Topology Appl. **82** (1998), 397-419.

[18] M. E. Rudin, *Zero Dimensionality and Monotone Normality*, Topology Appl. **85** (1998), 319-333.

[19] L. B. Treybig, *Concerning Continuous Images of Compact Ordered Spaces*, Proc. Amer. Math. Soc., **15** (1964), 866-871.

[20] L. B. Treybig, *Concerning Continua Which are Continuous Images of Compact Ordered Spaces*, Duke Math. J., **32** (1965), 417-422.

[21] L. B. Treybig, *Separation by Finite Sets in Connected Continuous Images of Ordered Compacta*, Proc. Amer. Math. Soc., **74** (1979), 326-328.

[22] L. B. Treybig, *Extending Maps of Arcs to Maps of Ordered Continua*, Topology and Order Structures, **142** (1979), 83-93.

[23] L. B. Treybig, *A Characterization of Spaces that are the Continuous Image of an Arc*, Top. and Its Appl., **24** (1986), 229-239.

[24] L. B. Treybig, *Arcwise Connectivity in Continuous Images of Ordered Compacta*, Glasnik, **41** (1986), 201-211.

[25] E. D. Tymchatyn, *The Hahn-Mazurkiewicz Theorem for Finitely Suslinian Continua*, Gen. Top. Appl., **7** (1977), 123-127.

[26] S. Willard, *General Topology*, Addison-Wesley, 1970.

Department of Mathematics Texas A&M University College Station, Texas, USA 77843

Current address: Department of Mathematics, Lamar University, P.O. Box 10047 Beaumont, Texas, USA 77710

E-mail address: `daniel@math.lamar.edu`