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QUASI-SPAN FOR CONTINUA

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Abstract

We introduce the concept of quasi-span for a pair (M, N) of continua and prove that if M and N are continua with span equal to zero then the pair (M, N) has quasi-span equal to zero. We also remark that if (M, M) has quasi-span equal to zero then M has surjective semispan equal to zero, with this we obtain a characterization of continua with span equal to zero. Using these results we generalize some theorems related to span.

1. Introduction

A continuum is a compact, connected, metric space. The concept of span of a metric space and some variations were introduced by A. Lelek in [2]and [5]. We recall that the span $\sigma(M)$ of a continuum M is the least upper bound of the set of non-negative real numbers ε for which there exists a subcontinuum Z of $M \times M$ such that

$$\pi_1(Z) = \pi_2(Z) \qquad (\sigma)$$

and $d(x, y) \geq \varepsilon$ for every $(x, y) \in Z$, where d is the metric in M and π_1 and π_2 denote the projection mappings from the product onto the first and second coordinate spaces, respectively.

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The concepts of semispan $\sigma_o(M)$, surjective semispan $\sigma_o^*(M)$, and surjective span $\sigma^*(M)$, of a continuum M are obtained when we change respectively the condition (σ) for each of the following conditions:

$$\pi_1(Z) \subset \pi_2(Z) \qquad (\sigma_o)$$
$$\pi_1(Z) \subset \pi_2(Z) = M \qquad (\sigma_o^*)$$
$$\pi_1(Z) = \pi_2(Z) = M \qquad (\sigma^*)$$

For a continuum M the span and semispan can be obtained by the formula

$$\sigma(M) = \sup\{\sigma^*(A) : A \text{ is a subcontinuum of } M\}$$
$$\sigma_o(M) = \sup\{\sigma_o^*(A) : A \text{ is a subcontinuum of } M\}.$$

The span has become an important concept in continuum theory, particularily with respect to chainability of continua. It is known, for instance, that chainable continua have zero span, see [2]. It is an open question as to whether or not continua which have zero span are chainable, see [3, Problem 1].

Let M and N be continua with metrics d_M and d_N respectively. We define the quasi-span $\sigma_q(M,N)$ of the pair (M,N) to be the least upper bound of all nonnegative real numbers ε for which there exist subcontinua H, K of the product $M \times N$ such that $\pi_1(H) = M$, $\pi_2(K) = N$ and $\mathbf{d}(H,K) \geq \varepsilon$, where $\mathbf{d}$ is the metric on $M \times N$ defined by $\mathbf{d}((x_1,y_1),(x_2,y_2)) = \max\{d_M(x_1,x_2), d_N(y_1,y_2)\}$. Notice that $\sigma_q(M,N) = \sigma_q(N,M)$. When $M = N$, $\sigma_q(M,M)$ is simply denoted by $\sigma_q(M)$ and is called the quasi-span of M.

Notice that $\sigma_q\ (M,N) = 0$ means that for every subcontinua H and K of $M \times N$ such that $\pi_1(H) = M$, $\pi_2(K) = N$ we have $H \cap K \neq \emptyset$, hence the property of having quasi-span zero does not depend on the choice of metrics on M and N.

The main results in this paper are: If M and N are continua such that $\sigma(M) = \sigma(N) = 0$ then $\sigma_q(M,N) = 0$, which is shown in Theorem 2.1. We also prove, in Theorem 2.2, that $\sigma_q(M) = 0$ implies $\sigma_o^*(M) = 0$ and so, quasi-span zero is an intermediate condition between span zero and surjective semispan zero. Then we obtain a characterization of span zero by establishing, in Theorem 2.3, that $\sigma(M) = 0$ if and only if for each subcontinuum A of M, $\sigma_q(A) = 0$.

Using the concept of quasi-span and these results, we obtain some applications: In Theorem 3.1, we prove that quasi-span zero is preserved for an irreducible map f provided that its product $f \times f$ is weakly confluent, which generalizes a theorem by Kawamura, see [7, Theorem 3.2, p. 337]. We also prove, in Theorem 3.3, that the product $M \times N$ is strictly non-mutually aposyndetic if and only if both M and N are indecomposable continua, provided that quasi-span of the pair (M, N) is equal to zero, which generalizes a theorem by Hagopian, see [1, Theorem 10, p. 621].

We recall that a map $f : M \to N$ is an ε-map, for a given $\varepsilon > 0$, provided that $diam f^{-1}(y) < \varepsilon$ for each $y \in N$. A continuum M is said to be chainable provided, for each $\varepsilon > 0$, there exists an ε-map from M onto the unit interval $I = [0, 1]$. A map between continua $f : M \to N$ is irreducible provided that it is onto and there exists no proper subcontinuum of M whose image under f is N. The map f is weakly confluent if for every subcontinuum C of N, there exists a subcontinuum B of M such that $f(B) = C$.

2. Main Theorems

Lemma 2.1. *The unit interval I has quasi-span equal to zero.*

Proof. Suppose H and K are disjoint subcontinua of $I \times I$ such that $\pi_1(H) = I = \pi_2(K)$. It is not difficult to prove that there exist disjoint locally connected subcontinua H' and K' of $I \times I$ such that $H \subseteq H'$ and $K \subseteq K'$. By considering arcs α and β contained in H' and K', respectively, such that $\alpha \cap (\{0\} \times I) \neq \emptyset \neq \alpha \cap (\{1\} \times I)$, and $\beta \cap (I \times \{0\}) \neq \emptyset \neq \beta \cap (I \times \{1\})$, we obtain a contradiction to the *θ-curve Theorem* see [8, Theorem 2, p. 511]. □

Lemma 2.2. *Let M and N be chainable continua. Then $\sigma_q(M,N)=0$.*

Proof. Suppose H and K are disjoint subcontinua of $M\times N$ such that $\pi_1(H)=M$ and $\pi_2(K)=N$. Let $\varepsilon>0$ such that $\varepsilon<\mathbf{d}(H,K)$ and let f_ε and g_ε be ε-maps from M and N onto I respectively. Then the product map $f_\varepsilon\times g_\varepsilon$ from $M\times N$ onto $I\times I$ is an ε-map. Let $H'=(f_\varepsilon\times g_\varepsilon)(H)$ and $K'=(f_\varepsilon\times g_\varepsilon)(K)$. Then H' and K' are disjoint subcontinua of $I\times I$. Since $\pi_1\circ(f_\varepsilon\times g_\varepsilon)=f_\varepsilon\circ\pi_1$ and $\pi_2\circ(f_\varepsilon\times g_\varepsilon)=g_\varepsilon\circ\pi_2$ we have that $\pi_1(H')=I=\pi_2(K')$. So we obtain a contradiction to Lemma 2.1. □

Theorem 2.1. *Let M and N be continua with span equal to zero. Then $\sigma_q(M,N)=0$.*

Proof. Suppose H and K are disjoint subcontinua of $M\times N$ such that $\pi_1(H)=M$ and $\pi_2(K)=N$. By [10, Corollary 6, p. 164], M and N are weakly chainable, that means there exist maps f and g from the pseudo-arc P onto M and N, respectively. Since the pseudo-arc is hereditarily equivalent, we may suppose that f and g are irreducible maps. It follows from [9, Theorem 3, p. 112] that the product map $f\times g$ from $P\times P$ onto $M\times N$ is weakly-confluent. So there exist disjoint subcontinua H' and K' of $P\times P$ such that $(f\times g)(H')=H$ and $(f\times g)(K')=K$. Since $\pi_1\circ(f\times g)=f\circ\pi_1$, $\pi_2\circ(f\times g)=f\circ\pi_2$ and f and g are irreducible maps, we have that $\pi_1(H')=P=\pi_2(K')$. Because the pseudo-arc P is a chainable continuum, we obtain a contradiction to Lemma 2.2. □

We notice that, if M is a continuum with $\sigma_q(M)=0$ and H is a subcontinuum of $M\times M$ such that $\pi_2(H)=M$, taking $K=\Delta_M$, where $\Delta_M=\{(x,x):x\in M\}$, we have that $H\cap\Delta_M\neq\emptyset$, so $\sigma_o^*(M)=0$ and we obtain the following.

Theorem 2.2. *Let M be a continuum. Then $\sigma(M)=0\Rightarrow\sigma_q(M)=0\Rightarrow\sigma_o^*(M)=0$.*

Theorem 2.3. *Let M be a continuum. Then $\sigma(M) = 0$ if and only if for each subcontinuum A of M, $\sigma_q(A) = 0$.*

Proof. Suppose $\sigma(M) = 0$. Let A be a subcontinuum of M, we have that $\sigma(A) = 0$. By the first implication in theorem 2.2, it follows that $\sigma_q(A) = 0$.

Conversely, suppose for each subcontinuum A of M, $\sigma_q(A) = 0$. It follows from the second implication in theorem 2.2 that for each subcontinuum A of M, $\sigma_o^*(A) = 0$. So, we have that $\sigma(M) = 0$. □

3. Applications

The following theorem was proved by K. Kawamura in [7, Theorem 3.2, p. 337].

Theorem. *Let $f : M \to N$ be an irreducible map which satisfies the following conditions:*

1) *For each map from the pseudo-arc onto N, $\alpha : P \to N$, there exists a continuum $Z \subset M \times P$ such that, $\pi_1(Z) = M$, $\pi_2(Z) = P$ and $f \circ \pi_1 |_Z = \alpha \circ \pi_2 |_Z$, and*

2) *$f \times f : M \times M \to N \times N$ is weakly confluent.*

If $\sigma(M) = 0$, then $\sigma^*(N) = 0$.

The following result generalizes the above theorem of Kawamura since we remove condition 1, we ask $\sigma_q(M) = 0$ which is weaker than $\sigma(M) = 0$, and obtain $\sigma_q(N) = 0$ which is stronger than $\sigma^*(N) = 0$.

Theorem 3.1. *Let M and N be continua and $f : M \to N$ be an irreducible map. Assume that the product map $f \times f$ is weakly confluent and $\sigma_q(M) = 0$. Then $\sigma_q(N) = 0$.*

Proof. Let H and K be subcontinua of $N \times N$ such that $\pi_1(H) = \pi_2(K) = N$. Since $f \times f$ is weakly confluent, there are subcontinua H' and K' of $M \times M$ such that $(f \times f)(H') = H$ and

$(f \times f)(K') = K$. Since $\pi_i \circ (f \times f) = f \circ \pi_i$, the irreducibility of f implies that $\pi_1(H') = \pi_2(K') = M$. It follows that $H' \cap K' \neq \emptyset$ and so $H \cap K \neq \emptyset$. □

It is known that open maps are weakly confluent and the product of open maps is open (see [4]). Therefore we have obtained the following result.

Corollary 3.1. *Let $f : M \to N$ be an open irreducible map from a continuum M onto a continuum N. Suppose $\sigma_q(M) = 0$, then $\sigma_q(N) = 0$.*

The referee suggested the following result.

Corollary 3.2. *Let $f : M \to N$ be an open map (not necessarily irreducible). If $\sigma_q(A) = 0$ for each subcontinuum A of M, then $\sigma_q(B) = 0$ for each subcontinuum B of N.*

Proof. First note that, by theorem 2.3, if $\sigma_q(A) = 0$ for each subcontinuum A of M, then $\sigma(M) = 0$. It is known that span zero is preserved by open maps (see [6, Theorem 1, p. 606]), hence $\sigma(N) = 0$. By theorem 2.3 $\sigma_q(B) = 0$ for each subcontinuum B of N. □

We recall that a continuum M is said to be strictly nonmutually aposyndetic if each pair of subcontinua in M which have the nonempty interior intersect, see [1].

Theorem 3.2. *Let M and N be indecomposable continua such that $\sigma_q(M, N) = 0$. Then the product $M \times N$ is strictly nonmutually aposyndetic.*

Proof. Suppose H and K are subcontinua of $M \times N$ with nonempty interior. So their projections $\pi_1(H)$ and $\pi_2(K)$ are subcontinua of M and N respectively, with nonempty interior. By the indecomposability of M and N we have that $\pi_1(H) =$

M and $\pi_2(K) = N$. Since $\sigma_q(M, N) = 0$ we have that $H \cap K \neq \emptyset$. □

C. Hagopian proved that the condition $M \times M$ is strictly non-mutually aposyndetic implies that M is indecomposable, see the proof of theorem 10 of [1, p. 621], in fact with slight modifications his proof will prove a more general result: if the product $M \times N$ is strictly non-mutually aposyndetic then both M and N are indecomposable continua.

He proved the following result when $M = N$ is a chainable continuum. Since chainability of M and N implies that the pair (M, N) has quasi-span equal to zero, the following result generalizes Hagopian's theorem.

Theorem 3.3. *Let M and N be continua such that $\sigma_q(M, N) = 0$. Then the product $M \times N$ is strictly non-mutually aposyndetic if and only if both M and N are indecomposable continua.*

From theorems 2.1 and 3.3 we obtain the following.

Corollary 3.3. *Let M and N be continua with span equal to zero. Then the product $M \times N$ is strictly non-mutually aposyndetic if and only if both M and N are indecomposable continua.*

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References

[1] C. L. Hagopian, *Mutual aposyndesis*, Proc. Amer. Math. Soc., 24 (1969) 615-622.

[2] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math. 55 (1964) 199-214.

[3] ______, *Some problems concerning curves*, Colloq. Math. 23 (1971) 93-98.

[4] ______, *Properties of mappings and continua theory*, Rocky Mountain J. Math., 6 (1976) 47-59.

[5] ______, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. 37 (1977) 35-45.

[6] K. Kawamura, *On some properties on span*, J. Math. Soc. Japan, Vol. 40, No. 4, (1988) 605-613.

[7] ______, *Span zero continua and the pseudo-arc*, Tsukuba J. Math. Vol. 14, No. 2 (1990) 327-341.

[8] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, N. Y., 1968.

[9] L. G. Oversteegen, *On products of confluent and weakly confluent mappings related to span*, Houston J. Math. Vol. 12, No. 1 (1986) 109-116.

[10] L.G. Oversteegen and E.D. Tymchatyn, *On span and weakly chainable continua,* Fund. Math. 122 (1984) 159-174.

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