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SPLITTABILITY OVER LINEAR ORDERINGS

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Abstract

A partial order X is splittable over a partial order Y if for every subset A of X there exists an order preserving mapping $f : X \to Y$ such that $f^{-1}f(A) = A$. We define a cardinal function $sc(X)$ (the 'splittability ceiling' for X) to be the least cardinal β such that the disjoint sum of β copies of X fails to split over a single copy of X. We allow $sc(X) = \infty$ to cover the case where arbitrarily many disjoint copies may be split. We investigate this cardinal function with respect to (linear) partial orders.

1. Introduction

A. V. Arhangel'skiĭ formulated and developed a range of definitions of splittability (or cleavability) in topology (see for example [1, 2]), of which the following are amongst the most basic.

Definition 1.1. *For topological spaces X and Y:*

– *X is splittable over Y along the subset A of X if there exists continuous $f : X \to Y$ such that:*

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(i) $f(A) \cap f(X \setminus A) = \emptyset$ *or, equivalently,*
(ii) $f^{-1}f(A) = A$.

– *X is splittable over Y if for every subset A of X there exists continuous $f : X \to Y$ such that $f^{-1}f(A) = A$.*

It quickly becomes apparent that splittability is not exclusively a topological idea. Indeed, only a routine translation into the language of the appropriate category is required for an analogous definition of splittability over other structures. (For example, splittability over semigroups is considered in [6].)

Definition 1.2. *Let X and Y be partially ordered sets (posets).*

– *A map f between partial orders is* increasing *(or* order preserving*) if $x \leq y$ implies $f(x) \leq f(y)$.*

– *X is splittable over Y along the subset A of X if there exists increasing $f : X \to Y$ such that $f^{-1}f(A) = A$.*

– *X is splittable over Y if for every subset A of X there exists increasing $f : X \to Y$ such that $f^{-1}f(A) = A$.*

The following result was obtained by D. J. Marron [4, 5]:

Theorem 1.3. *A poset X is splittable over the n-point chain if and only if:*

(i) X does not contain a chain of height greater than n, and

(ii) X does not contain two disjoint chains of height n.

Note 1.4. The previous result shows that it is not possible to split the (disjoint) sum of two copies of a finite chain over a single copy of the same finite chain. However, it is possible to disjointly embed two copies of ω (the positive integers with usual ordering) into a single copy of ω. Clearly, then, it is possible to split 'two disjoint copies' of ω over ω.

In general, suppose that α copies of a poset X can be disjointly embedded into a single copy of X. It is clear that the disjoint sum of α copies of X will split over a single copy of X. Indeed, if $(X \cdot \alpha)$ can be embedded into X, we can split the sum of α copies of X over X.

For notation and further information on linear orderings the interested reader is referred to [8].

Definition 1.5 (the 'splittability ceiling' for X)**.** *Let $sc(X)$ be the least cardinal β such that the (disjoint) sum of β copies of X fails to split over a single copy of X. We allow $sc(X) = \infty$ to cover the case where the sum of arbitrarily many disjoint copies may be split.*

Note 1.6. *The critical case for deciding $sc(X)$ is reached in attempting to split $2^{|X|}$ copies. If we have more than $2^{|X|}$ disjoint copies of X and split along some subset of their sum, then there must be copies which we are splitting along the same subset (since X has precisely $2^{|X|}$ subsets) and hence the 'same' map will do. In other words, if $sc(X) \geq 2^{|X|}$ then $sc(X) = \infty$.*

Definition 1.7. [8] *A cardinal number $\aleph_\alpha$ is said to be regular if it is not the sum of fewer than $\aleph_\alpha$ cardinal numbers smaller than $\aleph_\alpha$.*

Theorem 1.8. *For any partial order X, if $sc(X) \neq \infty$ then $sc(X)$ is a regular cardinal.*

Proof. Suppose $sc(X) = \lambda < \infty$ is not a regular cardinal; then λ can be expressed as the sum of α cardinals β_i each less than λ, where α is less than λ. Let $Y = \bigcup_{i \in \lambda} X_i$ be the disjoint union of λ copies of X. We can write $Y = \bigcup_{i \in \alpha} (\bigcup_{j \in \beta_i} X_j)$. For each $i \in \alpha$ we can split $\bigcup_{j \in \beta_i} X_j$ over a single copy X_{β_i} of X, since $\beta_i < \lambda$.

Likewise we can split $\bigcup_{i\in\alpha} X_{\beta_i}$ over a single copy of X, since $\alpha < \lambda$. Hence we can split λ copies of X (along any subset) over X - a contradiction. □

Proposition 1.9. *The splittability ceiling for the chain of positive integers ω is infinity (i.e. $sc(\omega) = \infty$).*

Proof. Given X, the (disjoint) sum of copies of ω and a subset A of X, we define a map $f : X \to \omega$ as follows:

$$f(x) = \begin{cases} x & \text{(if } x \in A \text{ and } x \text{ is odd) or } (x \notin A \text{ and } x \text{ is even}), \\ x+1 & \text{otherwise.} \end{cases}$$

It is clear that f is increasing and that $f(A)$ is a subset of the odds while $f(X \setminus A)$ is a subset of the evens. It follows that f splits X along A over ω as required. □

The corresponding result holds for the negative integers ω^* and for the integers $\omega^* + \omega$.

Proposition 1.10. *Let α be an ordinal (considered as a linear order). Then*

$$sc(\alpha) = \begin{cases} \infty & \text{if } \alpha \text{ is a limit ordinal} \\ 2 & \text{if } \alpha \text{ is a non-limit ordinal} \end{cases}$$

Proof. Note that each element in a limit ordinal has an immediate successor. The first part of the result follows from similar methods as employed for ω. If α is a non-limit ordinal we specify the subset A to contain the 'odd' ordinals less than α. Similarly we specify the subset B to contain the 'even' ordinals less than α. We can express $\alpha = \xi + n$ where ξ is a limit ordinal and n is finite. Now $f(x) \geq x$ for all $x \in \alpha$ and $f(x) = x$ for $x = \xi + i$ $(0 \leq i < n)$ whenever f is a map splitting α along A or B over α. Clearly it will not be possible to split the sum of two copies of α along A and B respectively over a single copy of α. □

Proposition 1.11. *The splittability ceiling for the chain of rationals η is infinity (i.e. $sc(\eta) = \infty$).*

Proof. Decompose η into two disjoint subsets C and D, each of which is dense in η. Enumerate both C and D in an arbitrary fashion. Given disjoint copies of η and a subset A to split along, define a map for each copy. Begin by enumerating the copy $X_1 = \{x_1, x_2, x_3, \ldots\}$. If $x_1 \in A$ ($\notin A$) map x_1 to the first point in the enumeration of C (D). The process continues inductively (using a method similar to that devised by Cantor to show that every countable linear order can be embedded into η). □

Proposition 1.12. *The splittability ceiling for the chain of the real numbers λ is $\mathfrak{c}^+$ (i.e. $sc(\lambda) = \mathfrak{c}^+$).*

Proof. We first note that it is possible to disjointly embed continuum-many copies of λ into λ. To prove the result we show that there are only continuum-many increasing maps from the reals into the reals. We know that there are only continuum-many maps from the rationals into the reals. Given increasing $f : \mathbb{R} \to \mathbb{R}$ consider its restriction to the rationals $f_{|\mathbb{Q}}$. For how many increasing maps $g : \mathbb{R} \to \mathbb{R}$ do we have $f_{|\mathbb{Q}} = g_{|\mathbb{Q}}$?

We can show that f and g can only differ at countably many points: for each irrational x select both a strictly decreasing sequence (a_n) and a strictly increasing sequence (b_n) of rationals, each converging to x. Now $(f(a_n))$ converges to some limit l while $(f(b_n))$ converges to some limit l'. If $l = l'$ then $f(x) = g(x) = l$, otherwise $f(x), g(x) \in [l', l]$ and $f(X) \cap [l', l] = \{f(x)\}$. Since there can only be countably many disjoint intervals in the reals, there can only be countably many points x where $f(x) \neq g(x)$.

It follows that there can only be continuum-many maps within each equivalence class. Hence there are at most continuum-many increasing maps from the reals to the reals. Clearly if we have more than continuum-many disjoint copies of λ and pick different subsets in them, then the union of these copies cannot be split over a single copy of λ along the union of these sets, due to the cardinality restriction on increasing maps. $\square$

Similar arguments can be used to locate an upper bound for the number of increasing maps from any linear order into itself. The most interesting case appears to be that of the countable linear orders. Moreover, unless the order is scattered it can be shown that the splittability ceiling will be infinity. This follows since any non-scattered linear order will contain a copy of η and we already know that $sc(\eta) = \infty$.

2. Countable Linear Orderings

Lemma 2.1. *Let X be a partial order. If $sc(X) > 2$ then $sc(X) \geq \aleph_0$.*

Proof. Let X_1, X_2, X_3 be disjoint copies of the partial order X, with subsets A_1, A_2, A_3 respectively. Let X_4 be a fourth copy of X. Since $sc(X) > 2$ we can split $X_1 \cup X_2$ along $A_1 \cup A_2$ over X_4 using an increasing map f (i.e. $f^{-1}f(A_1 \cup A_2) = A_1 \cup A_2$). Now split $X_3 \cup X_4$ along $B \cup A_3$ (where $B = f(A_1 \cup A_2)$) over X using an increasing map g (i.e. $g^{-1}g(f(A_1 \cup A_2) \cup A_3) = f(A_1 \cup A_2) \cup A_3$).

Define a map $h : X_1 \cup X_2 \cup X_3 \to X$ by

$$h(x) = \begin{cases} g \circ f(x) & \text{if } x \in X_1 \cup X_2, \\ g(x) & \text{if } x \in X_3; \end{cases}$$

then h splits $X_1 \cup X_2 \cup X_3$ along $A_1 \cup A_2 \cup A_3$ over X; for suppose $x \in X_1 \cup X_2 \cup X_3$ and

$$\begin{aligned} h(x) \in h(A_1 \cup A_2 \cup A_3) &= h(A_1 \cup A_2) \cup h(A_3) \\ &= g \circ f(A_1 \cup A_2) \cup g(A_3) \\ &= g(f(A_1 \cup A_2) \cup A_3). \end{aligned}$$

If $x \in X_3$ then $h(x) = g(x) \in g(f(A_1 \cup A_2) \cup A_3)$ so $x \in f(A_1 \cup A_2) \cup A_3$ and $x \in A_3$. If $x \in X_1 \cup X_2$ then $h(x) = g \circ f(x) \in g(f(A_1 \cup A_2) \cup A_3)$ so $f(x) \in f(A_1 \cup A_2) \cup A_3$.

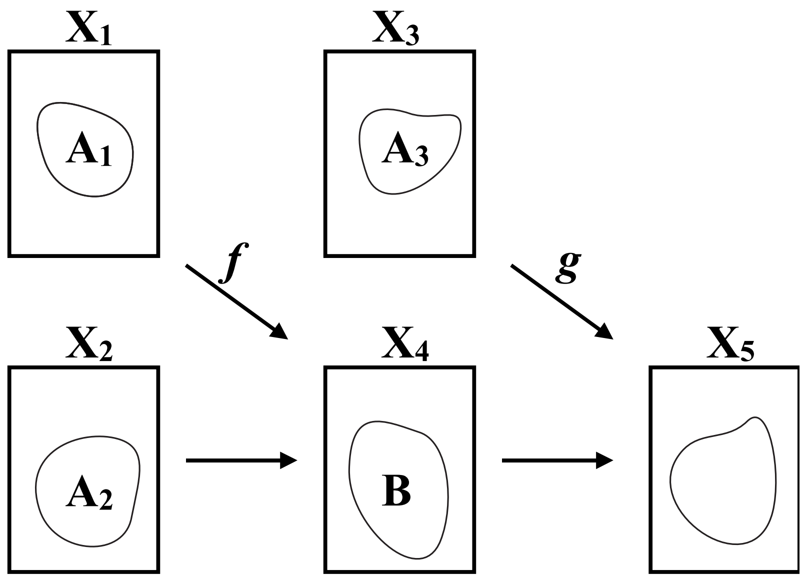

Fig. 1. Splitting 3 copies of X over a single copy of X

Hence $f(x) \in f(A_1 \cup A_2)$ and $x \in A_1 \cup A_2$.
Clearly this argument can be extended by induction so that $sc(X) > n$ for all $n \in \mathbb{N}$. □

Corollary 2.2. *Let X be a finite partial order; then $sc(X) = 2$ or ∞.*

We show now that the previous result extends to countable linear partial orders. To do so, we employ the notion of an 'order shuffling' and a result due to J. L. Orr.

Definition 2.3. [7] *Let A be a countable linearly ordered set. A function $f : A \to \mathbb{N}^+$ is called an* order shuffling on *A. A linearly ordered set B* shuffles into *(A, f) if there is an increasing surjection σ from B onto A such that the cardinality of $\sigma^{-1}\{a\}$ is at least $f(a)$ for all but finitely many $a \in A$. If this holds for all $a \in A$ then B shuffles into (A, f)* exactly.

Theorem 2.4. [7] *Let A be a countable scattered linear ordering and let f be an order shuffling on A; then A shuffles into (A, f).*

Lemma 2.5. *Let X be a countable scattered linear order; then there exist an order preserving surjection $\pi : X \to X$ and points $\{a_1, a_2, \ldots, a_n\}$ such that:*

(i) $|\pi^{-1}(x)| > 1$ *for each* $x \in X \setminus \{a_1, a_2, \ldots, a_n\}$*, and*

(ii) $\pi^{-1}(a_i) = \{a_i\}$ *for each* $i \in \{1, 2, \ldots, n\}$*.*

Proof. We use Theorem 2.4 to find an order preserving surjection $\pi : X \to X$ such that $|\pi^{-1}(x)| > 1$ for all but n elements $\{a_1, a_2, \ldots, a_n\}$. We assume that n is minimal and that $a_1 < a_2 < \ldots < a_n$.

Note that if $\pi^{-1}(\{a_1, a_2, \ldots, a_n\}) \subseteq \{a_1, a_2, \ldots, a_n\}$ then since π is order preserving $\pi^{-1}(a_i) = \{a_i\}$. Suppose that π does not exhibit property (ii); then there exists i such that the singleton pre-image of a_i under π is not contained in $\{a_1, a_2, \ldots, a_n\}$. Let $\rho = \pi \circ \pi$ and consider $\rho^{-1}(x)$ for some $x \in X$. If $x \notin \{a_1, a_2, \ldots, a_n\}$ then $|\pi^{-1}(x)| > 1$, hence $|\rho^{-1}(x)| > 1$.

If $x = a_j$ for $j \neq i$ then clearly $|\rho^{-1}(x)| \geq 1$, but if $x = a_i$ then we can find $y \in X \setminus \{a_1, a_2, \ldots, a_n\}$ such that $\pi(y) = x$. Now $|\pi^{-1}(y)| > 1$, so $|\rho^{-1}(x)| > 1$, but ρ now contradicts the minimality of n. □

Theorem 2.6. *Let X be a countable linear ordering; then $sc(X) = 2$ or $sc(X) = \infty$.*

Proof. We know that if X is not scattered, then X contains a copy of the rationals, so $sc(X) = \infty$. We also know that if $sc(X) > 2$ then $sc(X) \geq \aleph_0$, that is, we can split the sum of any finite number of copies of X over a single copy. Let X be a countable scattered linear order such that $sc(X) > 2$. Using Lemma 2.5 it is possible to find an increasing surjection $\pi : X \to X$ and points $\{a_1, a_2, \ldots, a_n\}$ such that:

(i) $|\pi^{-1}(x)| > 1$ for each $x \in X \setminus \{a_1, a_2, \ldots, a_n\}$ and

(ii) $\pi^{-1}(a_i) = \{a_i\}$ for each $i = 1, 2, \ldots, n$.

For each $x \in X \setminus \{a_1, a_2, \ldots, a_n\}$ choose $x_1, x_2 \in \pi^{-1}(x)$ with $x_1 < x_2$.
Let $Y = \bigcup_{i \in \beta} X_i$ be the disjoint union of β copies of X. Let $A = \bigcup_{i \in \beta} A_i$ where $A_i \subseteq X_i$. For each subset B of $\{a_1, a_2, \ldots, a_n\}$ let X_B be a copy of X. For each $i \in \beta$ let $C_i = A_i \cap \{a_1, a_2, \ldots, a_n\}$ and define a map $f_i : X_i \to X_{C_i}$ as follows:

$$f_i(x) = \begin{cases} a_i & \text{if } x = a_i, \\ x_1 & \text{if } x \in A_i \setminus \{a_1, a_2, \ldots, a_n\}, \\ x_2 & \text{if } x \notin A_i \cup \{a_1, a_2, \ldots, a_n\}. \end{cases}$$

These maps can be used to split Y along A over 2^n copies of X (using f say), which can in turn be split along $f(A)$ over a single copy of X. Hence we can split β copies of X over X, so $sc(X) = \infty$. □

Note 2.7. *Given a countable scattered linear order X, for $x, y \in X$ we set $x \equiv y$ if and only if there are only finitely many $z \in X$ such that $x < z < y$ or $y < z < x$, and thus obtain an equivalence relation on X. Let us denote the equivalence class of a point $x \in X$ by $e(x)$. Now we can determine a subset A of X such that between each two points in A we can find a point not in A and vice versa. The first step is to select a point x from each equivalence class. We assign a point $y \in e(x)$ to the set A if there are an even number of points between x and y (inclusive). We say that A and $X \setminus A$ alternate in X. Note that this only works because the order under consideration is scattered.*

Lemma 2.8. *Let X be a countable scattered linear order with $sc(X) > 2$. For each $x \in X$ there exists an order preserving injection $f : X \to X$ such that $x \notin f(X)$.*

Proof. Let $x \in X$, where X is a countable scattered linear order with $sc(X) > 2$. Choose a subset A of X that alternates in X

as described in Note 2.7. Let $Y = X_1 \cup X_2$ be the disjoint union of 2 copies of X. Let $B = A_1 \cup A_2$ where $A = A_1 \subseteq X_1$ and $X \setminus A = A_2 \subseteq X_2$. Choose f that splits Y along B over X and set $f_i = f_{|X_i}$ for $i = 1, 2$. The choice of A ensures that both f_1 and f_2 are order preserving injections. If $f_1(X_1)$ or $f_2(X_2)$ do not contain x we have found a suitable map. Otherwise we can find distinct $a_1, a_2 \in X$ such that $f_1(a_1) = f_2(a_2) = x$. Now $a_1 < a_2$ say, so define a map $g : X \to X$ by

$$g(z) = \begin{cases} f_2(z) & \text{for } z < a_2 \\ f_1(z) & \text{for } z \geq a_2. \end{cases}$$

This map is an order preserving injection and $x \notin g(X)$. □

Lemma 2.9. *Let X be a countable scattered linear order such that for each $x \in X$ there exists an order preserving injection $f : X \to X$ such that $x \notin f(X)$. If A is a finite subset of X there exists an order preserving injection $g : X \to X$ such that $A \cap g(X) = \emptyset$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of X. Suppose that there exists an order preserving injection $g : X \to X$ such that $g(X) \cap \{a_1, a_2, \ldots, a_k\} = \emptyset$. If $k < n$, then either $a_{k+1} \notin g(X)$ or there exists $b \in X$ such that $g(b) = a_{k+1}$. In the first case, let $h = g$, and in the second case, choose an order preserving injection $f : X \to X$ such that $b \notin f(X)$ and set $h = g \circ f$. Now h is an order preserving injection and $h(X) \cap \{a_1, a_2, \ldots, a_{k+1}\} = \emptyset$, and we repeat the above argument. When $k = n$, we are done. □

Theorem 2.10. *Let X be a countable linear order; then $sc(X) = \infty$ if and only if $2 \cdot X$ order embeds into X.*

Proof. We need only prove that if X is a countable linear order and $sc(X) = \infty$ then $2 \cdot X$ order embeds into X. If X is not scattered then X contains a subset isomorphic to the rationals. Since every countable linear order embeds into the rationals (see

[8]) clearly $2 \cdot X$ order embeds into X. We assume now that X is scattered. First find an increasing surjection $\sigma : X \to X$ such that $|\sigma^{-1}(x)| \geq 2$ for all $x \in X \setminus \{a_1, a_2, \ldots, a_m\}$. It is possible (via Lemmas 2.8 and 2.9) to find an order preserving injection $f : X \to X$ such that $a_i \notin f(X)$ for all i.

Set $Y = \sigma^{-1}(f(X)) \subseteq X$ and define $\pi : Y \to X$ by $\pi = f^{-1} \circ \sigma$. It follows that π is order preserving and that $|\pi^{-1}(x)| \geq 2$ for all $x \in X$.

Select, for each x, two points $x_0, x_1 \in \pi^{-1}(x)$ with $x_0 < x_1$. Define $\phi : \{0,1\} \times X \to X$ by

$$\phi(i,x) = \begin{cases} x_0 & \text{if } i = 0, \\ x_1 & \text{if } i = 1. \end{cases}$$

Clearly, ϕ order embeds $2 \cdot X$ into X. □

Lemma 2.11. *The following statements are equivalent for any linear order X:*

(i) $2 \cdot X$ *order embeds into* X*,*

(ii) $n \cdot X$ *order embeds into* X *for all* $n \in \mathbb{N}$*,*

(iii) $n \cdot X$ *order embeds into* X *for some* $n \in \mathbb{N}$ *where* $n > 1$*.*

Proof. We prove first that (i) implies (ii). Let X be a linear order such that $2 \cdot X$ order embeds into X; that is, there exists an order preserving injection $f : \{0,1\} \times X \to X$. Suppose that $k \cdot X$ order embeds into X for all $k < n$ for some $n \in \mathbb{N}$. Hence there exists an order preserving injection $g : \{0, 1, \ldots, k-2\} \times X \to X$. Define a map $h : \{0, 1, \ldots, k-1\} \times X \to 2 \cdot X$ as follows:

$$h(i,x) = \begin{cases} (0, g(i,x)) & \text{if } i < k-1, \\ (1, g(k-2,x)) & \text{if } i = k-1. \end{cases}$$

Now define $\pi : \{0, 1, \ldots, k-1\} \times X \to X$ as $\pi = f \circ h$. It follows that π is an order preserving injection so by induction we have shown that $n \cdot X$ order embeds into X for all $n \in \mathbb{N}$. That (ii) implies (iii) is trivial. Finally (iii) implies (i) since $2 \cdot X$ will clearly order embed into $n \cdot X$ for any $n > 1$. □

Theorem 2.12. *Let X be a countable linear order. Then $sc(X) = \infty$ if and only if $sc(n \cdot X) = \infty$ for all $n \in \mathbb{N}$.*

Proof. If $sc(X) = \infty$ then $2 \cdot X$ (and hence $k \cdot X$ for all $k \in \mathbb{N}$) will order embed into X by Lemma 2.11. It follows that $2n \cdot X$ will order embed into $n \cdot X$ and hence into X, a sufficient condition for $sc(n \cdot X) = \infty$.

If $sc(n \cdot X) = \infty$ then $2n \cdot X$ order embeds into $n \cdot X$ by Theorem 2.10. That is, we can find an order preserving injection $f : \{0, 1, \ldots, 2n-1\} \times X \to \{0, 1, \ldots, n-1\} \times X$. For any $x \in X$ we can find $x', x'' \in X$ such that:

$$f(0, x) \leq (0, x') < (0, x'') \leq f(2n-1, x).$$

Define a map $g : \{0, 1\} \times X \to X$ by

$$g(i, x) = \begin{cases} x' & \text{if } i = 0, \\ x'' & \text{if } i = 1. \end{cases}$$

Clearly g is an order preserving injection that order embeds $2 \cdot X$ into X, a sufficient condition for $sc(X) = \infty$ by Theorem 2.10. □

Theorem 2.13. *Let X be a countable linear order. Then $sc(X) = \infty$ if and only if $n \cdot X$ order embeds into X for all $n \in \mathbb{N}$.*

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