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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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PROPERTIES RELATED TO COMPACTNESS IN HYPERSPACES

L'. Holá and H.P. Künzi*

Abstract

Let X be a Hausdorff topological space and $CL(X)$ the family of all nonempty closed subsets of X. Countable compactness, ultracompactness and ω-boundedness are all different for the Vietoris (finite) topology on $CL(X)$. The Fell topology on $CL(X)$ is countably compact (ultracompact, ω-bounded) if and only if X is countably compact (ultracompact, ω–bounded). We also give a partial answer to a question in [Ke1].

1. Introduction

Let X be a T_1 topological space, $CL(X)$ the family of all nonempty closed subsets of X, τ_V the Vietoris topology on $CL(X)$, τ_F the Fell topology on $CL(X)$ and τ_K the topologization of the Kuratowski-Painleve convergence on $CL(X)$. In our paper we are interested in countable compactness and ω-boundedness of these topologies. As far as we know the only papers in which we can find some results concerning this subject are Keesling's articles [Ke1, Ke2, Ke3] and [Gi1,Gi2,Na].

In [Ke1] Keesling showed that $(CL(X), \tau_V)$ can be countably compact and noncompact, in the way that he presented some

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sufficient conditions for countable compactness of the Vietoris topology which fail to force it to be compact. But these conditions - X is a T_1 normal ω-bounded space - give simultaneously also ω-boundedness of the Vietoris topology.

We present an example of a non normal Tychonoff ω-bounded space, for which $(CL(X), \tau_V)$ is ω-bounded; this example also gives a partial answer to a question in [Ke1].

A natural question arises whether countable compactness and ω-boundedness are different for the Vietoris topology on $CL(X)$.

If X is a linearly ordered topological space, then countable compactness and ω-boundedness coincide on $(CL(X), \tau_V)$. We show that in general these two notions are different for the Vietoris topology on $CL(X)$; we show even more - that countable compactness, ultracompactness and ω- boundedness are all different for the Vietoris topology on $CL(X)$.

In our paper we also prove that the Fell topology on $CL(X)$ is countably compact if and only if X is countably compact and in the class of T_2 spaces the same is true also for ω- boundedness.

2. Preliminaries

Throughout the paper let X be a T_1 topological space, $CL(X)$ the family of all nonempty closed subsets of X and $K(X)$ the family of all nonempty compact subsets of X.

For a subset E of X, we associate the following subsets of $CL(X)$:

$$E^- = \{A \in CL(X) : \ A \cap E \neq \emptyset\}, E^+ = \{A \in CL(X) : \ A \subseteq E\}.$$

The Vietoris (finite) topology τ_V [Mi] on $CL(X)$ is the supremum of the lower Vietoris topology and the upper Vietoris topology on $CL(X)$, where the lower Vietoris topology is generated by all subcollections of the form G^-, where G is an open subset of X; the upper Vietoris topology is generated by all sets of the form $(K^c)^+$, where $K \in CL(X)$ and K^c is the complement of K.

If in the definition of the upper Vietoris topology K runs over all sets from $K(X) \cap CL(X)$, we obtain the so-called co-compact topology on $CL(X)$. The supremum of the lower Vietoris topology and the co-compact topology is the well-known Fell topology [Fe], a further classical hyperspace topology.

If $\{A_\sigma : \sigma \in \Sigma\}$ is a net in $CL(X)$ then by LiA_σ [Be] we denote the lower closed limit of $\{A_\sigma : \sigma \in \Sigma\}$ and by LsA_σ we denote the upper closed limit of $\{A_\sigma : \sigma \in \Sigma\}$. We declare $\{A_\sigma : \sigma \in \Sigma\}$ Kuratowski-Painleve convergent to A, and write $A = K - limA_\sigma$, provided $LiA_\sigma = LsA_\sigma = A$.

For every topological space X there is the strongest topology τ_K on $CL(X)$ such that Kuratowski-Painleve convergence of an arbitrary net $\{A_\sigma : \sigma \in \mathit{\Sigma}\}$ in $CL(X)$ to some $A \in CL(X)$ implies τ_K-convergence of $\{A_\sigma : \sigma \in \mathit{\Sigma}\}$ to A. This topology is often referred to as the topologization of the Kuratowski-Painleve convergence.

For every topological space X we have $\tau_F \subseteq \tau_K$, $\tau_F \subseteq \tau_V$ on $CL(X)$ and if X is regular then also $\tau_K \subseteq \tau_V$ on $CL(X)$.

3. Ultracompact Spaces and ω-Bounded Spaces

Definition 3.1. [Br] Let $\mathcal{D}$ be an ultrafilter over the positive integers I. An element x of X is a $\mathcal{D}$-limit point of (x_i) if and only if given any neighbourhood U of x, $\{i : x_i \in U\} \in \mathcal{D}$.

This definition is of interest only in the case that $\mathcal{D}$ is non-principal since if $\mathcal{D}$ is principal, say $\mathcal{D} = \{S \subseteq I : k \in S\}$ for some $k \in I$, then any sequence (x_i) has as $\mathcal{D}$-limit point simply x_k. If $\mathcal{D}$ is non-principal then any set in $\mathcal{D}$ is infinite so any neighbourhood about a $\mathcal{D}$-limit point x of (x_i) must contain an infinite number of the x_i; thus x is in particular a cluster point of (x_i).

Definition 3.2. [Br] A topological space X is $\mathcal{D}$-compact if and only if every sequence of points of X has a $\mathcal{D}$-limit point.

Clearly if X is $\mathcal{D}$-compact for some non-principal ultrafilter $\mathcal{D}$ then X is in particular countably compact. (Note that for principal $\mathcal{D}$, any space is $\mathcal{D}$- compact.)

Definition 3.3. [Br] A topological space X is ultracompact if and only if given any ultrafilter $\mathcal{D}$ over I, X is $\mathcal{D}$-compact.

Proposition 3.4. *A topological space X is ultracompact if and only if every net in X with a countable range has a cluster point.*

Proof. Suppose first that X is ultracompact. Let $\{a_\sigma : \sigma \in \Sigma\}$ be a net in X with a countable range. Let $\{c_1, c_2, ...c_n, ...\}$ be an enumeration of its values. For every $\sigma \in \Sigma$ put

$$B(\sigma) = \{n \in I : a_\eta = c_n \text{ for some } \eta \geq \sigma\}.$$

Then $\mathcal{L} = \{B(\sigma) : \sigma \in \Sigma\}$ forms a filter-base in I; let $\mathcal{F}_\mathcal{L}$ be a filter generated by the filterbase $\mathcal{L}$ and let $\mathcal{H}$ be an ultrafilter which contains $\mathcal{F}_\mathcal{L}$. By the assumption there is an $\mathcal{H}$-limit point, say x of $\{c_1, c_2, ...c_n, ...\}$. We claim that x is also a cluster point of $\{a_\sigma : \sigma \in \Sigma\}$. Let V be an open neighbourhood of x and $\sigma_0 \in \Sigma$. We claim there is $\sigma \geq \sigma_0$ such that $a_\sigma \in V$. By the assumption $\{n \in I : c_n \in V\} \in \mathcal{H}$. Thus $B(\sigma_0) \cap \{n \in I : c_n \in V\} \in \mathcal{H}$; since also $B(\sigma_0) \in \mathcal{L}$. Thus $B(\sigma_0) \cap \{n \in I : c_n \in V\} \neq \emptyset$. Let $k \in B(\sigma_0) \cap \{n \in I : c_n \in V\}$. Then $c_k \in V$ and also $c_k = a_\eta$ for some $\eta \geq \sigma_0$; i.e. $a_\eta \in V$.

Suppose now that every net in X with a countable range has a cluster point. We prove that X is ultracompact. Let $\mathcal{D}$ be an ultrafilter over I. We have to prove that every sequence of points of X has a $\mathcal{D}$-limit point. Let $\{c_n : n \in I\}$ be a sequence of points of X. Let "$\leq$" be the partial order on $\mathcal{D}$ corresponding to the reverse inclusion. For every $A \in \mathcal{D}$ let $n(A)$ be the first element from A and put $x_A = c_{n(A)}$. Now, by the assumption $\{x_A : A \in \mathcal{D}\}$ has a cluster point, say $x \in X$. We claim that x is also a $\mathcal{D}$-limit point of $\{c_n : n \in I\}$. Let V be an open neighbourhood of x. It is easy to verify that for every $A \in \mathcal{D}$ we have $\{i : c_i \in V\} \cap A \neq \emptyset$. (For $A \in \mathcal{D}$ there is $B \in \mathcal{D}$ such that $B \geq A$, $x_B \in V, x_B = c_{n(B)}$ and $n(B) \in B \subseteq A$.) Thus the set $\{i : c_i \in V\}$ belongs to $\mathcal{D}$, since $\mathcal{D}$ is ultrafilter and $\{i : c_i \in V\} \cap A \neq \emptyset$ for every $A \in \mathcal{D}$. □

Definition 3.5. [GFW, Va] A topological space X is ω-bounded provided for every sequence in X its range is contained in a compact subset of X.

Of course every ω-bounded space is ultracompact and every ultracompact space is countably compact. In T_2 spaces ω-boundedness is equivalent to the condition that every countable set has compact closure (such spaces are called strongly countably compact in [Ke1]); in general these two conditions are different.

The following proposition shows that in regular spaces the notions of ultracompactness and ω-boundedness coincide and in the next section we present an example of a T_2 ultracompact space which fails to be ω-bounded.

Proposition 3.6. *Let X be a regular space. If X is ultracompact then X is ω-bounded.*

Proof. Suppose that X is not ω-bounded. Let A be a countable subset of X whose closure $B = \overline{A}$ is not compact. Then, B has an open cover $\mathcal{V}$ such that $B \setminus \cup\mathcal{V}_0 \neq \emptyset$ for every finite $\mathcal{V}_0 \subseteq \mathcal{V}$. Since X is regular, there exists an open cover $\mathcal{W}$ of B such that $\mathcal{V}$ is refined by $\{\overline{W} : W \in \mathcal{W}\}$. Let Σ be the set of all finite subsets of $\mathcal{W}$, and let "$\leq$" be the partial order on Σ corresponding to the usual set-theoretical inclusion "$\subseteq$". Now, by assumption, for every $\sigma \in \Sigma$ there exists a point $a_\sigma \in A \setminus \overline{\cup\sigma}$. Thus, we get a net $\{a_\sigma : \sigma \in \Sigma\}$ with a countable range. Hence, by the assumption, it has a cluster point $x \in B$. Since $\mathcal{W}$ covers B, there exists $\sigma_0 \in \Sigma$ such that $x \in \cup\sigma_0$. Since x is a cluster point, there also exists $\sigma \geq \sigma_0$ such that $a_\sigma \in \cup\sigma_0 \subseteq \cup\sigma$. However, by the construction, $a_\sigma \in A \setminus \cup\sigma$. We have reached a contradiction. □

For X completely regular the above proposition was proved in [Br] (with entirely different proof).

The property of being ultracompact has very interesting consequences. It is productive, closed hereditary, and preserved under continuous mappings.

From Theorem 2.1 in [Gi1] we can deduce that a Hausdorff space X is ultracompact if and only if $(CL(X), \tau_V)$ is ultracompact. It is easy to verify that the same is true also for T_1 space.

Proposition 3.7. *Let X be a T_1 topological space. Then X is ultracompact if and only if $(CL(X), \tau_V)$ is ultracompact.*

Of course the claim of Proposition 3.7 holds for τ_F, τ_K and for every topology on $CL(X)$ which is weaker than the Vietoris topology and finer than the lower Vietoris topology.

4. Countable Compactness and ω-Boundedness in Hyperspaces

There is no satisfactory characterization of countable compactness of the Vietoris topology on $CL(X)$ in the literature. The following Proposition offers such a characterization for countable compactness of the Fell topology.

Proposition 4.1. *Let X be a T_1 topological space. The following are equivalent:*

(1) X is countably compact;

(2) $(CL(X), \tau_K)$ is countably compact;

(3) $(CL(X), \tau_F)$ is countably compact.

Proof. (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are clear. To prove (1) $\Rightarrow$ (2) let $\{C_n : n \in I\}$ be a sequence in $CL(X)$. We show that there is a τ_K-cluster point of $\{C_n : n \in I\}$. For every $n \in I$ choose $c_n \in C_n$. The countable compactness of X implies that there is a cluster point c of $\{c_n : n \in I\}$. Thus there is a net $\{l_m : m \in M\}$ which is a subnet of $\{c_n : n \in I\}$ convergent to c (i.e. there is a directed set M and a function $f : M \to I$ with $l_m = c_{f(m)}$ for $m \in M$ and such that for every $n_0 \in I$ there is $m \in M$

with $f(m) \geq n_0$ whenever $m \geq m_0$). For every $m \in M$ put $L_m = C_{f(m)}$. Then $\{L_m : m \in M\}$ is a subnet of $\{C_n : n \in I\}$. By Mrowka's theorem [Mr] there is a subnet of $\{L_m : m \in M\}$ which is Kuratowski-Painleve convergent to a closed set A. Since $c \in LiL_m$ and $LiL_m \subseteq A$, $A \neq \emptyset$. Thus the nonempty closed set A is a τ_K-cluster point of $\{C_n : n \in I\}$. □

Concerning ω-boundedness of the Fell topology we have the following result:

Proposition 4.2. *Let X be a T_2 topological space. The following are equivalent:*

(1) X is ω-bounded;

(2) $(CL(X), \tau_K)$ is ω-bounded;

(3) $(CL(X), \tau_F)$ is ω-bounded.

Proof. (2) $\Rightarrow$ (3) is clear. To prove that (3) $\Rightarrow$ (1) realize that the Hausdorffness of X implies that X is a closed subspace of $(CL(X), \tau_F)$. Thus also X is ω-bounded as a closed subspace of an ω-bounded space.

(1) $\Rightarrow$ (2) Now we prove that ω-boundedness of X implies ω-boundedness of $(CL(X), \tau_K)$. Let $\{C_n : n \in I\}$ be a sequence in $CL(X)$. For every $n \in I$ take $c_n \in C_n$. By ω-boundedness of X there is a compact set K in X containing all c_n. We prove that K^- is τ_K-compact. Then we are done, since every $C_n \in K^-$. Let $\{F_\sigma : \sigma \in \Sigma\}$ be a net in K^-. For every $\sigma \in \Sigma$ let $f_\sigma \in F_\sigma \cap K$. By compactness of K there is a cluster point $x \in K$ of the net $\{f_\sigma : \sigma \in \Sigma\}$. Thus there is a net $\{l_m : m \in M\}$ which is a subnet of $\{f_\sigma : \sigma \in \Sigma\}$ convergent to x. Now we can use the same argument as above to show that there is a subnet $\{L_m : m \in M\}$ of the net $\{F_\sigma : \sigma \in \Sigma\}$ such that $l_m \in L_m$ for every $m \in M$. By Mrowka's theorem [Mr] there is a subnet of $\{L_m : m \in M\}$ which is Kuratowski-Painleve convergent to a closed set A. Since $x \in A$, we can deduce that there is a subnet of $\{F_\sigma : \sigma \in \Sigma\}$ which τ_K-converges to $A \in K^-$. □

Notice that to prove (1) $\Rightarrow$ (2) it is sufficient to suppose that X is only T_1.

Since for linearly ordered topological spaces countable compactness and ω-boundedness are equivalent notions [GFW], we have also the coincidence of these notions for $(CL(X), \tau_K)$ as well as for $(CL(X), \tau_F)$.

The same holds also for the Vietoris topology.

Proposition 4.3. *Let X be a linearly ordered topological space. The following are equivalent:*

(1) $(CL(X), \tau_V)$ is countably compact;

(2) $(CL(X), \tau_V)$ is ω-bounded.

Proof. (1) $\Rightarrow$ (2) Countable compactness of $(CL(X), \tau_V)$ implies countable compactness of X. By [GFW] X is also ω-bounded. Since every linearly ordered topological space is T_1 normal, by Theorem 5 in [Ke1] we have that $(CL(X), \tau_V)$ is ω-bounded. □

Of course, in general these two notions are not equivalent for the Vietoris topology; more precisely there is a topological space X for which $(CL(X), \tau_V)$ is countably compact but fails to be ultracompact and there is a topological space X for which $(CL(X), \tau_V)$ is ultracompact but fails to be ω-bounded.

Consider a topological space X from Example 4 in [Br]. Then X is a separable completely regular space which is $\mathcal{D}$-compact for a given non-principal ultrafilter $\mathcal{D}$ but which is not compact. Of course, X cannot be ultracompact by Proposition 3.6 (X is regular, not ω-bounded). For this topological space X $(CL(X), \tau_V)$ is $\mathcal{D}$-compact by Theorem 2.1 in [Gi1] thus also countably compact which fails to be ultracompact, since X is not ultracompact (Proposition 3.7).

In this section we give an example of a T_2 ultracompact space X, which is not ω-bounded. Such an example of a space is missing in the literature. For such a space X $(CL(X), \tau_V)$ is ultracompact (see Proposition 3.7), but $(CL(X), \tau_V)$ fails to be ω-bounded, since X is a closed subspace of $(CL(X), \tau_V)$ and ω-boundedness is closed hereditary. Of course such a space X cannot be regular (see Proposition 3.6). There are also non regular T_2 ω-bounded spaces for which $(CL(X), \tau_V)$ is ω-bounded (see Remark 5.2 below).

If X is a Hausdorff space, the situation in $(K(X), \tau_V)$ is different; countable compactness and ω-boundedness coincide in $(K(X), \tau_V)$ ([Ml]). In fact it is equivalent to the condition that in X the countable union of compact sets has a compact closure. It is readily seen that for spaces X, where every $K \in K(X)$ is separable, ω-boundedness of X is equivalent to ω-boundedness of $(K(X), \tau_V)$; the space $X = \Sigma(0)$ - the Σ-product of $[0,1]^{\omega_1}$ is an example of an ω-bounded space, for which $(K(X), \tau_V)$ is not ω-bounded.

Example 4.4. Consider the one-point compactification X of the locally compact T_2 space Z from the Example 7.1 in [D]. (Facts important for our analysis: Z can be expressed as $k \cup \omega$, where k is a cardinal of uncountable cofinality, k is a closed subset of Z whose subspace topology is the usual order topology on k and ω is a countable dense set of isolated points in the topology of Z.) We define a new space Y on the carrier set of X by modifying the neighbourhoods of ∞ as follows: A basic neighbourhood of ∞ in Y is $(G \cap \omega) \cup \{\infty\}$ where G is an open set containing ∞ in X. Since the topology of Y is strictly finer than the one of X, Y is a Hausdorff noncompact space. Therefore Y is not even ω-bounded, since ω is a countable dense subspace of Y. To show that Y is ultracompact let $\{x_\sigma : \sigma \in \Sigma\}$ be a net without a cluster point in Y. Then necessarily $\{x_\sigma : \sigma \in \Sigma\}$ converges to ∞ in the compact space X; otherwise it would have a cluster point different from ∞ in X and thus in Y. Since ∞ is not a cluster point of $\{x_\sigma : \sigma \in \Sigma\}$ in Y, there are an open neighbourhood G of ∞ in X and $\sigma_0 \in \Sigma$

such that $x_\sigma \notin (G \cap \omega) \cup \{\infty\}$ whenever $\sigma \in \Sigma$ and $\sigma \geq \sigma_0$. Since the net $\{x_\sigma : \sigma \geq \sigma_0\}$ converges in X to ∞, we conclude that $\{x_\sigma : \sigma \in \Sigma\}$ cannot have a countable range, since k is a cardinal of uncountable cofinality. Thus we showed that every net without a cluster point in Y cannot have a countable range. Hence Y is ultracompact.

5. ω-Boundedness of the Vietoris Topology

Keesling proved that if X is a T_1 normal ω-bounded space, then $(CL(X), \tau_V)$ is ω-bounded too, and he mentioned that it would be interesting to know whether the normality of X in this result could be weakend [Ke1]. In the last part of our paper we present an example of a non normal Tychonoff ω-bounded space X, for which $(CL(X), \tau_V)$ is still ω-bounded; this example offers also a partial answer to a question in [Ke1] after Theorem 5.

Example 5.1. Let $X = (\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ be equipped with the usual topology. Let $\{R_n : n \in \omega\}$ be a sequence in $CL(X)$. Let $\{F_d : d \in D\}$ be a net in the τ_V-closure of $\{R_n : n \in \omega\}$ which Kuratowski converges to a nonempty closed set C (for 'nonempty' see the beginning of the proof of Theorem 5.3). We show that C is a τ_V-cluster point of $\{F_d : d \in D\}$. Suppose not. Thus there are $F \in CL(X)$ and $d_* \in D$ with $C \cap F = \emptyset$ and $F_d \cap F \neq \emptyset$ for every $d \geq d_*$. Hence F is not compact and for every $d \geq d_*$ the net $\{F_l : l \geq d\}$ cannot have a countable range (by Propositions 3.4 and 3.7). Put

$$U = \omega_1 \times \{\omega_1\}, \quad L = \{\omega_1\} \times \omega_1$$

and put

$$E = C \quad \text{if} \quad C \cap U \text{ and } C \cap L \text{ are both bounded},$$

$$E = C \cup L \quad \text{if} \quad C \cap L \text{ is unbounded and } C \cap U \text{ is bounded},$$

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$$E = C \cup L \cup U \quad \text{if} \quad C \cap U \text{ and } C \cap L \text{ are both unbounded}.$$

If $F \cap L$ is bounded, let (ω_1, ϵ_2) be an upper bound and if $F \cap U$ is bounded, let (ϵ_1, ω_1) be an upper bound. Let

$$\mathcal{R}_1 = \{R_n \colon n \in \omega, \overline{R_n \setminus E} \text{ is compact}\};$$

if $C \cap U$ is bounded, let

$$\mathcal{R}_{2U} = \{R_n \colon n \in \omega, \overline{R_n \setminus E} \cap U \text{ is not compact}\};$$

if $C \cap L$ is bounded, let

$$\mathcal{R}_{2L} = \{R_n \colon n \in \omega, \overline{R_n \setminus E} \cap L \text{ is not compact}\}.$$

Finally let

$$\mathcal{R}_3 = \{R_n \colon n \in \omega\} \setminus (\mathcal{R}_1 \cup \mathcal{R}_{2U} \cup \mathcal{R}_{2L}),$$

where of course only the collections defined are used in this difference.

For any $\alpha \in \omega_1$ set $A_\alpha = ((\omega_1 + 1) \times [0, \alpha]) \cup ([0, \alpha] \times (\omega_1 + 1))$. Let α_0 be such that $\overline{R \setminus E} \subsetneqq A_{\alpha_0}$ if $R \in \mathcal{R}_1$.

If $C \cap U$ is bounded, choose $\alpha_1 \in \omega_1$ such that $R \in \{R_n : n \in \omega\} \setminus \mathcal{R}_{2U}$ implies $((\alpha_1, \omega_1) \times \{\omega_1\}) \cap \overline{R \setminus E} = \emptyset$. If $C \cap L$ is bounded, choose $\alpha_2 \in \omega_1$ such that $R \in \{R_n : n \in \omega\} \setminus \mathcal{R}_{2L}$ implies that $(\{\omega_1\} \times (\alpha_2, \omega_1)) \cap \overline{R \setminus E} = \emptyset$. If $C \cap U$ is bounded, choose $c_1 = (\delta_1, \omega_1) \in (\cap\{\overline{R \setminus E} : R \in \mathcal{R}_{2U}\} \setminus C)$. If $C \cap L$ is bounded, choose $c_2 = (\omega_1, \delta_2) \in (\cap\{\overline{R \setminus E} : R \in \mathcal{R}_{2L}\} \setminus C)$. Choose $k \in \omega_1$ such that $k > \max\{\alpha_0, \alpha_1, \alpha_2, \epsilon_1, \epsilon_2, \delta_1, \delta_2\}$ (of course, only for the defined ones!)

Note that if $S \in \mathcal{R}_3$, then $S \setminus (E \cup A_k)$ does not contain any point with coordinate ω_1: if $s \in S \setminus (E \cup A_k)$ and s has ω_1 as the first coordinate, then $C \cap L$ is bounded by the definition of E. Thus $S \in \mathcal{R}_3$ implies that $s \in \overline{S \setminus E} \cap (\{\omega_1\} \times \omega_1) \subsetneqq \{\omega_1\} \times [0, \alpha_2] \subsetneqq A_k$, a contradiction. The other case is similar.

For each $\alpha \in \omega_1$ there is $d_\alpha \in D$ such that $d_\alpha \geq d_*$ and $F_d \cap ([(A_\alpha \cup A_k) \cap F] \cup \{c_i : c_i \text{ defined}\}) = \emptyset$ whenever $d \geq d_\alpha$ (by compactness of $[(A_\alpha \cup A_k) \cap F] \cup \{c_i : c_i \text{ defined}\}$ and since this latter set is disjoint from C). Without loss of generality we can suppose $F_{d_\alpha} \neq F_{d_\beta}$ for any $\alpha, \beta \in \omega_1$, $\alpha \neq \beta$, since $\{F_l : l \geq d\}$ cannot have a countable range for every $d \geq d_*$.

Thus for any $\alpha \in \omega_1$ we can choose $f_\alpha \in (F \cap F_{d_\alpha}) \setminus (A_\alpha \cup A_k)$. There is a cofinal subset T of ω_1 such that $\{F_{d_\alpha} : \alpha \in T\} \cap \{R_n : n \in \omega\} = \emptyset$ and $t \in T$ implies $t > k$. Since $F \cap C = \emptyset$, $f_\alpha \notin C$. In fact $f_\alpha \notin E$. Suppose for instance that $C \cap U$ is unbounded; then $F \cap U$ is bounded, but $f_\alpha \notin [0, \epsilon_1] \times \{\omega_1\}$. Thus $f_\alpha \notin U$. The other cases are similar.

Fix $\alpha \in T$. For any neighbourhood $G(f_\alpha)$ of f_α we have $F_{d_\alpha} \in \mathcal{U} = (\{c_i : c_i \text{ defined }\}^c)^+ \cap (G(f_\alpha) \setminus (E \cup A_k))^-$. Thus $R_n \in \mathcal{U}$ for infinitely many R_n. Hence $\emptyset \neq R_n \setminus (E \cup A_k)$ for any such R_n. Clearly such an R_n cannot belong to $\mathcal{R}_1$, since $k > \alpha_0$. If $C \cap L$ is bounded, then because $c_2 \notin \overline{R_n \setminus E}$, $R_n \notin \mathcal{R}_{2L}$. Similarly, if $C \cap U$ is bounded, then $c_1 \notin \overline{R_n \setminus E}$, thus $R_n \notin \mathcal{R}_{2U}$. Therefore $R_n \in \mathcal{R}_3$.

No coordinate of f_α is equal to ω_1. Suppose the first coordinate of f_α is equal to ω_1. By the structure of the neighbourhood base of f_α, $f_\alpha \in \overline{R_m \setminus E}$ for some $R_m \in \mathcal{R}_3$. If $C \cap L$ is bounded, then $\overline{(R_m \setminus E)} \cap L \subseteqq \{\omega_1\} \times [0, \alpha_2]$, since $R_m \in \mathcal{R}_3$; thus $f_\alpha \in (\{\omega_1\} \times [0, \alpha_2]) \subseteqq A_k$, but $f_\alpha \notin A_k$. If $C \cap L$ is unbounded, then $F \cap L$ is bounded. Then $f_\alpha \in F \cap L \subseteq \{\omega_1\} \times [0, \epsilon_2] \subseteqq A_k$, a contradiction. Analogously we can show that the second coordinate of f_α cannot be equal to ω_1.

Let $\{S_n : n \in \omega\}$ be an enumeration of the infinitely many members of $\mathcal{R}_3$. Inductively choose a sequence $\{\beta_n : n \in \omega\}$ in $X \setminus A_k$ such that for all $i \in \omega$, $(\beta_{2i,1}, \beta_{2i,2})$ is equal to some f_α ($\alpha \in T$), $\beta_{2i+1} = (\beta_{2i+1,1}, \beta_{2i+1,2}) \in S_i \setminus E$ and $\max\{\beta_{j,1}, \beta_{j,2}\} < \min\{\beta_{j+1,1}, \beta_{j+1,2}\}$ whenever $j \in \omega$. Then $\{\beta_i : i \in \omega\} \longrightarrow (\sup\beta_{i,1}, \sup\beta_{i,2}) = (\gamma, \delta)$. Thus $K = \{\beta_i : i \in \omega\} \cup \{(\gamma, \delta)\}$ is compact and $(\gamma, \delta) \in F$. Furthermore $C \cap K = \emptyset$. Suppose that there is $d \in D$ with $d \geq d_0$ and $F_d \cap K = \emptyset$. By the same argument as above $F_d \in \mathcal{V} = (X \setminus (A_k \cup E))^- \cap (\{c_i : c_i \text{ defined}\}^c)^+ \cap (K^c)^+$. There is $m \in \omega$ such that $R_m \in \mathcal{V}$. We conclude as above that $R_m \in \mathcal{R}_3$, a contradiction. Thus

$F_d \cap K \neq \emptyset$ whenever $d \geq d_0$. Thus $C \cap K \neq \emptyset$ by compactness of K, another contradiction.

We leave to the interested reader to verify the following remark; its proof can be based on ideas that are similar to, but simpler than the ones used above.

Remark 5.2. The space obtained from the space W of all ordinal numbers $\leq \omega_1$ by making the set Z of all countable limit ordinals closed, is a non regular T_2 ω- bounded space for which $(CL(X), \tau_V)$ is ω-bounded.

We finish our paper with a new direct proof (without using the embedding and compactification) of Theorem 5 in [Ke1], which shows the importance of normality of the basic space.

Theorem 5.3. Let X be a T_1 normal ω-bounded space. Then $(CL(X), \tau_V)$ is ω-bounded.

Proof. Let $\{R_n : n \in \omega\}$ be a sequence in $CL(X)$. Consider a net $\{F_d : d \in D\}$ in the τ_V-closure of $\{R_n : n \in \omega\}$. Then by Mrowka's theorem [Mr] there is a subnet $\{F_{d'} : d' \in D'\}$ of $\{F_d : d \in D\}$ which Kuratowski-Painleve converges to a closed set C. We show that $C \neq \emptyset$. For every $n \in \omega$ choose $k_n \in R_n$. Set $N = \overline{\{k_n : n \in \omega\}}$. The ω-boundedness of X implies that N is compact. Let $d' \in D'$. Then $F_{d'} \cap N = \emptyset$ implies that there must exist $n_0 \in \omega$ with $R_{n_0} \cap N = \emptyset$, a contradiction. Therefore $F_{d'} \cap N \neq \emptyset$ for each $d' \in D'$. Choose $f_{d'} \in F_{d'} \cap N$ for every $d' \in D'$. Let x be a cluster point of $\{f_{d'} : d' \in D'\}$. Then $x \in C$ and thus $C \neq \emptyset$.

We want to show that C is a τ_V-cluster point of $\{F_{d'} : d' \in D'\}$. Then of course C is a τ_V-cluster point of $\{F_d : d \in D\}$.

It suffices to show that if $F \in CL(X)$ with $C \cap F = \emptyset$, then $F_{d'} \cap F = \emptyset$ frequently. Suppose not. Then there is $d_0' \in D'$ such that $F_{d'} \cap F \neq \emptyset$ for each $d' \geq d_0'$. Since X is normal, there is an open set $H \subseteq X$ such that $C \subseteq H$ and $\overline{H} \cap F = \emptyset$. If $d' \in D'$ and $d' \geq d_0'$, then $F_{d'} \in (X \setminus \overline{H})^-$. Let $\mathcal{K} = \{R_n : n \in \omega$ and $R_n \in (X \setminus \overline{H})^-\}$. For each $K \in \mathcal{K}$ choose some $l_K \in K \cap (X \setminus \overline{H})$

and put $M = \overline{\{l_K : K \in \mathcal{K}\}}$. Note that M is compact. Let $d' \in D'$ with $d' \geq d_0'$. If $F_{d'} \cap M = \emptyset$, then $F_{d'} \in (M^c)^+ \cap (X \setminus \overline{H})^-$. Thus there is $K \in \mathcal{K}$ such that $K \in (M^c)^+ \cap (X \setminus \overline{H})^-$. Hence $K \subseteq X \setminus M$, a contradiction. Therefore $F_{d'} \cap M \neq \emptyset$ whenever $d' \in D'$ and $d' \geq d_0'$. Choose for every $d' \in D'$ such that $d' \geq d_0'$ $e_{d'} \in F_{d'} \cap M$. Let $m \in M$ be a cluster point of the net $\{e_{d'} : d' \in D', d' \geq d_0'\}$. Then $m \in C$, but $M \cap H = \emptyset$, a contradiction. □

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Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK–814 73 Bratislava, SLOVAKIA

E-mail address: hola@mau.savba.sk

Mathematical Institute, University of Berne, Sidlerstrasse 5, CH-3012 Berne, Switzerland

E-mail address: kunzi@math-stat.unibe.ch