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A CHARACTERIZATION OF DENDRITES WITH THE PERIODIC-RECURRENT PROPERTY

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Abstract

In this paper we prove that a dendrite X contains a topological copy of the Gehman dendrite if and only if there is a map $f : X \to X$ such that the closure of the set of periodic points of f is properly contained in the closure of the set of recurrent points of f.

1. Introduction

A *continuum* is a compact connected metric space. A *dendrite* is a locally connected continuum containing no simple closed curve. A *tree* is a one-dimensional compact connected acyclic polyhedron. A *map* is a continuous function. The set of positive integers is denoted by $\mathbb{N}$.

Let X be a continuum and let $f : X \to X$ be a map. Denote by f^n the n-th iteration of f. A point $p \in X$ is said to be:

- a *periodic point of* f, provided that there exists $n \in \mathbb{N}$ such that $f^n(p) = p$,

- a *recurrent point of* f, provided that for every neighborhood U of p there exists $n \in \mathbb{N}$ such that $f^n(p) \in U$.

The set of periodic points and recurrent points of a map $f : X \to X$ are respectively denoted by $P(f)$ and $R(f)$. Clearly $P(f) \subset R(f)$.

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A continuum X is said to have ([1, Definition 1.4]) the *periodic-recurrent property* (shortly, *PR-property*) provided that for every map $f : X \to X$ the following equality holds:

$$cl_X P(f) = cl_X R(f).$$

The notions of periodic and recurrent point are important in dynamical systems. In [3], E. M. Coven and G. A. Hedlund proved that the unit closed interval $[0, 1]$ has the PR-property. This result was generalized by X. D. Ye ([12]) who showed that trees have the PR-property. H. Kato showed ([7]) that this result is not true for dendrites. He proved that for the Gehman dendrite G ([5, example on p. 42]; see also [9, p. 422-423] for a detailed description, and [11, Figure 1, p. 203] for a picture). Recall that G can be characterized as the only dendrite whose set of end points is homeomorphic to the Cantor set, and whose ramification points are of order 3 only (See [10, p. 100]).

Recently J. J. Charatonik ([1]) and J. J. Charatonik and W. J. Charatonik ([2]) have studied continua with the PR-property.

In particular, J. J. Charatonik proved the following theorem.

Theorem 1. ([1, Theorem 3.3]) *If a continuum contains the Gehman dendrite, then it does not have the PR-property.*

In a private conversation, J. J. Charatonik asked to the author if the converse of Theorem 1 is true for dendrites. The purpose of this paper is to give a positive answer to this question by proving the following theorem.

Theorem 2. *A dendrite X contains a Gehman dendrite if and only if X does not have the PR-property.*

2. Proof of Theorem 2

We need the following conventions.

From now on X will denote a dendrite with metric d. If $\varepsilon > 0$ and $p \in X$ let $B(\varepsilon, p) = \{q \in X : d(p, q) < \varepsilon\}$. Given two points

p and q in X, denote by $pq = \{p\}$, if $p = q$ and denote by pq the unique arc in X joining p and q, if $p \neq q$.

If A is an arc contained in X, let $r_A : X \to A$ be the function given by

$$r_A(p) = \text{the unique point } a \in A \text{ such that } ap \cap A = \{a\}.$$

It is known that r_A is a continuous retraction from X onto A ([8, Lemma 10.25]). Since A is an arc, we can fix a natural total order $<_A$ in A. Then define the following order in X: $p \prec_A q$ if and only if $r_A(p) \leq_A r_A(q)$. We write simply r and $\prec$ instead of r_A and $\prec_A$, respectively, when there is no possibility of confusion.

Let $C(X)$ denote the hyperspace of subcontinua of X, metrized with the Hausdorff metric. A *Whitney map* is a map $\mu : C(X) \to \mathbb{R}$ such that $\mu(\{p\}) = 0$ for every $p \in X$, and $\mu(B) < \mu(D)$ if $B, D \in C(X)$ and B is properly contained in D. For the existence of Whitney maps see [8, 4.33].

Lemma 1. *Let A be an arc in the dendrite X. Let $p \in X - A$. Then there is an open arcwise connected neighborhood U of p such that $r(U) = \{r(p)\}$.*

Proof. Since dendrites are locally arcwise connected ([8, Theorem 8.25]), there exists an open arcwise connected neighborhood U of p such that $U \cap A = \emptyset$. Let $q \in U$. Then $pq \subset U$. By definition $pr(p) \cap A = \{r(p)\}$. Then $(pq \cup pr(p)) \cap A = \{r(p)\}$. Since $pq \cup pr(p)$ is arcwise connected, $qr(p) \subset pq \cup pr(p)$. Then $qr(p) \cap A = \{r(p)\}$. By the definition of $r(q)$, we conclude that $r(q) = r(p)$. This proves the lemma. □

Lemma 2. *Let A be an arc in the dendrite X. Let $p \in A$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $r^{-1}(A \cap B(\delta, p) - \{p\}) \subset B(\varepsilon, p)$.*

Proof. Suppose on the contrary that there is no such δ. Then for each $n \in \mathbb{N}$, there exists a point $x_n \in r^{-1}(A \cap B(\frac{1}{n}, p) - \{p\}) - B(\varepsilon, p)$. Taking a subsequence if necessary, we may assume that

$x_n \to x$ for some $x \in X - B(\varepsilon, p)$. Since $d(r(x_n), p) \to 0$, $r(x) = p$. Since $x \neq p$, $x \notin A$. By Lemma 1, there exists $N \in \mathbb{N}$ such that $r(x_N) = r(x) = p$. This contradicts the choice of x_N and completes the proof of the lemma. □

Lemma 3. *Let A be an arc in the dendrite X. Let $f : X \to X$ be a map. Let $p_0 < q_0$ be two points in A such that $p_0 < r(f(p_0))$ and $r(f(q_0)) < q_0$. Then there exists a fixed point z of f such that $p_0 < r(z) < q_0$.*

Proof. Let $p, q \in p_0q_0$ be such that $p_0 < p < q < q_0$, $p < r(f(p))$ and $r(f(q)) < q$. Consider the map $r \circ f|A : A \to A$. Since $p < r(f(p))$ and $r(f(q)) < q$, there exists $y \in pq$ such that $r(f(y)) = y$.

Let $B = \{x \in X : x \in f(x)r(f(x))\} \cap r^{-1}(pq)$. Notice that $y \in B$, so $B \neq \emptyset$. In order to see that B is closed in X, take a sequence $\{x_n\}_{n=1}^{\infty} \subset B$, which converges to a point $x \in X$. Since X is a dendrite and $f(x_n) \to f(x)$ and $r(f(x_n)) \to r(f(x))$, we have $f(x_n)r(f(x_n)) \to f(x)r(f(x))$. Thus $x \in f(x)r(f(x))$. Hence $x \in B$. Therefore, B is closed.

Let $\mathcal{B} = \{xr(f(x)) \in C(X) : x \in B\}$. We will prove that $\mathcal{B}$ is closed in $C(X)$. Suppose that $\{x_nr(f(x_n))\}_{n=1}^{\infty}$ is a sequence in $\mathcal{B}$ and $x_nr(f(x_n)) \to D$ for some $D \in C(X)$. We may assume that $x_n \to x$ for some $x \in B$. Since $x_nr(f(x_n)) \to xr(f(x))$, we conclude that $D \in \mathcal{B}$. Therefore, $\mathcal{B}$ is closed in $C(X)$.

Let $\mu : C(X) \to \mathbb{R}$ be a Whitney map. Since $\mathcal{B}$ is compact, there exists $z \in B$ such that $\mu(zr(f(z))) \geq \mu(xr(f(x)))$ for every $x \in B$.

We will show that $f(z) = z$. Suppose on the contrary that $f(z) \neq z$. Since $z \in f(z)r(f(z))$, we have $f(z) \notin zr(f(z))$ and $f(z) \notin A$. Thus there exists an open arcwise connected subset U of X such that $f(z) \in U$ and $cl_X(U) \cap (A \cup zr(f(z))) = \emptyset$.

Since $zr(f(z)) \cap cl_X(U) = \emptyset$, there exists $z_1 \in zf(z) - \{z, f(z)\}$ such that $zz_1 \cap cl_X(U) = \emptyset$, and $f(zz_1) \subset U$. Notice that the set $f(z)r(f(z)) \cup cl_X(U)$ is a continuum of X which contains $f(z_1)$ and $r(f(z))$. Then $f(z_1)r(f(z)) \subset f(z)r(f(z)) \cup cl_X(U)$. This implies that $f(z_1)r(f(z)) \cap A = \{r(f(z))\}$. Thus $r(f(z_1)) = r(f(z))$.

On the other hand, $f(z_1)r(f(z_1)) \subset [z_1r(f(z_1))] \cup [z_1f(z) \cup cl_X(U)]$, $[z_1r(f(z_1))] \cap [z_1f(z) \cup cl_X(U)] = \{z_1\}$, $r(f(z_1)) \in z_1r(f(z_1))$ and $f(z_1) \in z_1f(z) \cap cl_X(U)$. Thus, the connectedness of $f(z_1)r(f(z_1))$ implies that $z_1 \in f(z_1)r(f(z_1))$. This proves that $z_1 \in B$. By the choice of z, we have $\mu(z_1r(f(z_1))) \leq \mu(zr(f(z)))$. This is a contradiction since $zr(f(z))$ is properly contained in $z_1r(f(z_1))$. This contradiction completes the proof that $f(z) = z$.

Since $p \leq r(z) \leq q$, we conclude that $p_0 < r(z) < q_0$. □

Lemma 4. *Suppose that the dendrite X contains a nonempty, dense in itself subset S with the property that, for every arc B in X, the set $B \cap S$ is discrete, then X contains a homeomorphic copy of the Gehman dendrite.*

Proof. First, we will prove the following claim.

(1) Suppose that $p \neq q$ are two points in X such that $q \in S$. Then there exists a point $z \in X$ such that $q \in pz$, $z \in cl_X(pz \cap S)$ and, for each $s \in S$, $z \notin ps - \{s\}$.

In order to prove (1), let $T = \{x \in X : x \in cl_X(px \cap S)$ and $q \in px\}$. Then $q \in T$.

Let $\ll$ be the order defined in T by:

if x, y are ponts in T, then $x \ll y$ if and only if $px \subset py$.

Clearly, $\ll$ is a partial order for T. We will prove that there are maximal elements in T, with respect to the order $\ll$. By Brower's Maximal Principle ([6, p. 161]), it is enough to check that if $\{x_n\}_{n=1}^{\infty}$ is a sequence of elements in T such that $x_1 \ll x_2 \ll ...$, then there exists $x \in T$ such that $x_n \ll x$ for each $n \in \mathbb{N}$. Since $px_1 \subset px_2 \subset ...$ and every increasing sequence of arcs in a dendroid tends to an arc, then $px_n \to px$ for some $x \in X$. In fact, $px = cl_X(\bigcup\{px_n : n \in \mathbb{N}\})$. Clearly, $x \in T$ and $x_n \ll x$ for every $n \in \mathbb{N}$. This completes the proof that there exists a maximal element z of $(T, \ll)$.

If there exists $s \in S$ such that $z \in ps - \{s\}$. Then $q \in pz \subset ps$. Therefore, $s \in T$, $z \ll s$ and $z \neq s$. This contradicts the maximality of z and proves that, for each $s \in S$, $z \notin ps - \{s\}$. This ends the proof of (1).

Fix two points p^* and q^* in S and fix a point $x_0 \in p^*q^* - \{p^*, q^*\}$. A point $x \in X$ is said to be *terminal with respect to* S provided that $x \in cl_X(x_0x \cap S)$ and, for each $s \in S$, $x \notin x_0s - \{s\}$.

For each $n \in \mathbb{N}$, we will inductively construct sets $E_n = \{p(\alpha) : \alpha \in \{0,1\}^n\}$ and $R_n = \{q(\alpha) : \alpha \in \{0,1\}^n\}$ of points of X with the following properties:

(a) $x_0 \in p(0)p(1)$,

(b) for each $\alpha \in \{0,1\}^n$, $p(\alpha)$ is terminal with respect to S,

(c) if $n \geq 2$ and $\alpha \in \{0,1\}^{n-1}$, then $p(\alpha, 0) = p(\alpha)$ $(\in E_{n-1} \cap E_n)$,

(d) if $n \in \mathbb{N}$ and $\alpha = (a_1, ..., a_n) \in \{0,1\}^n$, then $q(\alpha) \in x_0p(\alpha) - \{p(\alpha), x_0\}$ and $\text{diam}(p(\alpha), q(\alpha)) < \frac{1}{2^n}$. If $n \geq 2$, then $q(a_1, ..., a_{n-1}) \in x_0q(\alpha) - \{q(\alpha)\}$

(e) if $n \geq 2$ and $\alpha = (a_1, ..., a_{n-1}) \in \{0,1\}^{n-1}$, then $x_0p(\alpha, 0) \cap x_0p(\alpha, 1) = x_0q(\alpha)$.

The first step of the induction goes as follows: Apply (1) to x_0 and $p^* \in S$. Then there exists a point $p(0) \in X$ such that $p^* \in x_0p(0)$, $p(0) \in cl_X(x_0p(0) \cap S)$ and, for each $s \in S$, $p(0) \notin x_0s - \{s\}$. Now apply (1) to x_0 and $q^* \in S$. Then there exists a point $p(1) \in X$ such that $q^* \in x_0p(1)$, $p(1) \in cl_X(x_0p(1) \cap S)$ and, for each $s \in S$, $p(1) \notin x_0s - \{s\}$. Clearly, $p(0)$ and $p(1)$ are terminal with respect to S.

Now, suppose that $n \in \mathbb{N}$ and the sets E_n and R_{n-1} have been constructed with the required properties.

Let $\alpha = (a_1, ..., a_n) \in \{0,1\}^n$ and $\beta = (b_1, ..., b_n) \in \{0,1\}^n$ be such that $\alpha \neq \beta$. We will see that $p(\alpha) \neq p(\beta)$. From the way we constructed $p(0)$ and $p(1)$, $p(0) \neq p(1)$. Then we may assume that $n > 1$.

Suppose on the contrary that $p(\alpha) = p(\beta)$. Let $r \in \{1, ..., n\}$ be such that $(a_1, ..., a_{r-1}) = (b_1, ..., b_{r-1})$ and $a_r \neq b_r$. Let $\gamma = (a_1, ..., a_{r-1}) = (b_1, ..., b_{r-1})$. Assume that $r < n$, the case $r = n$ is easier. By (c) and (d), $q(\gamma, a_r), q(\gamma, b_r) \in x_0p(\alpha) = x_0p(\beta)$. That is, $q(\gamma, 0), q(\gamma, 1) \in x_0p(\alpha)$. Thus, we may assume that $x_0q(\gamma, 0) \subset x_0q(\gamma, 1)$. Then $x_0q(\gamma, 0) \subset x_0p(\gamma, 0) \cap x_0p(\gamma, 1) = x_0q(\gamma)$ (in the case that $r = 1$, this implies that $q(\gamma, 0) = x_0$ which is a contradiction). Hence, $x_0q(\gamma, 0) \subset x_0q(\gamma)$. This contradicts (d) and proves that $p(\alpha) \neq p(\beta)$.

Let $\alpha, \beta \in \{0, 1\}^n$ be such that $\alpha \neq \beta$. We will check that $p(\alpha) \notin x_0p(\beta)$. Suppose on the contrary that $p(\alpha) \in x_0p(\beta)$. Since $p(\beta) \in cl_X(x_0p(\beta) \cap S)$ and $p(\alpha) \neq p(\beta)$, there exists $s \in x_0p(\beta) \cap S$ such that $s \notin x_0p(\alpha)$. Then $s \in p(\alpha)p(\beta) - \{p(\alpha)\}$. Thus, $p(\alpha) \in x_0s - \{s\}$. This contradicts the terminality with respect to S of $p(\alpha)$ and ends the proof that $p(\alpha) \notin x_0p(\beta)$.

For each $\alpha \in \{0, 1\}^n$, fix a point $q \in x_0p(\alpha)$ such that $\text{diam}(qp(\alpha)) < \frac{1}{2^n}$, $qp(\alpha) \cap (\bigcup\{x_0p(\beta) : \beta \in \{0, 1\}^n - \{\alpha\}\}) = \emptyset$ and $q \neq p(\alpha)$. Since $p(\alpha)$ is terminal with respect to S, there exists a point $s \in qp(\alpha) \cap S - \{q\}$. By hypothesis s is an isolated point of the set $x_0p(\alpha) \cap S$. Since X is a dendrite, there exists an arcwise connected neighborhood U of s such that $U \cap x_0q = \emptyset$ and $U \cap (x_0p(\alpha) \cap S) = \{s\}$. By hypothesis there is a point $z \in (U \cap S) - \{s\}$. Thus $z \notin x_0p(\alpha)$. Let $q(\alpha) \in x_0p(\alpha)$ be the point with the property that $zq(\alpha) \cap x_0p(\alpha) = \{q(\alpha)\}$.

Since U is arcwise connected and $s \in x_0p(\alpha)$, $q(\alpha) \in qp(\alpha)$. Thus $\text{diam}(q(\alpha)p(\alpha)) < \frac{1}{2^n}$. If $q(\alpha) = p(\alpha)$, then $p(\alpha) \in x_0z - \{z\}$. This contradicts the terminality with respect to S of $p(\alpha)$. Hence $q(\alpha) \neq p(\alpha)$. Therefore, $q(\alpha) \in x_0p(\alpha) - \{x_0, p(\alpha)\}$.

Applying (1) to x_0 and $z \in S$, there exists a point $p(\alpha, 1) \in X$ such that $z \in x_0p(\alpha, 1)$ and $p(\alpha, 1)$ is terminal with respect to S.

We complete the definition by making $p(\alpha, 0) = p(\alpha)$. Then we have defined $q(\alpha)$ for every $\alpha \in \{0,1\}^n$ and $p(\beta)$ for every $\beta \in \{0,1\}^{n+1}$. From the construction it is clear that properties (a), (b), (c), (e) and the first part of (d) are satisfied.

We will only check that if $\alpha = (a_1, ..., a_{n-1}) \in \{0,1\}^{n-1}$ and $a_n \in \{0,1\}$, then $q(\alpha) \in x_0q(\alpha, a_n) - \{q(\alpha, a_n)\}$. By induction hypothesis, $q(\alpha) \in x_0p(\alpha, a_n) \cap x_0p(\alpha, 1-a_n)$ and from the choice of the point q in the construction of $q(\alpha, a_n)$, $qp(\alpha, a_n) \cap x_0p(\alpha, 1-a_n) = \emptyset$. Therefore $q(\alpha) \in x_0p(\alpha, a_n) - qp(\alpha, a_n)$. By construction, $q(\alpha, a_n) \in qp(\alpha, a_n)$. Thus $q(\alpha) \in x_0q(\alpha, a_n) - \{q(\alpha, a_n)\}$.

This completes the inductive construction.

Define

$$\Lambda = \bigcup \{\{0,1\}^n : n \in \mathbb{N}\}$$

and

$$G = cl_X(\bigcup \{x_0p(\alpha) : \alpha \in \Lambda\}).$$

Then G is a dendrite.

We will prove that G is homeomorphic to the Gehman dendrite. This will be done by proving a series of claims.

(A) If $\alpha = (a_1, ..., a_m) \neq (b_1, ..., b_m) = \beta$ are elements in Λ and r is the first index such that $a_r \neq b_r$, then $x_0p(\alpha) \cap x_0p(\beta) = x_0q'$, where $q' = q(a_1, ..., a_{r-1})$ if $r \geq 2$ and $q' = x_0$ if $r = 1$.

We will only analize the case $r \geq 2$. The analysis for the case $r = 1$ is similar. Let $\gamma = (a_1, ..., a_{r-1})$. By (d), $q(\gamma) \in x_0q(\alpha) \cap x_0q(\beta) \subset x_0p(\alpha) \cap x_0p(\beta)$. Then $x_0q(\gamma) \subset x_0p(\alpha) \cap x_0p(\beta)$. Now suppose that there exists a point $y \in x_0p(\alpha) \cap x_0p(\beta) - x_0q(\gamma)$. Then $x_0q(\gamma) \subsetneq x_0y \subset x_0p(\alpha)$. By (d), $q(\gamma) \in x_0q(\gamma, a_r) - \{q(\gamma, a_r)\} \subset x_0p(\alpha)$. Then we may assume that $y \in x_0q(\gamma, a_r)$. Similarly, we may assume that $y \in x_0q(\gamma, b_r)$. This implies that $y \in x_0q(\gamma, a_r) \cap x_0q(\gamma, b_r) = x_0q(\gamma)$ which is a contradiction. This completes the proof of (A).

(B) $G = (\bigcup \{x_0p(\alpha) : \alpha \in \Lambda\}) \cup cl_X(\{p(\alpha) : \alpha \in \Lambda\})$.

We only need to prove that if $x \in G$ and $x \notin \bigcup\{x_0p(\alpha) : \alpha \in \Lambda\}$, then $x \in cl_X(\{p(\alpha) : \alpha \in \Lambda\})$.

Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$. Let $\delta > 0$ be such that $B(\delta, x) \cap (\bigcup\{x_0p(\alpha) : \alpha \in \{0,1\}^r$ for some $r \leq n\}) = \emptyset$ and $\delta < \frac{\varepsilon}{2}$. Let U be an arcwise connected open subset of X such that $x \in U \subset B(\delta, x)$. Let $\alpha, \beta \in \Lambda$ be such that $U \cap x_0p(\alpha) \neq \emptyset$ and $(U - x_0p(\alpha)) \cap x_0p(\beta) \neq \emptyset$. By (c), we may assume that there is $m \in \mathbb{N}$ such that $\alpha, \beta \in \{0,1\}^m$. Notice that $\alpha \neq \beta$ and $n < m$. Choose points $y \in U \cap x_0p(\alpha)$ and $z \in (U - x_0p(\alpha)) \cap x_0p(\beta)$, and let w be the unique point in $x_0p(\alpha)$ such that $zw \cap x_0p(\alpha) = \{w\}$. Since U is arcwise connected, $w \in zy \subset U$. Since $z \in x_0p(\beta)$, we have $x_0p(\alpha) \cap x_0p(\beta) = x_0w$. Since $x_0 \notin U$, $x_0 \neq w$. Then, by (A) and the choice of δ, $w = q(\gamma)$ for some $\gamma \in \Lambda - \bigcup\{\{0,1\}^r : r \leq n\}$. Thus $\gamma \in \{0,1\}^k$ for some $k > n$. Then $d(p(\gamma), x) \leq d(p(\gamma), q(\gamma)) + d(q(\gamma), x) < \varepsilon$. Therefore, $B(\varepsilon, x) \cap (\bigcup\{p(\eta) : \eta \in \Lambda\}) \neq \emptyset$. This completes the proof of (B).

(C) Let $G_0 = \bigcup\{x_0p(\alpha) - \{p(\alpha)\} : \alpha \in \Lambda\}$ and $E = cl_X(\{p(\alpha) : \alpha \in \Lambda\})$. Then $E \cap G_0 = \emptyset$.

Suppose on the contrary that there is a point $x \in G_0 \cap E$. Suppose that $x \in x_0p(\alpha) - \{p(\alpha)\}$. Let $\alpha = (a_1, ..., a_n)$. Notice that the points of the sequence

$$Q = \{x_0, q(a_1), q(a_1, a_2), ..., q(a_1, ..., a_n), \\ q(a_1, ..., a_n, 0), q(a_1, ..., a_n, 0, 0, ...), ...\}$$

are pairwise different, they belong to $x_0p(\alpha)$ and they tend to $p(\alpha)$. Then there exists an arcwise connected neighborhood U of x such that U contains at most one element of this sequence and $p(\alpha) \notin U$.

Let $\beta \in \Lambda$ be such that $p(\beta) \in U$. Then $p(\alpha) \neq p(\beta)$. The terminality with respect to S of the points $p(\alpha)$ and $p(\beta)$ implies that $p(\beta) \notin x_0p(\alpha)$.

Let $\beta = (b_1, ..., b_m)$. Adding zeroes if necessary, we may assume that $m > n$. Let $a_{n+1} = ... = a_m = 0$ and let $\alpha' =$

$(a_1, ..., a_m)$. Let $r \in \{1, ..., m\}$ be the first index such that $a_r \neq b_r$. Let $q' = q(a_1, ..., a_{r-1})$ if $r \geq 2$ and $q' = x_0$ if $r = 1$. By (A), $x_0p(\alpha) \cap x_0p(\beta) = x_0q'$.

Since U is arcwise connected, $xp(\beta) \subset U$. Then $q' \in U$. Thus q' is the unique point in the sequence Q such that $q' \in U$. Let $q_1 = q(b_1, ..., b_r)$. Then $q' \in x_0q_1 - \{q_1\} \subset x_0p(\beta)$. Thus $q_1 \notin x_0q'$ and $q_1 \notin x_0p(\alpha)$.

Let V be an arcwise connected neighborhood of x such that $V \subset U - \{q_1\}$. Let $\gamma \in \Lambda$ be such that $p(\gamma) \in V$. Suppose that $\gamma = (c_1, ..., c_k)$. Adding zeroes if necessary, we may assume that $k > n$. Since $p(\gamma) \neq p(\alpha)$, we have $(c_1, ..., c_k) \neq (a_1, ..., a_k)$, where $a_i = 0$ for $i > n$. Let $j \in \{1, ..., k\}$ be the first index such that $c_j \neq a_j$. Let $q_2 = x_0$ if $j = 1$ and $q_2 = q(a_1, ..., a_{j-1})$ if $j \geq 2$. Proceeding as before, q_2 is the unique point in the sequence Q such that $q_2 \in V$. Thus, $q' = q_2$. In particular, $j = r$.

Let $q_3 = q(c_1, ..., c_r) = q(a_1, ..., a_{r-1}, c_r)$. Since $q_1 = q(b_1, ..., b_r) = q(a_1, ..., a_{r-1}, b_r)$ and $c_r \neq a_r \neq b_r$, we conclude that $c_r = b_r$. Then $q_3 = q_1$. Since $q_3 \in x_0p(\gamma) \subset x_0x \cup xp(\gamma) \subset x_0p(\alpha) \cup xp(\gamma)$ and $q_1 \notin x_0p(\alpha)$, $q_3 \in xp(\gamma) \subset V \subset U - \{q_1\}$. Thus $q_3 \neq q_1$. This contradiction proves (C).

(D) The set E is totally disconnected.

Suppose on the contrary that there exists a nondegenerate component A of E. Since E is closed, A is a dendrite. Then A contains a nondegenerate arc B. Let x be the unique point in B such that $x_0x \cap B = \{x\}$. Choose a point $y \in B - \{x\}$. Let U be an arcwise connected neighborhood of y with the property that $U \cap x_0x = \emptyset$. Let $\alpha \in \Lambda$ be such that $p(\alpha) \in U$.

Since $x_0x \cup B \cup yp(\alpha)$ is connected, $x_0p(\alpha) \subset x_0x \cup B \cup yp(\alpha)$. By (C), $x_0p(\alpha) - \{p(\alpha)\} \subset x_0x \cup yp(\alpha)$. Since $x_0p(\alpha) - \{p(\alpha)\}$ is connected and $x_0x \cap yp(\alpha) = \emptyset$, $x_0p(\alpha) - \{p(\alpha)\} \subset x_0x$. This implies that $p(\alpha) \in x_0x$. Since $p(\alpha) \in U$ and $U \cap x_0x = \emptyset$, this is a contradiction. The proof of (D) is now complete.

(E) $E = \{p \in G : p \text{ is an end point of } G\}$.

Clearly, no point in G_0 is an end point of G. Then $\{p \in G : p$ is an end point of $G\} \subset E$. Now take a point $x \in E$ and suppose that x is not an end point of G. Then there exist points $y, z \in G$ such that $x \in yz - \{y, z\}$. By (D), yx and xz are not contained in E. Then there exist points $u \in yx - E \subset G_0$ and $v \in xz - E \subset G_0$. Then $x \neq u$ and $x \neq v$. Clearly, G_0 is arcwise connected. Then $x \in uv \subset G_0$. This contradicts (C) and ends the proof of (E).

(F) E is homemorphic to the Cantor set.

According [8, Theorem 7.14], by (D), we only need to prove that E does not have isolated points. Since $\{p(\alpha) : \alpha \in \Lambda\}$ is dense in E, it is enough to show that the set $\{p(\alpha) : \alpha \in \Lambda\}$ does not have isolated points.

Take $\alpha = (a_1, ..., a_n) \in \Lambda$ and let $\varepsilon > 0$. Apply Lemma 2 to the arc $x_0p(\alpha)$ and the point $p(\alpha)$, and let $\delta > 0$ be as in such lemma.

Since the sequence $q(\alpha, 0), q(\alpha, 0, 0), ...$ tends to $p(\alpha)$, there exists $\beta = (\alpha, 0, ..., 0) \in \Lambda$ such that $d(p(\alpha), q(\beta)) < \delta$. By (A), $x_0p(\beta, 0) \cap x_0p(\beta, 1) = x_0q(\beta)$. Since $q(\beta) \neq p(\beta, 0)$, $p(\beta, 0) \neq p(\beta, 1)$. Notice that $p(\beta, 0) = p(\alpha)$ and $x_0p(\alpha) \cap p(\beta, 1)q(\beta) = \{q(\beta)\}$. By the definition of $r_{x_0p(\alpha)}$ at the beginning of this section, $r_{x_0p(\alpha)}(p(\beta, 1)) = q(\beta) \in x_0p(\alpha) \cap B(\delta, p(\alpha)) - \{p(\alpha)\}$. By the choice of δ we conclude that $p(\beta, 1) \in B(\varepsilon, p(\alpha))$.

This proves that $\{p(\alpha) : \alpha \in \Lambda\}$ does not have isolated points. Therefore, E is homeomorphic to the Cantor set.

(G) All the ramification points of G have order 3.

Suppose on the contrary that there is a ramification point x of G such that the order of x in G is at least 4.

Then there are four points $x_1, x_2, x_3, x_4 \in G - \{x\}$ such that $xx_i \cap xx_j = \{x\}$ if $i \neq j$. By (E), $x \notin E$. Since E is closed, we may assume that $(xx_1 \cup xx_2 \cup xx_3 \cup xx_4) \cap E = \emptyset$. Let y_0 be the unique point in $xx_1 \cup xx_2 \cup xx_3 \cup xx_4$ such that $x_0y_0 \cap (xx_1 \cup xx_2 \cup xx_3 \cup xx_4) = \{y_0\}$. Suppose that $y_0 \in xx_4$. Notice that

$x \in x_0x_1 \cap x_0x_2 \cap x_0x_2$, and if $i, j \in \{0,1,2,3\}$ and $i \neq j$, then $xx_i \cap xx_j = \{x\}$.

Since $x_1, x_2, x_3 \notin E$, there exist $\alpha, \beta, \gamma \in \Lambda$ such that $x_1 \in x_0p(\alpha)$, $x_2 \in x_0p(\beta)$ and $x_3 \in x_0p(\gamma)$. Adding zeroes if necessary, we may put $\alpha = (a_1, ..., a_m)$, $\beta = (b_1, ..., b_m)$ and $\gamma = (c_1, ..., c_m)$.

If $\alpha = \beta$, then $\{x, x_1, x_2\} \subset x_0p(\alpha)$. This implies that $xx_1 \subset xx_2$ or $xx_2 \subset xx_1$, which is a contradiction. Hence, $\alpha \neq \beta$. Let $r \in \{1, ..., m\}$ be the minimum index such that $a_r \neq b_r$. Let $q' = x_0$ if $r = 1$ and $q' = q(a_1, ..., a_{r-1})$ if $r \geq 2$. By (A), $x_0p(\alpha) \cap x_0p(\beta) = x_0q'$.

Notice that $x_0p(\alpha) = x_0x_1 \cup x_1p(\alpha) = x_0x \cup xx_1 \cup x_1p(\alpha)$ and $x_0p(\beta) = x_0x \cup xx_2 \cup x_2p(\beta)$. Then $x_0p(\alpha) \cap x_0p(\beta) = x_0x$. Thus $x = q'$ and $xp(\alpha) \cap xp(\beta) = \{x\}$.

Similarly, $\beta \neq \gamma$. Let $k \in \{1, ..., m\}$ be the first index such that $b_k \neq c_k$. Let $q'' = x_0$ if $k = 1$ and $q'' = q(b_1, ..., b_{k-1})$ if $k \geq 2$. Proceeding as before it follows that $q'' = x$ and $xp(\beta) \cap xp(\gamma) = \{x\}$. Then $q(b_1, ..., b_{k-1}) = q(b_1, ..., b_{r-1})$. Thus, $k = r$.

With a similar argument, it can be obtained that $xp(\alpha) \cap xp(\gamma) = \{x\}$.

By (d), $q(a_1, ..., a_r) \in x_0p(\alpha) - x_0q' \subset xp(\alpha) - \{x\}$, $q(b_1, ..., b_r) \in xp(\beta) - \{x\}$ and $q(c_1, ..., c_r) \in xp(\gamma) - \{x\}$. But $(a_1, ..., a_{r-1}) = (b_1, ..., b_{r-1}) = (c_1, ..., c_{r-1})$ and $\{a_r, b_r, c_r\} \subset \{0, 1\}$. Hence two of the three finite sequences $(a_1, ..., a_r)$, $(b_1, ..., b_r)$, $(c_1, ..., c_r)$ are equal. This is a contradiction since $xp(\alpha) - \{x\}$, $xp(\beta) - \{x\}$ and $xp(\gamma) - \{x\}$ are pairwise disjoint. Then the proof of (G) is complete.

(H) G is homeomorphic to the Gehman dendrite.

According [10, p. 100], (H) is a consequence of (F) and (G).$\square$

Proof of Theorem 2.

By Theorem 1, we only need to prove the sufficiency. Then suppose that there exists a map $f : X \to X$ such that $cl_X(P(f)) \neq cl_X(R(f))$. Thus $R(f) \nsubseteq cl_X(P(f))$.

Choose a point $p \in R(f) - cl_X(P(f))$. Let

$$S = \{f^n(p) : n \in \mathbb{N}\}.$$

We will show that S satisfies the hypothesis of Lemma 4. This will be done in 4 steps.

(1) $S \subset R(f)$.

Let $n \in \mathbb{N}$. Let $\varepsilon > 0$. Let $\delta > 0$ be such that $d(p,q) < \delta$ implies that $d(f^n(p), f^n(q)) < \varepsilon$. Since $p \in R(f)$, there exists $m \in \mathbb{N}$ such that $d(p, f^m(p)) < \delta$. Then $d(f^n(p), f^n(f^m(p))) < \varepsilon$. Therefore, $f^n(p) \in R(f)$.

(2) $S \cap cl_X(P(f)) = \emptyset$.

Let $n \in \mathbb{N}$. Let $\varepsilon > 0$ be such that $B(\varepsilon, p) \cap P(f) = \emptyset$. Let $m \in \mathbb{N}$ be such that $f^m(p) \in B(\varepsilon, p)$. Since $p \notin P(f)$, we may assume that $m > n$. Let $\rho > 0$ be such that $B(\rho, f^m(p)) \subset B(\varepsilon, p)$. By the continuity of f^{m-n}, there exists $\delta > 0$ such that $d(f^n(p), q) < \delta$ implies that $d(f^{m-n}(f^n(p)), f^{m-n}(q)) < \rho$.

We claim that $B(\delta, f^n(p)) \cap P(f) = \emptyset$.

Suppose on the contrary that there exists a point $q \in B(\delta, f^n(p)) \cap P(f)$. Then $f^{m-n}(q) \in B(\varepsilon, p)$. This is a contradiction since $f^{m-n}(q) \in P(f)$. Thus $B(\delta, f^n(p)) \cap P(f) = \emptyset$. Therefore, $f^n(p) \notin cl_X(P(f))$. This ends the proof of (2).

(3) S is dense in itself.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. By (1) there exists $m \in \mathbb{N}$ such that $f^m(f^n(p)) \in B(\varepsilon, f^n(p))$. By (2), $f^n(p) \notin P(f)$. Then $f^{m+n}(p) \neq f^n(p)$. Therefore, S is dense in itself.

(4) Let B be an arc in X. Then $B \cap S$ is a discrete subset of X.

Suppose on the contrary that there exists a point $x \in B \cap S$ such that x is not an isolated point of $B \cap S$. Let $x = f^n(p)$. By (2), there exists $\varepsilon > 0$ such that $B(\varepsilon, x) \cap P(f) = \emptyset$. Let $r = r_B$ be defined as at the beginning of this section for the arc B. By Lemma 2, there exists $\delta > 0$ such that $r^{-1}(B \cap B(\delta, x) - \{x\}) \subset B(\varepsilon, x)$.

Let $m > n$ be such that $f^m(p) \in B(\delta, x) \cap B - \{x\}$ and $xf^m(p) \subset B(\delta, x)$. Assume, for example, that $x < f^m(p)$, where

$<$ is a preasigned order for B. Let $g : X \to X$ be defined by $g = f^{m-n}$. Then $x < g(x)$.

We will inductively prove that $y \leq r(g^k(y))$ for each $y \in xg(x)$ and each $k \in \mathbb{N}$.

For $k = 1$, suppose on the contrary that there exists $y \in xg(x)$ such that $r(g(y)) < y$.

Since $x < g(x) = r(g(x))$, $x < y$. By Lemma 3, there exists a fixed point z of g such that $x < r(z) < y \leq g(x)$. Then $z \in r^{-1}(B(\delta, x) \cap B - \{x\}) \subset B(\varepsilon, x)$. But $z = f^{m-n}(z)$ implies that $z \in P(f)$. This contradicts the choice of ε and completes the first step of the induction.

Now, suppose that $y \leq r(g^k(y))$ for every $y \in xg(x)$. In particular, $g(x) \leq r(g^{k+1}(x))$. Thus $x < r(g^{k+1}(x))$. Reasoning as in the first step of the induction, with the map g^{k+1} instead of g, it can be proved that $y \leq r(g^{k+1}(y))$ for each $y \in xg(x)$.

This completes the induction.

In particular, $g(x) \leq r(g^k(x))$ for each $k \in \mathbb{N}$. Thus $r^{-1}(\{w \in B : w < g(x)\})$ is an open neighborhood of x in X which does not intersect the set $\{g^k(x) : k \in \mathbb{N}\}$. This proves that x is not a recurrent point of the map $g = f^{m-n}$.

However, Theorem I of [4] says that $R(f) = R(f^{m-n})$. Thus $x \notin R(f)$. This contradicts (1) and completes the proof of (4).

Finally, a direct aplication of Lemma 4 implies that X contains a homeomorphic copy of the Gehman dendrite. □

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