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INDECOMPOSABLE CONTINUA ARISING IN INVERSE LIMITS ON $[0, 1]$

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Abstract

In this paper we survey theorems whose conclusion is the existence of indecomposable subcontinua in inverse limits on $[0, 1]$. Particular attention is paid to inverse limits using a single unimodal bonding map.

0. Introduction

In this paper we consider the existence of indecomposable subcontinua which arise in inverse limits on $[0, 1]$. We pay particular attention to inverse limits using a single unimodal bonding mapping. Numerous examples are provided to illustrate the theorems.

By a *continuum* we mean a compact, connected subset of a metric space.By a *mapping* we mean a continuous function. A continuum is said to be *decomposable* if it is the union of two of its proper subcontinua and is called *indecomposable* if it is not decomposable. If $X_1, X_2, X_3, \cdots$ is a sequence of topological spaces and $f_1, f_2, f_3, \cdots$ is a sequence of mappings such that, for each positive integer i, $f_i : X_{i+1} \to X_i$, then by the *inverse limit* of the inverse sequence $\{X_i, f_i\}$ we mean the subset of

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$\prod_{i>0} X_i$ to which the point x belongs if and only if $f_i(x_{i+1}) = x_i$ for $i = 1, 2, 3, \cdots$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is denoted $\varprojlim\{X_i, f_i\}$. It is sometimes convenient to denote $f_i \circ f_{i+2} \circ \cdots \circ f_{j-1}$ by f_i^j and the inverse system by $\{X_i, f_i^j\}$.

It is well known that if each factor space, X_i is a continuum, the inverse limit is a continuum. In case we have a single factor space, M, and a single bonding map, f, we denote the inverse limit by $\varprojlim\{M, f\}$. We denote the projection of the inverse limit into the nth factor space by π_n. If K is a subcontinuum of the inverse limit, we denote $\pi_n[K]$ by K_n. If $f : M \to M$ is a mapping and $f[M] = M$ then we write $f : M \twoheadrightarrow M$.

1. Indecomposability

In this section we present some of the basic definitions and theorems. We begin with a definition and a fundamental theorem.

Definition. Suppose $\{X_i, f_i\}$ is an inverse sequence such that, for each positive integer i, X_i is a continuum. The inverse sequence is called an *indecomposable inverse sequence* provided that, for each positive integer i, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_i[A_{i+1}] = X_i$ or $f_i[B_{i+1}] = X_i$.

Theorem 1. [13, page 21], [14] *If* $\{X_i, f_i\}$ *is an indecomposable inverse limit sequence, then* $\varprojlim\{X_i, f_i\}$ *is an indecomposable continuum.*

Proof Suppose $\{X_i, f_i\}$ is an indecomposable inverse limit sequence and $M = \varprojlim\{X_i, f_i\}$. If $M = A \cup B$ where A and B are proper subcontinua then there is a positive integer N such that, if $n \geq N$, then $A_n \neq X_n$ and $B_n \neq X_n$. Suppose j is an integer not less than N. Note that $A_{j+1} \cup B_{j+1} = X_{j+1}$ but $f[A_{j+1}] \neq X_j$ and $f[B_{j+1}] \neq X_j$ contrary to the hypothesis that $\{X_i, f_i\}$ is an indecomposable inverse limit sequence. □

Corollary. [3, page 38] *Suppose* $[a, b]$ *is an interval and* $f : [a, b] \twoheadrightarrow [a, b]$ *is a mapping with the property that there exist two non-overlapping subintervals* α *and* β *of* $[a, b]$ *such that* $f[\alpha] = f[\beta] = [a, b]$. *Then,* $\varprojlim\{[a, b], f\}$ *is an indecomposable continuum.*

We refer to the hypothesis of the corollary as the "two-pass" condition. It is often the case that a mapping does not satisfy the two-pass condition but some finite composition of the map with itself does satisfy the condition. Since $\varprojlim\{M, f\}$ is homeomorphic to $\varprojlim\{M, f^n\}$ for any positive integer n, [13, Exercise 2.7, page 33], it is sufficient to look for the two-pass condition in composites of a bonding map with itself. A simple example suffices to illustrate this.

Example 1. Let g be defined by $g(x) = x + \frac{1}{2}$ if $0 \leq x \leq \frac{1}{2}$ and $g(x) = 2(1 - x)$ if $\frac{1}{2} \leq x \leq 1$.

The map g clearly does not satisfy the two-pass condition, however, the map g^2 does. Pictures of the maps g and g^2 are shown in Figure 1.

It is possible to get an indecomposable inverse limit without any composite of the map satisfying the hypothesis of Theorem 1 as can be seen by the following example first shown to the author by D. P. Kuykendall [11, pp. 16–17].

Example 2. Suppose $a_1, a_2, a_3, \cdots$ is an increasing sequence of numbers in the open interval $(0, 1)$ with limit 1 and let $b_1, b_2, b_3, \cdots$ be a sequence such that, for each positive integer n, $a_n < b_n < a_{n+1}$. Let $a_0 = 0$ and define $f : [0, 1] \twoheadrightarrow [0, 1]$ by $f(x) = 0$ for each x in $[0, a_1]$, $f(1) = 1$, for each positive integer n, $f(a_n) = a_{n-1}$ and $f(b_n) = b_n$, and f is linear on $[a_n, b_n]$ as well as on $[b_n, a_{n+1}]$ for each n. Then, $\varprojlim\{[0, 1], f\}$ is indecomposable.

A picture of such a mapping, f, where $a_n = 1 - 2^{-n}$ and $b_n = \frac{a_n + a_{n+1}}{2}$ is shown in Figure 2. To see that $M = \varprojlim\{[0, 1], f\}$ is indecomposable, suppose A and B are proper subcontinua of M

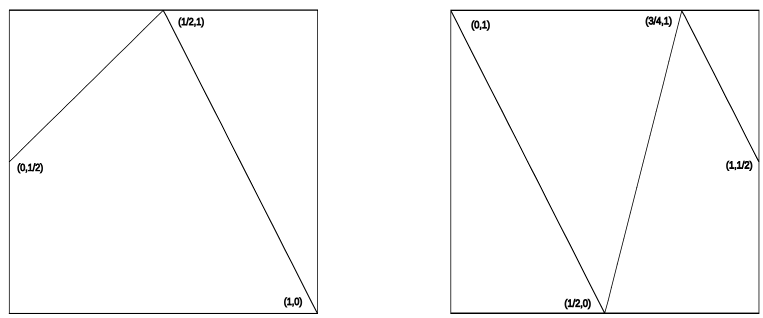

Fig. 1.

such that $M = A \cup B$, $(0,0,0,\cdots)$ is a point of A and $(1,1,1,\cdots)$ is a point of B. Since A and B are proper subcontinua of M, there is a positive integer N such that if $n \geq N$ then $A_n \neq [0,1]$ and $B_n \neq [0,1]$. There exist a positive integers k and m such that if $j \geq m$ then A_j does not contain b_k for if not, then A_N contains b_k for each k and thus $A_N = [0,1]$. It follows that, for each n, B_n contains b_i for all $i \geq m$. Thus, b_m is in B_{N+m+1}, so a_{m+1} is a point of B_{N+m+1}. Since $f^{m+1}(a_{m+1}) = 0$, it follows that $B_N = [0,1]$.

In his doctoral dissertation at the University of Houston, Dan Kuykendall gave a characterizing condition for an inverse limit to be indecomposable in terms of the factor spaces and the bonding maps.

Theorem 2. [12, Theorem 2] *Suppose* $\{X_i, f_i^j\}$ *is an inverse system and* M *is its inverse limit. The continuum* M *is indecomposable if and only if it is true that if* n *is a positive integer and* ε *is a positive number, there are a positive integer* m, $m > n$, *and three points of* X_m *such that if* K *is a subcontinuum of* X_m *containing two of them then* $d_n(x, f_n^m[K]) < \varepsilon$ *for each* x *in* X_n.

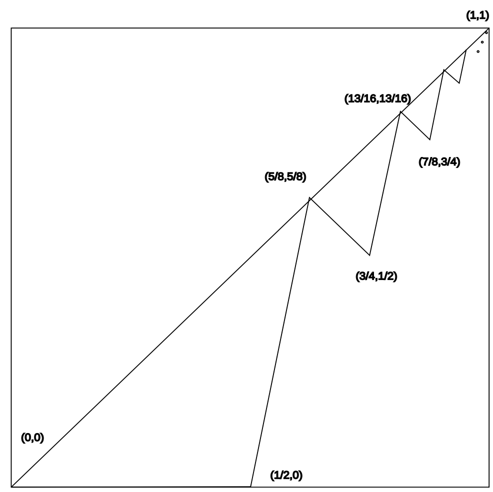

Fig. 2.

It is also possible to use Theorem 2 to argue that the inverse limit in Example 2 is indecomposable.

2. Periodicity

We employ Theorem 2 to argue that periodicity influences the existence of indecomposable subcontinua in inverse limits of intervals. A rotation on the simple triod shows that Theorem 3 is dependent on the nature of the factor spaces. If $f : X \to X$ is a mapping of a space X into itself, a point x of X is called a *periodic point* of period n if $f^n(x) = x$ and $f^j(x) \neq x$ if $0 < j < n$.

Theorem 3. *Suppose f is a mapping of $[0,1]$ into itself and f has a periodic point of period 3. Then, $\varprojlim\{[0,1], f\}$ contains an indecomposable continuum.*

Proof. Suppose x is a periodic point of period 3 for f. Let I_1

denote the interval with end points x and $f(x)$ and, inductively, let $I_j = f[I_{j-1}]$ for $j > 1$. Consider $H = cl \bigcup_{j\geq 1} I_j$ (cl denotes the closure). Then, H is a continuum such that $f : H \twoheadrightarrow H$. It is easy to see that if L is a subcontinuum of H containing two of the three points $x, f(x)$ and $f^2(x)$, $H = cl \bigcup_{j\geq 0} f^j[L]$. From Kuykendall's Theorem it follows that $\varprojlim\{H, f|H\}$ is indecomposable. □

From the proof of Theorem 3 we obtain another argument that $\varprojlim\{[0,1], g\}$ where g is the mapping from Example 1 is indecomposable. The point 0 is periodic of period 3 and the continuum $H = [0,1]$. Theorem 3 has a very nice corollary.

Corollary. [1], [6] *If f is a mapping of an interval I into itself and f has a periodic point whose period is not a power of* 2*, then $\varprojlim\{I, f\}$ contains an indecomposable continuum.*

Proof. If f has a periodic point of period $2^j(2k+1)$ where $j \geq 0$ and $k > 0$, then f^{2^j} has a periodic point of odd period, $2k+1$. By Sarkovskii's Theorem [4, Theorem 10.2], f^{2^j} has a periodic point of period 6. Therefore, $(f^{2^j})^2$ has a periodic point of period 3. By Theorem 3, $\varprojlim\{I, f^{2^{j+1}}\}$ contains an indecomposable continuum. But, $\varprojlim\{I, f^{2^{j+1}}\}$ is homeomorphic to $\varprojlim\{I, f\}$. □

In [7] the author generalized Theorem 3 replacing the interval with an atriodic and hereditarily unicoherent continuum and period 3 by odd period greater than one. The proof of this theorem makes extensive use of Kuykendall's Theorem. A continuum is a *triod* if it contains a subcontinuum whose complement has three components. A continuum is *atriodic* if it contains no triod. The statement that a continuum M is *hereditarily unicoherent* means if H and K are subcontinua of M with a common point then $H \cap K$ is connected. It is well known that inverse limits on intervals are atriodic and hereditarily unicoherent. A continuum which is homeomorphic to an inverse limit on intervals is called *chainable.*

Theorem 4. [7] *If M is an atriodic and hereditarily unicoherent continuum and f is a mapping of M into M which has a periodic point of period n where $n = 2k+1$ for some $k > 0$ then* $\varprojlim\{M, f\}$ *contains an indecomposable continuum.*

Corollary. *If h is a homeomorphism of an hereditarily decomposable chainable continuum and h has a periodic point of period n then there is a non-negative integer j such that $n = 2^j$.*

One can see from Example 2 that odd periodicity is not necessary for indecomposability since if f is a map as in Example 2, $f(x) \leq x$ for each x in $[0,1]$ so f has no periodic points except fixed points. In fact, George Henderson [5] gives a map of the interval so that the inverse limit is the pseudo-arc and the map has no periodic points except for two fixed points. However, for unimodal maps, the story is different. A mapping is called *monotone* provided all point-inverses are connected. A mapping $f : [a, b] \twoheadrightarrow [a, b]$ is called *unimodal* provided it is not monotone and there is a point c, $a < c < b$, such that $f(c)$ belongs to $\{a, b\}$ and f is monotone on $[a, c]$ and on $[c, b]$.

Theorem 5. [8] *Suppose $f : [a, b] \twoheadrightarrow [a, b]$ is a unimodal mapping such that $f(b) = a$ and q is the first fixed point for f^2 in $[c, b]$. Then, the following are equivalent:*

(1) $\varprojlim\{[a, b], f\}$ *is indecomposable.*

(2) f has a periodic point of odd period greater than 1.

(3) $f(a) < q$.

For unimodal mappings for which $f(a) = a$ some anolomies occur when f has fixed points in $[a, c]$ other than a. In Figure 3 we see a unimodal mapping with an extra fixed point in $[0, \frac{1}{2}]$. By choosing the three points to be 0, $\frac{1}{2}$ and 1 and using Kuykendall's Theorem, one can see that the inverse limit on $[0, 1]$ using this bonding map produces an indecomposable continuum.

On the other hand, in Figure 4 we see a unimodal mapping with an extra fixed point in $[0, \frac{1}{2}]$ but for which the inverse limit

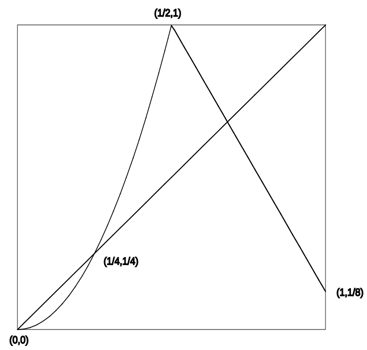

Fig.3

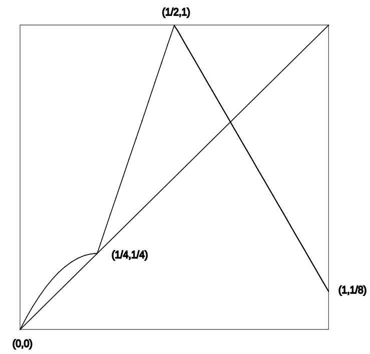

Fig.4

is decomposable. In this case, the inverse limit is the union of an arc and an indecomposable continuum. The indecomposable continuum is $\varprojlim\{[\frac{1}{8}, 1], g|[\frac{1}{8}, 1]\}$ and it can be seen to be indecomposable by observing that $(g|[\frac{1}{8}, 1])^2$ satisfies the Corollary to Theorem 1. If f is a unimodal mapping of $[a, b]$ into itself and $f(b)$ is not less than the last fixed point for f between a and c, then the interval $[f(b), b]$ is mapped into itself by f. In case $[f(b), b]$ is mapped into itself by f, we call $\varprojlim\{[f(b), b], f|[f(b), b[\}$ the *core* of the inverse limit. The analog of Theorem 5 for unimodal maps which fix a is the following.

Theorem 6. [8] *Suppose $f : [a, b] \twoheadrightarrow [a, b]$ is a unimodal mapping such that $f(a) = a$, f has no fixed point between a and c and q is the first fixed point for f^2 in $[c, b]$. The following are equivalent:*

(1) the core of $\varprojlim\{[a, b], f\}$ is indecomposable

(2) f has a periodic point of odd period greater than 1

(3) $f^2(b) < q$.

Theorems 5 and 6 give the full story for unimodal maps of an interval since each unimodal map for which $f(a) = b$ is topologically conjugate to one for which $f(b) = a$ while each unimodal map for which $f(b) = b$ is conjugate to one for which $f(a) = a$. Topologically conjugate maps yield homeomorphic inverse limits.

3. Families of Unimodal Maps

The author has investigated inverse limits arising from inverse limits on intervals using a single bonding mapping chosen from a family of mappings. These include the *tent family* given by

$$f_m(x) = \begin{cases} mx & \text{if } 0 \leq x \leq \frac{1}{m} \\ 2 - mx & \frac{1}{m} \leq x \leq 1 \end{cases} \text{ where } 1 \leq m \leq 2$$

the family, **F**, given by

$$f_t(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-t)(1-x) + t & \frac{1}{2} \leq x \leq 1 \end{cases} \text{ where } 0 \leq t \leq 1,$$

the family, $\mathbf{G} = \{g_t \mid g_t(x) = f_t(1-x)$ for x in $[0,1]$ and $0 \leq t \leq 1\}$ and the *logistic family* given by $f_\lambda(x) = 4\lambda x(1-x)$ where $0 \leq x \leq 1$ and $0 \leq \lambda \leq 1$. For the tent family, we find that the inverse limit contains an indecomposable continuum for $m > 1$, [9]. For the family **F** we find that the core of the core of the inverse limit is an indecomposable continuum for $t < \frac{1}{2}$. Specifically, f_t^4 satisfies the two-pass condition on $[f_t^2(1), 1]$, [9]. For the family **G** we find that the inverse limit is an indecomposable continuum if and only if $t < \frac{2}{3}$, [9]. The maps in the logistic family produce an indecomposable subcontinuum if and only if $\lambda > \lambda_c$ where λ_c is the Feigenbaum limit (the limit of the first period doubling sequence of parameter values, $\lambda_c \approx 0.89249$, [1, Section 4]).

Except for the results on the logistic family, all of the results mentioned above are subsumed by the following theorem.

Choose numbers b and c with $0 \leq b \leq 1$ and $0 < c < 1$ and denote by g_{bc} the mapping of $[0,1]$ onto itself which passes through the points $(0,b)$, $(c,1)$ and $(1,0)$ and is linear on the intervals $[0,c]$ and $[c,1]$. The map g_{bc} is given by

$$g_{bc}(x) = \begin{cases} \frac{1-b}{c}x + b & \text{if } 0 \leq x \leq c \\ \frac{x-1}{c-1} & c \leq x \leq 1. \end{cases}$$

Theorem 7. [10] *If* $b < c^2 - c + 1$ *then* $\underleftarrow{\lim}\{[0,1], g_{bc}\}$ *contains an indecomposable continuum.*

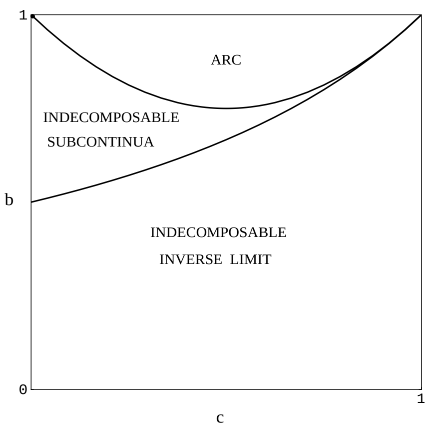

Fig.5

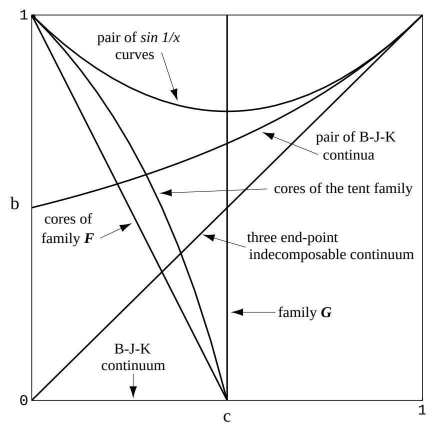

Fig.6

In addition, we show in [10] that if $b > c^2 - c + 1$, $\underleftarrow{\lim}\{[0,1], g_{bc}\}$ is an arc while if $b = c^2 - c + 1$ then the inverse limit is the union of two $\sin\frac{1}{x}$-curves intersecting at the end points of their rays. These results are most clearly illustrated by the pictures in Figures 5 and 6. The curve separating the region of parameter space where indecomposable subcontinua occur and the region where the inverse limit is indecomposable is given by $b = \frac{1}{2-c}$. Along this curve, the inverse limit is the union of two Brouwer-Janiszewski-Knaster (B-J-K) continua intersecting at their end-points. (The B-J-K continuum is the result of the inverse limit on $[0,1]$ using, for example, the tent map, f_m, with $m = 2$.)

In Figure 6, the family **G** is represented by the curve $c = \frac{1}{2}$ while the cores of the tent family are represented in the picture by the family of maps of $[0,1]$ onto itself given by $mx + (2 - m)$ on $[0, \frac{1}{1-m}]$ and $m(1 - x)$ on $[\frac{1}{1-m}, 1]$. The members of this family are conjugate to the cores of the tent maps and lie along the curve $b = \frac{2c-1}{c-1}$ where $c = \frac{1}{1-m}$. The cores of the family **F** are represented in the picture by a family of maps of $[0,1]$ onto itself topologically conjugate to the maps $f_t|[f_t(1), 1]$. The curve is given by $b = 1 - 2c$ where $b = \frac{t}{1-t}$. Along the curve $b = c$, the inverse limit is the three end-point indecomposable chainable continuum constructed by choosing three points in the plane and constructing the continuum as the common part of a sequence of chains of open disks which in an alternating fashion begin at one of the three points and end at another, [13, page 8].

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