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SPACES OF SEMICONTINUOUS FORMS

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Abstract

The space $C_k(X)$ of continuous real-valued functions on X under the compact-open topology is extended to the space, $D_k^{\#}(X)$, of locally bounded semicontinuous forms on X. When X is locally compact, this space is a locally convex linear topological space, and is completely metrizable if X is also σ-compact. Conditions are given for two such spaces to have the same density, thus ensuring that they are homeomorphic. Other results about $D_k^{\#}(X)$ include a characterization of its network weight and an Ascoli-type theorem.

1. Introduction

Let $C(X)$ denote the set of all real-valued continuous functions on the topological space X. With the compact-open topology, this function space is denoted by $C_k(X)$, and is a locally convex linear topological space. This space has been used extensively, and its properties have been thoroughly investigated. For example, if X is a hemicompact k-space, then $C_k(X)$ is completely metrizable. In such a space, the Baire category theorem holds and gives a useful way to argue the existence of certain kinds of continuous functions.

Our goal is to extend $C(X)$ to the space of semicontinuous functions on X in such a way that the many useful theorems on $C_k(X)$ have analogs to the larger space of semicontinuous

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functions. To this end, we start with the vector space, $SC(X)$, generated by the set of all upper semicontinuous and lower semicontinuous functions on X. Each member of $SC(X)$ can be written as the sum of an upper semicontinuous function and a lower semicontinuous function. We call such a function a semicontinuous function on X.

To achieve our goal, in section 2, we make two modifications to the vector space $SC(X)$. By restricting the semicontinuous functions on X to those that are locally bounded, we can ensure that, for locally compact X, addition and scalar multiplication are continuous operations. Also, to avoid having semicontinuous functions that differ only at inessential points of discontinuity, we put an appropriate equivalence relation on $SC(X)$, which gives us the space $D(X)$ of densely continuous forms.

In section 3, the topology of uniform convergence on compact sets is introduced for this space as it is done in [5], denoted by $D_k^{\#}(X)$, and is shown to make this space a locally convex linear topological space whenever X is locally compact. In addition, if X is σ-compact, $D_k^{\#}(X)$ is completely metrizable, which is shown in section 4. Section 5 includes an examination of the cardinal functions weight, network weight, density, and cellularity on $D_k^{\#}(X)$; which are shown to be equivalent whenever X is hemicompact. Further, assuming the continuum hypothesis, if X is a nondiscrete locally compact second countable space, then these cardinal functions on $D_k^{\#}(X)$ all take on the cardinality of the continuum, c. So for any two such spaces X and Y, $D_k^{\#}(X)$ and $D_k^{\#}(X)$ are homeomorphic. The last section characterizes the compact subsets of $D_k^{\#}(X)$ with an Ascoli-type theorem.

The space of real numbers with the usual topology and the set of positive integers are denoted by $\mathbf{R}$ and $\mathbf{N}$, respectively. For a reference to definitions and facts about various topological properties, see [4].

2. Semicontinuous and Densely Continuous Functions

A real-valued function f on a space X is *upper* (*lower*, respectively) *semicontinuous at* x provided that for each $\varepsilon > 0$, x has a neighborhood U such that $f(u) < f(x) + \varepsilon$ ($f(u) > f(x) - \varepsilon$, respectively) for all $u \in U$. Then f is *upper* (*lower*, respectively) *semicontinuous* whenever it is upper (lower, respectively) semicontinuous at every point in X. This is equivalent to saying that $f^{-1}(-\infty, r)$ ($f^{-1}(r, \infty)$, respectively) is open for every $r \in \mathbf{R}$.

The sum of two upper semicontinuous functions on X is upper semicontinuous, and the sum of two lower semicontinuous functions on X is lower semicontinuous. The product of an upper semicontinuous function and a positive number is upper semicontinuous, whereas the product of an upper semicontinuous function and a negative number is lower semicontinuous. The same is true with "upper" and "lower" interchanged.

Let $SC(X)$ denote the linear span in the vector space of all real-valued functions on X of the union of the set of upper semicontinuous functions on X and the set of lower semicontinuous functions on X. This linear space can be described as the set of $f + g$ where f is an upper semicontinuous function on X and g is a lower semicontinuous function on X. We call the members of $SC(X)$ *semicontinuous functions* on X.

For any function f into a metric space, the set of points in the domain at which the function is continuous is a G_δ-subset of the domain. Let $C(f)$ denote this set of points of continuity of f.

For $f \in SC(X)$, it is possible that $C(f) = \emptyset$. For example, let X equal the space of rational numbers and take $f(p/q) = 1/q$ where p/q is in lowest terms and $q > 0$; then f is upper semicontinuous, but not continuous at any point. The reason for this example is that X is of first category in itself. When X is a Baire space, the next proposition shows that $C(f)$ must be a dense G_δ-subset of X for all $f \in SC(X)$.

Proposition 2.1. *If X is a Baire space and $f \in SC(X)$, then $C(f)$ is dense in X.*

Proof. We show that $C(f)$ is dense when f is upper semicontinuous. A similar argument would show that $C(g)$ is dense when g is lower semicontinuous. Then $C(f+g)$ is dense because $C(f) \cap C(g) \subseteq C(f+g)$ and because the intersection of two dense G_δ-subsets of a Baire space is dense.

For each $x \in X$, let

$$\mathrm{osc}(g,x) = \inf\{\mathrm{diam} f(U) : U \text{ is a neighborhood of } x\}.$$

Then f is continuous at x if and only if $\mathrm{osc}(f,x) = 0$. For each $n \in \mathbf{N}$, let

$$G_n = \{x \in X : \mathrm{osc}(f,x) < \frac{1}{n}\}.$$

Then $\cap_{n=1}^{\infty} G_n = C(f)$. Because X is a Baire space, it remains only to show that each G_n is open and dense in X.

To show that G_n is open, let $x \in G_n$. Then $\mathrm{osc}(f,x) < \frac{1}{n}$, so that x has an open neighborhood U with $\mathrm{diam} f(U) < \frac{1}{n}$. But U is a neighborhood of each of its elements, and hence $U \subseteq G_n$.

Finally, to show that G_n is dense in X, suppose, by way of contradiction, that G_n is not dense. Then there is a nonempty open set U disjoint from G_n. Since f is upper semicontinuous, we may assume that $f(U)$ is bounded above by some number b. Let $\varepsilon = \frac{1}{2n}$.

Define U_1 to be the set of all x in U for which there exists a y in U with $f(x) < f(y) - \varepsilon$. Note that U_1 is open because f is upper semicontinuous. Also U_1 is dense in U since for every nonempty open subset V of U, $f(V)$ has diameter greater than ε. Continuing by induction, there exists a nested sequence (U_k) of open dense subsets of U such that for each $x \in U_{k+1}$ there is a $y \in U_k$ with $f(x) < f(y) - \varepsilon$.

Because X is a Baire space, there exists some x in $\cap_{n=1}^{\infty} U_n$. For each $k > 1$, $x \in U_k$, so that there are $y_{k-1} \in U_{k-1}, \ldots, y_1 \in U_1$, $y_0 \in U$ with

$$f(x) < f(y_{k-1}) - \varepsilon < \ldots < f(y_1) - (k-1)\varepsilon < f(y_0) - k\varepsilon \leq b - k\varepsilon.$$

So for all k, $f(x) < b - k\varepsilon$, which is a contradiction because f is finite valued. Therefore each G_n must be dense, and this finishes the proof. ☐

Having our functions continuous at a dense set of points plays a crucial role in our theory. For this reason we restrict our attention to the linear subspace of $SC(X)$ given by

$$DSC(X) = \{f \in SC(X) : C(f) \text{ is dense in } X\}.$$

Proposition 2.1 says that if X is a Baire space, then $DSC(X) = SC(X)$.

Whether X is a Baire space or not, the converse of Proposition 2.1 is not true. For an example, let X be the interval [0,1], and let E be the subset of endpoints of the middle third intervals (whose deletion results in the usual Cantor set). Define f on X by $f(x) = 1$ for $x \in E$ and $f(x) = 0$ for $x \in X \setminus E$. Then $C(f)$ is dense in X, but f is not in $SC(X)$.

Nevertheless, by putting an equivalence relation on the set of f with dense $C(f)$, we can equate such functions with the semicontinuous functions.

We define the set of densely continuous real-valued functions on X to be the set, $DC(X)$, of all real-valued functions f on X such that $C(f)$ is dense in X. In particular, $DSC(X) \subseteq DC(X)$. For each $f \in DC(X)$, think of the restriction of f to $C(f)$, $f|_{C(f)}$, as a subset of $X \times \mathbf{R}$. Then for such f, let $\overline{f}$ denote the closure of $f|_{C(f)}$ in $X \times \mathbf{R}$. This definies an equivalence relation on $DC(X)$ by relating f to g whenever $\overline{f} = \overline{g}$.

Now define the set of densely continuous forms on X to be the set

$$D(X) = \{\overline{f} : f \in DC(X)\}.$$

This space with an appropriate function space topology is introduced in [5], and is also studied in [6] under a hyperspace topology.

The members of $D(X)$ can be considered as multifunctions from X to $\mathbf{R}$. For each $x \in X$, let

$$\overline{f}(x) = \{y \in \mathbf{R} : (x, y) \in \overline{f}\}.$$

Then each $\overline{f}(x)$ is a closed subset of $\mathbf{R}$. In a sense, the formation of the multifunction $\overline{f}$ from the densely continuous function f eliminates all the inessential points of discontinuity of f; that is, $\overline{f}(x)$ is a singleton subset of $\mathbf{R}$ whenever x is an inessential point of discontinuity of f (such as a removable point of discontinuity).

The graph of each function in $C(X)$ is closed in $X \times \mathbf{R}$, so that $C(X)$ can be identified with a subset of $D(X)$.

The next two lemmas show us how to identify elements in $D(X)$.

Lemma 2.1. *Let $f, g \in DC(X)$. If $\overline{f} = \overline{g}$, then $f(x) = g(x)$ for every $x \in C(f) \cap C(g)$.*

Proof. Let $x \in C(f) \cap C(g)$. Now $\mathrm{osc}(f, x) = 0 = \mathrm{osc}(g, x)$, so that $\overline{f}(x)$ and $\overline{g}(x)$ must be singleton sets. Since $\overline{f} = \overline{g}$, $\{f(x)\} = \overline{f}(x) = \overline{g}(x) = \{g(x)\}$; and therefore $f(x) = g(x)$. □

In general, the $C(f) \cap C(g)$ in Lemma 2.1 may be empty. However, if X is a Baire space, the intersection of two (even countably many) dense G_δ-subsets is dense, so that $C(f) \cap C(g)$ is dense in this case. Because of this, if X is a Baire space, the converse of Lemma 2.1 is true. More generally, we have the following lemma.

Lemma 2.2. *Let $f, g \in DC(X)$. If $f(x) = g(x)$ for every x in some dense subset of X, then $\overline{f} = \overline{g}$.*

Proof. Suppose that $f(x) = g(x)$ for every $x \in Z$, where Z is dense in X. Let $x \in X$ and let $t \in \overline{f}(x)$. For every neighborhood U of x and every $m \in \mathbf{N}$, there exists an $x_{U,m} \in U \cap Z$ such that $|f(x_{U,m}) - t| < \frac{1}{m}$. This defines a net $(x_{U,m})$ directed on the set of such pairs (U, m), ordered by: $(U, m) \leq (V, n)$ if and only if $V \subseteq U$ and $m \leq n$. Then $(x_{U,m})$ converges to x in X, and $(f(x_{U,m}))$ converges to t in $\mathbf{R}$. By hypothesis, each $g(x_{U,m}) = f(x_{U,m})$, so that $(g(x_{U,m}))$ also converges to t in $\mathbf{R}$. This means that $t \in \overline{g}(x)$, and hence $\overline{f}(x) \subseteq \overline{g}(x)$. Similarly, $\overline{g}(x) \subseteq \overline{f}(x)$, so that $\overline{f}(x) = \overline{g}(x)$. Since this is true for all $x \in X$, $\overline{f} = \overline{g}$. □

The final proposition in this section shows that the forms in $D(X)$ can be genereated from semicontinuous functions on X.

Proposition 2.2. *The set* $D(X) = \{\overline{f} : f \in DSC(X)\}$.

Proof. Let $f \in DC(X)$. We need to find an $f_0 \in DSC(X)$ such that $\overline{f_0} = \overline{f}$. To this end, define $f_0 = f_+ + f_-$, where

$$f_+(x) = \max\Big\{0, \sup\{\inf\{f(x') \,:\, x' \in U \cap C(f)\} : U \text{ is a neighborhood of } x\}\Big\},$$

$$f_-(x) = \min\Big\{0, \inf\{\sup\{f(x') \,:\, x' \in U \cap C(f)\} : U \text{ is a neighborhood of } x\}\Big\}.$$

First we show that f_- is upper semicontinuous; a similar argument shows that f_+ is lower semicontinuous. Fix x_0 and let $\varepsilon > 0$. Because $f_-(x) \leq 0$ for all x, we may suppose, without loss of generality, that $f_-(x_0) < 0$. Then there is a neighborhood U_0 of x_0 such that $\sup\{f(x) : x \in U_0 \cap C(f)\} < 0$.

Suppose, by way of contradiction, that for every neighborhood U of x_0 contained in U_0 there exists an $x_U \in U$ with $f_-(x_0) + \varepsilon \leq f_-(x_U)$. Then for each such U,

$$f_-(x_0) + \varepsilon \leq f_-(x_U) \leq \inf\Big\{\sup\ \{\ f(x) : x \in U' \cap C(f)\} : U' \text{ is a neighborhood of } x_U\Big\}$$

$$\leq \sup\{f(x) : x \in U \cap C(f)\}.$$

This means that

$$f_-(x_0) + \varepsilon \leq \inf\Big\{\sup\ \{\ f(x) : x \in U' \cap C(f)\} : \\ U' \text{ is a neighborhood of } x_0\Big\} = f_-(x_0),$$

which is a contradiction. Therefore x_0 has a neighborhood U such that for each $x \in U$, $f_-(x) < f_-(x_0) + \varepsilon$; showing that f_- is upper semicontinuous at x_0.

We now know that f_0 is in $SC(X)$. It remains to show that f_0 is densely continuous and that $\overline{f_0} = \overline{f}$. To do this, we first show that $f_0(x) = f(x)$ for all $x \in C(f)$; so let $x \in C(f)$. Because f is continuous at x,

$$\begin{aligned} f(x) &= \inf\Big\{\sup\{f(x') : x' \in U \cap C(f)\} : U \text{ is a neighborhood of } x\Big\} \\ &= \sup\Big\{\inf\{f(x') : x' \in U \cap C(f)\} : U \text{ is a neighborhood of } x\Big\}. \end{aligned}$$

Therefore $f_0(x) = \max\{0, f(x)\} + \min\{0, f(x)\} = f(x)$.

Next we show that $C(f) \subseteq C(f_0)$; so let $x \in C(f)$. To see that f_0 is continuous at x, let $\varepsilon > 0$. Then x has a neighborhood U such that $f(U) \subseteq (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. So for every $x' \in U$,

$$\max\{0, f(x') - \frac{\varepsilon}{2}\} \leq f_+(x') \leq \max\{0, f(x')\},$$

$$\min\{0, f(x')\} \leq f_-(x') \leq \min\{0, f(x') + \frac{\varepsilon}{2}\}.$$

Therefore

$$\begin{aligned} f_0(x) - \varepsilon = f(x) - \varepsilon &< f(x') - \frac{\varepsilon}{2} \leq f_+(x') + f_-(x') \\ &\leq f(x') + \frac{\varepsilon}{2} \\ &< f(x) + \varepsilon = f_0(x) + \varepsilon, \end{aligned}$$

so that $f_0(U) \subseteq (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$; and hence $x \in C(f_0)$.

We now have that $C(f) \cap C(f_0) = C(f)$, which is dense in X. Since $f_0(x) = f(x)$ for all $x \in C(f)$, it follows from Lemma 2.2 that $\overline{f_0} = \overline{f}$. ☐

In particular, if X is a Baire space, Propositions 2.1 and 2.2 imply that $D(X) = \{\overline{f} : f \in SC(X)\}$. An appropriate topology is given for this space $D(X)$ in the next section. We call this the space of *semicontinuous forms* on X (or if X is not a Baire space, this might more technically be called the space of densely continuous semicontinuous forms on X).

3. Spaces of Semicontinuous Forms

The topology of uniform convergence on compact sets for the space $D(X)$ can be defined by considering each of the members of $D(X)$ as a function from X into the hyperspace, $2^{\mathbf{R}}$, of closed subsets of $\mathbf{R}$. Let $F(X, 2^{\mathbf{R}})$ be the set of such functions. Let H be the Hausdorff metric on $2^{\mathbf{R}}$ obtained from the usual metric on $\mathbf{R}$ defined for nonempty $A, B \in 2^{\mathbf{R}}$ by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\},$$

where $d(s, T) = \inf\{|s - t| : t \in T\}$; and $H(A, \emptyset) = \infty$ if $A \neq \emptyset$.

For each compact subset A of X, let p_A be the (extended-valued) pseudometric on $F(X, 2^{\mathbf{R}})$ defined by

$$p_A(\phi, \psi) = \sup\{H(\phi(x), \psi(x)) : x \in A\}$$

for each $\phi, \psi \in F(X, 2^{\mathbf{R}})$. Then the topology of uniform convergence on compact sets for the space $F(X, 2^{\mathbf{R}})$ is the topology generated by the pseudometrics p_A over all compact subsets A of X. The Hausdorff metric has been used in [8] in a similar, but not exactly the same, way to define a function space topology when $X = \mathbf{R}$. See [2] for many references pertaining to set-valued functions.

Now for the space $D(X)$, the topology of uniform convergence on compact sets is obtained by considering $D(X)$ as a subspace of $F(X, 2^{\mathbf{R}})$, where the latter space has this topology. We denote this space by $D_k(X)$. A basic open set in this topology looks like

$$\langle \overline{f}, A, \varepsilon \rangle = \{\overline{g} \in D(X) : p_A(\overline{f}, \overline{g}) < \varepsilon\}$$

for $\overline{f} \in D(X)$, A compact in X, and $\varepsilon > 0$.

If we use finite subsets of X instead of compact sets A, we obtain the space $D_p(X)$ having the topology of pointwise convergence. Also if X is used instead of compact sets A, we get the space $D_u(X)$ having the topology of uniform convergence. This latter space, $D_u(X)$, is a (extended-valued) metric space with metric p_X. It is shown in [5] that p_X is a complete metric. The spaces $D_k(X)$ and $D_p(X)$ are completely regular Hausdorff spaces because the pseudometrics generating the topology form a Hausdorff uniform structure on $F(X, 2^{\mathbf{R}})$.

The spaces $C_k(X)$, $C_p(X)$ and $C_u(X)$ of continuous functions with corresponding topologies are subspaces of $D_k(X)$, $D_p(X)$ and $D_u(X)$, respectively. Our goal is to develop analogs for the theorems about the spaces of continuous functions that extend to the spaces of semicontinuous forms. Our emphasis is on $D_k(X)$, but the other topologies play a role.

It is reasonable to ask first whether $D_k(X)$ is a linear topological space, as $C_k(X)$ is. If X is a Baire space, the set $D(X)$ does have a natural vector space structure defined by $\overline{f} + \overline{g} = \overline{f+g}$ and $a\overline{f} = \overline{af}$ for $\overline{f}, \overline{g} \in D(X)$ and $a \in \mathbf{R}$. To see that these operations are well-defined, suppose that $\overline{f_1} = \overline{f_2}$ and $\overline{g_1} = \overline{g_2}$. Because X is a Baire space, $C(f_1) \cap C(f_2) \cap C(g_1) \cap C(g_2)$ is dense in X. The fact that $\overline{f_1 + g_1} = \overline{f_2 + g_2}$ and $\overline{af_1} = \overline{af_2}$ now follows from Lemmas 2.1 and 2.2.

Unfortunately, the vector space operations are not continuous on $D_k(X)$, even if X is compact. The problem is that members of $D(X)$ may be unbounded on compact sets and their neighborhoods. So in order to have a space where these operations are continuous, we restrict our attention to the members of $D(X)$ that are locally bounded in the following sense.

If $\overline{f} \in D(X)$ and $A \subseteq X$, we say that $\overline{f}$ is *bounded on A* provided that the set $\overline{f}(A) \equiv \cup\{\overline{f}(x) : x \in A\}$ is a bounded subset of $\mathbf{R}$. Then $\overline{f}$ is *locally bounded* provided that each point of X has a neighborhood on which $\overline{f}$ is bounded. Note that if $\overline{f}$ is locally bounded, then it must be bounded on every compact set.

Now define $D^{\#}(X)$ to be the set of members of $D(X)$ that are locally bounded. Then $D^{\#}_k(X)$ denotes this space as a subspace of $D_k(X)$.

Observe that for each $\overline{f} \in D^{\#}(X)$ and each $x \in X$, $\overline{f}(x) \neq \emptyset$. This means that the topology on $D^{\#}_k(X)$ is obtained by using the Hausdorff metric as a finite-valued metric on the space of nonempty closed subsets of $\mathbf{R}$.

Theorem 3.1. *If X is locally compact, $D^{\#}_k(X)$ is a locally convex linear topological space.*

Proof. To show that addition is continuous, let $\overline{f_1}, \overline{f_2} \in D^{\#}_k(X)$, let A be compact in X, and let $\varepsilon > 0$. Take B to be a compact set in X containing A in its interior. Let $\overline{g_1} \in \langle \overline{f_1}, B, \frac{\varepsilon}{4} \rangle$ and $\overline{g_2} \in \langle \overline{f_2}, B, \frac{\varepsilon}{4} \rangle$. We need to show that $\overline{g_1 + g_2} \in \langle \overline{f_1 + f_2}, A, \varepsilon \rangle$.

Let x be an arbitrary element of A. To show that $H(\overline{(f_1 + f_2)}(x), \overline{(g_1 + g_2)}(x)) \leq \frac{3\varepsilon}{4}$, let $s \in \overline{(f_1 + f_2)}(x)$. It suffices to find a $t \in \overline{(g_1 + g_2)}(x)$ such that $|s - t| \leq \frac{3\varepsilon}{4}$ (since a similar argument also works in the other direction). Then because x is arbitrary, we would have $p_A(\overline{f_1 + f_2}, \overline{g_1 + g_2}) < \varepsilon$.

Let $C = C(f_1) \cap C(f_2) \cap C(g_1) \cap C(g_2)$, and let $\mathcal{U}$ be the directed set of neighborhoods of x contained in the interior of B. Then for each $U \in \mathcal{U}$, there is an $x_U \in U \cap C$ such that $|s - (f_1 + f_2)(x_U)| < \frac{\varepsilon}{4}$. Because f_1, f_2, g_1, g_2 are bounded on B, by passing to subnets, we may assume that the nets $(f_1(x_U))$, $(f_2(x_U))$, $(g_1(x_U))$, $(g_2(x_U))$ converge; say to s_1, s_2, t_1, t_2, respectively. Then the net $((f_1 + f_2)(x_U))$ converges to $s_1 + s_2$, so that $|s - (s_1 + s_2)| \leq \frac{\varepsilon}{4}$.

By choice of $\overline{g_1}$ and $\overline{g_2}$, for each $U \in \mathcal{N}$, $|f_1(x_U) - g_1(x_U)| < \frac{\varepsilon}{3}$ and $|f_2(x_U) - g_2(x_U)| < \frac{\varepsilon}{4}$. Therefore $|s_1 - t_1| \leq \frac{\varepsilon}{4}$ and $|s_2 - t_2| \leq \frac{\varepsilon}{4}$. Define $t = t_1 + t_2$. Then

$$|s - t| \leq |s - (s_1 + s_2)| + |s_1 - t_1| + |s_2 - t_2| \leq \frac{3\varepsilon}{4}.$$

Finally, to check that $t \in \overline{(g_1 + g_2)}(x)$, observe that

$$t = t_1 + t_2 = \lim_{U \in \mathcal{U}} g_1(x_U) + \lim_{U \in \mathcal{U}} g_2(x_U) = \lim_{U \in \mathcal{U}} (g_1 + g_2)(x_U).$$

This ends the argument that addition is continuous.

To show that scalar multiplication is continuous, let $\overline{f} \in D_k^{\#}(X)$, let $a \in \mathbf{R}$, let A be compact in X, and let $\varepsilon > 0$. Take B to be a compact set in X containing A in its interior. Let M be an upper bound in $\mathbf{R}$ of the set $\{|s| + \frac{\varepsilon}{4} : s \in \overline{f}(x)$ for some $x \in B\} \cup \{|a| + 1\}$. Let $\overline{g} \in \langle \overline{f}, B, \frac{\varepsilon}{4M} \rangle$ and let b be in the interval $(a - \frac{\varepsilon}{4M}, a + \frac{\varepsilon}{4M})$. We need to show that $\overline{bg} \in \langle \overline{af}, A, \varepsilon \rangle$.

Let x be an arbitrary element of A. To show that $H(\overline{af}(x), \overline{bg}(x)) \leq \frac{3\varepsilon}{4}$, let $s \in \overline{af}(x)$. It suffices to find a $t \in \overline{bg}(x)$ such that $|s - t| \leq \frac{3\varepsilon}{4}$ (since a similar argument works in the other direction). Then since x is arbitrary, we would have $p_A(\overline{af}, \overline{bg}) < \varepsilon$.

Let $C = C(f) \cap C(g)$, and let $\mathcal{U}$ be the directed set of neighborhoods of x contained in the interior of B. Now for each $U \in \mathcal{U}$, there is an $x_U \in U \cap C$ such that $|s - af(x_U)| < \frac{\varepsilon}{4}$. Because f and g are bounded on B, by passing to subnets, we may assume that the nets $(f(x_U))$ and $(g(x_U))$ converge; say to s' and t', respectively. Then the net $(af(x_U))$ converges to as', so that $|s - as'| \leq \frac{\varepsilon}{4}$.

By choice of $\overline{g}$, for each $U \in \mathcal{U}$, $|f(x_U) - g(x_U)| < \frac{\varepsilon}{4M}$. Therefore $|s' - t'| \leq \frac{\varepsilon}{4M}$. Define $t = bt'$. Note that $M \geq 1$, so that $\frac{\varepsilon}{4M} \leq \frac{\varepsilon}{4}$. Then for each $U \in \mathcal{U}$,

$$|g(x_U)| \leq |f(x_U)| + \frac{\varepsilon}{4M} \leq |f(x_U)| + \frac{\varepsilon}{4} \leq M.$$

It follows that $|t'| \leq M$, and hence

$$|s-t| \leq |s-as'| + |a||s'-t'| + |t'||a-b| \leq \frac{\varepsilon}{4} + M \cdot \frac{\varepsilon}{4M} + M \cdot \frac{\varepsilon}{4M} = \frac{3\varepsilon}{4}.$$

Finally, to check that $t \in \overline{bg}(x)$, observe that

$$t = bt' = b \lim_{U \in \mathcal{U}} g(x_U) = \lim_{U \in \mathcal{U}} bg(x_U).$$

This finishes the argument that scalar multiplication is continuous.

To show that the topological vector space $D_k^{\#}(X)$ is locally convex, it suffices to show that $\langle \overline{f_0}, A, \varepsilon \rangle$ is convex, where f_0 is the constant zero function on X, A is compact in X, and $\varepsilon > 0$. Since X is locally compact, we may assume that A is equal to the closure of its interior; say U is its interior. We need only show that the set $\{t\overline{f} + (1-t)\overline{f_0} : 0 \leq t \leq 1\}$ is in $\langle \overline{f_0}, A, \varepsilon \rangle$ for any $\overline{f} \in \langle \overline{f_0}, A, \varepsilon \rangle$. Let $0 \leq t \leq 1$, let $g = tf + (1-t)f_0 = tf$, and let $x \in A$. For every $x' \in U \cap C(f)$,

$$|g(x')| = |tf(x')| \leq |t||f(x')| \leq |f(x')|,$$

and hence the diameter of $\overline{g}(x)$ is less than or equal to the diameter of $\overline{f}(x)$. Therefore

$$\begin{aligned} p_A(\overline{g}, \overline{f_0}) &= \sup\{H(\overline{g}(x), \{0\}) : x \in A\} \\ &\leq \sup\{H(\overline{f}(x), \{0\}) : x \in A\} = p_A(\overline{f}, \overline{f_0}), \end{aligned}$$

so that $\overline{g} \in \langle \overline{f_0}, A, \varepsilon \rangle$. □

Theorem 3.1 is not true if the hypothesis that X be locally compact is dropped. This is illustrated by the next example.

Example 3.1 Let X be $\mathbf{R}^2$ with $\mathbf{R} \times \{0\}$ identified with a point, and let $p : \mathbf{R}^2 \to X$ be the natural projection. Let X have the quotient topology relative to p, and let x_0 be the point $p(\mathbf{R} \times \{0\})$. Then X is locally compact at every point except x_0. The following argument shows that addition is not continuous on $D_k^{\#}(X)$.

Let $U = p(\mathbf{R} \times (0, \infty))$ and $V = p(\mathbf{R} \times (-\infty, 0))$. Take $\overline{f}, \overline{g} \in D_k^{\#}(X)$ so that $f(x) = 1$ if $x \in U$ and $f(x) = -1$ if $x \in V$, and so that $g(x) = -1$ if $x \in U$ and $g(x) = 1$ if $x \in V$. Then $\overline{f+g} = \overline{f_0}$, where f_0 is the zero function on X. Let A be any compact subset of X containing x_0, and let $\varepsilon > 0$ be arbitrary. We need to find a $\overline{g_0} \in \langle \overline{g}, A, \varepsilon \rangle$ such that $\overline{f + g_0} \notin \langle \overline{f+g}, \{x_0\}, 1 \rangle$.

Let A' be a compact subset of $\mathbf{R}^2$ such that $p(A') = A$, let B' be a compact subset of $\mathbf{R}^2$ containing A' in its interior, and let $B = p(B')$. Now define $\overline{g_0}$ by taking $g_0(x) = g(x)$ if $x \in B$ and $g_0(x) = -g(x)$ if $x \in X \setminus B$. To show that $\overline{g_0} \in \langle \overline{g}, A, \varepsilon \rangle$, let $x \in A$. If $x \neq x_0$, then x is in the interior of B, so that $\overline{g_0}(x) = \overline{g}(X)$. If $x = x_0$, then $\overline{g_0}(x) = \{1, -1\} = \overline{g}(x)$. Finally to show that $\overline{f + g_0} \notin \langle \overline{f + g}, \{x_0\}, 1 \rangle$, note that $x_0 \in \overline{U \setminus B}$. So there is a net (x_α) in $(U \setminus B) \cap C(f) \cap C(g_0)$ that converges to x_0. But each $(f + g_0)(x_\alpha) = f(x_\alpha) + g_0(x_\alpha) = 1 + 1 = 2$. Therefore $2 \in \overline{f + g_0}(x_0)$, while $\overline{f + g}(x_0) = \{0\}$. This finishes the argument that addition is not continuous on $D_k^{\#}(X)$.

Question 3.1. *For any space X, if addition is continuous on $D_k^{\#}(X)$, must X be locally compact?*

The remainder of the paper will be concerned with properties of this space $D_k^{\#}(X)$ of locally bounded semicontinuous forms on X that imitate and generalize corresponding properties of the space $C_k(X)$ of continuous functions. In order to prove some of these results, we use properties of $D_k(X)$ established in [5]. In particular, we need to know when $D_k^{\#}(X)$ is closed in $D_k(X)$, which is the topic of the next section.

4. Closed Subspaces and Metrizability

We first observe that $C(X)$ is closed in all of the spaces of semicontinuous forms that we have discussed. This is because of the following fact.

Proposition 4.1. *The set $C(X)$ is closed as a subset of $D_p(X)$.*

Proof. Let $\overline{f} \in D_p(X) \setminus C(X)$. Then there is an $x \in X$ such that $\operatorname{osc}(f, x) > 0$. So either there exist $s, t \in \overline{f}(x)$ with $s \neq t$ or $\overline{f}(x) = \emptyset$. Suppose the former. Then let $\varepsilon = \frac{1}{2}|s - t|$. If $\overline{g} \in \langle \overline{f}, \{x\}, \varepsilon \rangle$, $\overline{g}(x)$ must contain at least two elements, so that $\overline{g} \notin C(X)$. On the other hand, suppose $\overline{f}(x) = \emptyset$. Then if $\overline{g} \in \langle \overline{f}, \{x\}, 1 \rangle$, $\overline{g}(x) = \emptyset$, so that $\overline{g} \notin C(X)$ in this case as well. □

Note that the above proposition shows that $C(X)$ is closed in $D_p^{\#}(X)$, and hence in $D_k^{\#}(X)$. The more interesting problem is to determine when $D^{\#}(X)$ is closed in $D_k(X)$.

We define a space X to be a *weak k-space* provided that for every x in X, every sequence of regular closed subsets of X that cofinally intersects every neighborhood of x, cofinally intersects some compact subset of X. The next proposition and following example justifies the name that we have given to this property.

Proposition 4.2. *Every k-space is a weak k-space.*

Proof. Let X be a k-space, and let (C_n) be a sequence of regular closed sets in X which cofinally intersect every neighborhood of x. Suppose, by way of contradiction, that (C_n) does not cofinally intersect any compact subset of X. Then we may assume, without loss of generality, that x is not in any C_n. If C is the union of the C_n, then $x \in \overline{C} \setminus C$, and hence C is not closed in X. On the other hand, each compact subset of X intersects C in a compact set, which contradicts X being a k-space. □

Example 4.1 Let $X = \mathbf{R}$, where the open sets in X are all subsets not containing 0 and all subsets containing 0 that have countable complement. Since every compact set in X is finite, X is not a k-space. However, X is a weak k-space because every sequence of regular closed subsets of X that cofinally intersects every neighborhood of point x, cofinally intersects the compact set $\{x\}$.

This property of X being a weak k-space is precisely the property that characterizes $D^{\#}(X)$ being closed in $D_k(X)$.

Theorem 4.1. *As a subset, $D^{\#}(X)$ is closed in $D_k(X)$ if and only if X is a weak k-space.*

Proof. Suppose that X is a weak k-space. Let $\overline{f} \in D_k(X) \setminus D^{\#}(X)$. Then $\overline{f}$ is not locally bounded at some $x_0 \in X$. For each n, define $R_n = (n-1, n+1) \cup (-n-1, -n+1)$, define U_n

to be the union of all open U in X such that $f(U) \subseteq R_n$, and define C_n to be the regular closed set $\overline{U_n}$.

Since the sequence (C_n) cofinally intersects every neighborhood of x_0, it cofinally intersects some compact set A in X. So there is a subsequence (C_{n_k}) of (C_n) such that $C_{n_k} \cap A \neq \emptyset$; say $x_k \in C_{n_k} \cap A$. For each k, $x_k \in \overline{U_{n_k}}$ and $f(U_{n_k})$ is contained in the bounded set R_{n_k}, so that $\overline{f(x_k)} \cap \overline{R_{n_k}} \neq \emptyset$.

Let $\overline{g} \in \langle \overline{f}, A, 1 \rangle$. Then for each k, $\overline{g}(x_k) \cap ([n_k - 2, \infty) \cup (-\infty, -n_k + 2]) \neq \emptyset$. Since A is compact, the sequence (x_k) has a cluster point x' in A. We now see that $\overline{g}$ is not locally bounded a x', and therefore $\overline{g} \notin D^{\#}(X)$.

For the converse, suppose that X is not a weak k-space. Then there exist an $x_0 \in X$ and a sequence (C_n) of regular closed subsets of X that cofinally intersects every neighborhood of x_0 but does not cofinally intersect any compact subset of X. We may assume without loss of generality that for all n, $x_0 \notin C_n$.

Each $C_n = \overline{U_n}$ for some open subset U_n of X. Define pairwise disjoint sequence (V_n) of open subsets of X by induction as follows. Let $V_1 = U_1$, and for every $n > 1$, let $V_n = U_n \setminus (\overline{V_1} \cup \cdots \cup \overline{V_{n-1}})$.

To show that (V_n) cofinally intersects every neighborhood U of x_0, let $n \in \mathbf{N}$. Define $V = U \setminus (C_1 \cup \cdots \cup C_n)$, which is also a neighborhood of x_0. Let m be the first integer such that $V \cap C_m \neq \emptyset$. Then $m > n$ and $V \cap U_m \neq \emptyset$, and hence $V \cap V_m \neq \emptyset$.

Define $f : X \to \mathbf{R}$ by $f(x) = n$ if $x \in V_n$ for some n, $f(x) = 0$ otherwise. then f is densely continuous, so that $\overline{f} \in D(X)$. Also $\overline{f}$ is not locally bounded at x_0, and thus $\overline{f} \notin D^{\#}(X)$.

It remains to show that $\overline{f}$ is a limit point of the set $D^{\#}(X)$ in $D_k(X)$. So let A be compact in X and let $\varepsilon > 0$. Because each $V_n \subseteq C_n$, there exists an n such that $\overline{V_k} \cap A = \emptyset$ for all $k \geq n$. Define $g : X \to \mathbf{R}$ by $g(x) = k$ if $x \in V_k$ for $1 \leq k \leq n$, and $g(x) = 0$ otherwise. Now g is densely continuous and bounded, so that $\overline{g} \in D^{\#}(X)$. But also $\overline{g} \in \langle \overline{f}, A, 1 \rangle$, which shows that $\overline{f}$ is a limit point of $D^{\#}(X)$. Therefore $D^{\#}(X)$ is not closed in $D_k(X)$. ∎

An even more difficult problem, for which we have no answer, is to determine the spaces X for which $D^{\#}(X)$ is a G_{δ}-subset of $D_k(X)$.

It is shown in [5] that whenever X is locally compact and σ-compact, then $D_k(X)$ is completely metrizable. Using this fact, we obtain an analog to the theorem that $C_k(X)$ is completely metrizable if and only if X is a hemicompact k-space. Note that a hemicompact space having point countable type is locally compact and σ-compact.

Theorem 4.2. *Let X have point countable type. Then the following are equivalent.*

(a) $D_k^{\#}(X)$ is completely metrizable.

(b) $D_k^{\#}(X)$ is first countable.

(c) X is hemicompact.

(d) X is locally compact and σ-compact.

Proof. (b) implies (c) because $C_k(X)$ is a subspace of $D_k^{\#}(X)$, and $C_k(X)$ is first countable if and only if X is hemicompact [7]. (d) implies (a) because that is shown to be true for the space $D_k(X)$ in [5], and $D_k^{\#}(X)$ is closed in $D_k(X)$ by Theorem 4.1. □

Corollary 4.1. *If X is locally compact, then $D_k^{\#}(X)$ is a Fréchet-Urysohn space if and only if $D_k^{\#}(X)$ is (completely) metrizable.*

Proof. If $D_k^{\#}(X)$ is a Fréchet-Urysohn space, so is it's subspace $C_k(X)$. This means that X is a $k\gamma$-space (see [9]), which is a property that when combined with locally compact implies hemicompact. □

A hemicompact k-space need not be locally compact (cf. Example 3.1). So we cannot replace the hypothesis of point countable type in Theorem 4.2 by the weaker hypothesis of k-space. However, in general, the conditions (b) and (c) in Theorem 4.2 are equivalent, as expressed by the next proposition, whose proof is analogous to the proof of the corresponding result for $C_k(X)$.

Proposition 4.3. *The space $D_k^{\#}(X)$ is first countable if and only if X is hemicompact.*

Now $C_k(X)$ is metrizable if and only if it is first countable because $C_k(X)$ is a topological group under addition. However, $D_k^{\#}(X)$ is not a topological group under addition, unless X is locally compact. This suggests the following question.

Question 4.1. *Is $D_k^{\#}(X)$ metrizable whenever it is first countable? If not, is local compactness of X necessary for $D_k^{\#}(X)$ to be metrizable?*

The space X in Example 3.1 is hemicompact. We do not know whether $D_k^{\#}(X)$ is metrizable. On the other hand, the argument in [5] used to show that (d) implies (a) in Theorem 4.2 also shows that $D_k^{\#}(X)$ is submetrizable. The thing that is needed for this is a continuous function from a locally compact hemicompact space Z (in this case, $\mathbf{R}^2$) onto X so that the inverse image of each dense G_δ-set is dense. Such a function induces a continuous injection from $D_k^{\#}(X)$ into $D_k^{\#}(Z)$.

5. Properties Equivalent in a Metric Space

In this section, we consider the following cardinal functions applied to $D_k^{\#}(X)$. The *density*, $d(Z)$, of space Z is the minimum cardinality of a dense subset of Z. The *weight*, $w(Z)$, of Z is the minimum cardinality of a base for Z. The *network weight*, $nw(Z)$, of Z is the minimum cardinality of a network for Z. The *cellularity*, $c(Z)$, of Z is the supremum of the cardinalities of all pairwise disjoint families of nonempty open subsets of Z. Finally, let $|Z|$ denote the cardinality of Z. In general for a space Z, we have

$$c(Z) \leq d(Z) \leq nw(Z) \leq w(Z).$$

If Z is metrizable, then these cardinal numbers are all equal. Two references to cardinal functions on the function spaces $C_p(X)$ and $C_k(X)$ are [1] and [7].

Although we have not shown that $D_k^{\#}(X)$ is metrizable whenever X is hemicompact, we show at least, in the next theorem, that for such X, the above cardinal functions are equal on $D_k^{\#}(X)$.

Theorem 5.1. *If X is hemicompact, then*

$$c(D_k^{\#}(X)) = d(D_k^{\#}(X)) = nw(D_k^{\#}(X)) = w(D_k^{\#}(x)).$$

Proof. Let $\mathcal{A} = \{A_n : n \in \mathbf{N}\}$ be an increasing sequence of compact subsets of X such that each compact subset of X is contained in some A_n. Let $m = c(D_k^{\#}(X))$. It suffices to find a base for $D_k^{\#}(X)$ with cardinality m.

For each $n \in \mathbf{N}$, by Zorn's lemma, there exists a pairwise disjoint family $\mathcal{W}_n$ of basic open subsets of $D_k^{\#}(X)$ of the form $\langle \overline{f}, A, \varepsilon \rangle$, where $\overline{f} \in D_k^{\#}(X)$, A is compact and contains A_n and $0 < \varepsilon < \frac{1}{n}$, such that $\cup \mathcal{W}_n$ is dense in $D_k^{\#}(X)$. Then for each n, define

$$F_n = \{\overline{f} \in D_k^{\#}(X) : \langle \overline{f}, A, \varepsilon \rangle \in \mathcal{W}_n \text{ for some } A \text{ and some } \varepsilon\},$$

and

$$\mathcal{B}_n = \Big\{\langle \overline{f}, A, r \rangle \;:\; \overline{f} \in F_n,\; A \in \mathcal{A}, \\ \text{and } r \text{ is a positive rational number}\Big\}.$$

Then set $\mathcal{B} = \cup\{\mathcal{B}_n : n \in \mathbf{N}\}$, and note that $|\mathcal{B}| \leq m$.

It remains to show that $\mathcal{B}$ is a base for $D_k^{\#}(X)$. To this end, let $\overline{g} \in D_k^{\#}(X)$, let B be compact in X, and let $\delta > 0$. Choose $n \in \mathbf{N}$ such that $B \subseteq A_n$ and $n > \frac{6}{\delta}$.

Suppose, by way of contradiction, that $F_n \cap \langle \overline{g}, A_n, \frac{3}{n} \rangle = \emptyset$. Then $\langle \overline{g}, A_n, \frac{1}{n} \rangle \cap \langle \overline{f}, A, \varepsilon \rangle = \emptyset$ for every $\langle \overline{f}, A, \varepsilon \rangle \in \mathcal{W}_n$. But then $\mathcal{W}_n \cup \{\langle \overline{g}, A_n, \frac{1}{n} \rangle\}$ is larger than $\mathcal{W}_n$, which contradicts the maximality of $\mathcal{W}_n$. So let $\overline{f} \in F_n \cap \langle \overline{g}, A_n, \frac{3}{n} \rangle$. Then $\langle \overline{f}, A_n, \frac{3}{n} \rangle \in \mathcal{B}_n \subseteq \mathcal{B}$, and one can check that $\overline{g} \in \langle \overline{f}, A_n, \frac{3}{n} \rangle \subseteq \langle \overline{g}, B, \delta \rangle$. $\square$

We now give a characterization (at least when X is locally compact) of the network weight of $D_k^{\#}(X)$. To do this we need to introduce a new kind of network on X.

Define a family, $\mathcal{P}$, of subsets of X to be a *peripheral k-network* for X provided that for every regular open subset U of X and every compact subset A of $\overline{U}$, there exists a $P \in \mathcal{P}$ such that $P \subseteq U$ and every net in U that clusters at some point of A is cofinally in P. Now the *peripheral k-network weight* of X is the minimum cardinality of a peripheral k-network for X.

Proposition 5.1. *For every space X, if $\mathcal{P}$ is a peripheral k-network for X, then the family of the interiors of the members of $\mathcal{P}$ is also a peripheral k-network for X.*

Proof. Let U be a regular open subset of X and let A be a compact subset of $\overline{U}$. There is a $P \in \mathcal{P}$ such that $P \subseteq U$ and every net (x_α) in U with cluster point x in A is cofinally in P. We need only show that such a net (x_α) is cofinally in $\operatorname{Int} P$. Suppose not. Then we may assume that no x_α is in $\operatorname{Int} P$. For every neighborhood V of x, there is some x_α in V; so that there exists a $y_V \in U \cap V \setminus P$. This defines a net (y_V) in U which converges to x. But this contradicts the fact that (y_V) must be cofinally in P. □

Proposition 5.2. *For every regular space X, if $\mathcal{P}$ is a peripheral k-network for X, then the family of the interiors of the closures of the members of $\mathcal{P}$ is a base for X.*

Proof. Let U be a regular open subset of X and let $x \in U$. There exists a $P \in \mathcal{P}$ such that $P \subseteq U$ and every net in U which clusters at x is cofinally in P. The argument in the proof of Proposition 5.1 shows that every such net is, in fact, cofinally in $\operatorname{Int} P$, and hence cofinally in $\operatorname{Int} \overline{P}$. Define $B = \operatorname{Int} \overline{P}$. Since U is regular open, $B \subseteq U$. A trivial net (x_α) with $x_\alpha = x$ for every α converges to x, and is hence cofinally in B. This means that $x \in B$, as needed. □

Propositions 5.1 and 5.2 have the following corollary.

Corollary 5.1. *For every regular space X,*

$$w(X) \leq pknw(X) \leq |\mathcal{T}(X)|,$$

where $\mathcal{T}(X)$ is the topology on X.

As will be seen later in this section, $pknw(X)$ is strictly larger than $w(X)$ for a large class of spaces. However, we have no example showing that $pknw(X)$ may be stricly smaller than $|\mathcal{T}(X)|$. In any case, $pknw(X)$ can now be used to determine the network weight of $D_k^{\#}(X)$.

Theorem 5.2. *For every space, $nw(D_k^{\#}(X)) \leq pknw(X)$.*

Proof. Let $\mathcal{P}$ be a peripheral k-network for X, and let $\mathcal{B}$ be a countable base for $\mathbf{R}$ that is closed under finite unions. Define $\mathcal{W}$ to be the family of all

$$W(P_1, \ldots, P_n, B_1, \ldots, B_n) \equiv \Big\{ \overline{f} \in D_k^{\#}(X) \ : \ \overline{f}(P_i) \subseteq B_i \\ \text{for } i = 1, \ldots, n \Big\},$$

for all pairs of collections $P_1, \ldots, P_n \in \mathcal{P}$ and $B_1, \ldots, B_n \in \mathcal{B}$. Note that $|\mathcal{W}| = |\mathcal{P}|$.

To show that $\mathcal{W}$ is a network for $D_k^{\#}(X)$, let $\overline{f} \in D_k^{\#}(X)$, let A be compact in X, and let $\varepsilon > 0$. Since $\overline{f}(A)$ is a bounded set in $\mathbf{R}$, there are open sets $V_1, \ldots, V_n$ in $\mathbf{R}$ such that $\overline{f}(A) \subseteq V_1 \cup \cdots \cup V_n$ and the diameter of each V_i is less than $\frac{\varepsilon}{2}$. For each $i = 1, \ldots, n$, let $B_i \in \mathcal{B}$ have diameter less than $\frac{\varepsilon}{2}$ and contain $\overline{V_i}$. Also for each i, define W_i to be the union of all open W in X such that $f(W) \subseteq V_i$; and let U_i be the regular open set $\operatorname{Int} \overline{W_i}$.

We now check that $\overline{f}(U_i) \subseteq \overline{V_i}$. Let $x \in U_i$, let $t \in \overline{f}(x)$, and let V be a neighborhood of t. There exists a $y \in U_i \cap C(f)$ such that $f(y) \in V$. Then there is a neighborhood W of y contained in U_i so that $f(W) \subseteq V$. Now there is a $z \in W \cap W_i \cap C(f)$. Then $f(z) \in V_i \cap V$, showing that $t \in \overline{V_i}$.

We next show that $A \subseteq \overline{U_1} \cup \cdots \cup \overline{U_n}$; so let $x \in A$. Then $\overline{f}(x) \cap V_i \neq \emptyset$ for some i. To see that $x \in \overline{U_i}$, let U be a neighborhood of x. Now there is a $y \in U \cap C(f)$ such that $f(y) \in V_i$. So y has a neighborhood W such that $f(W) \subseteq V_i$. Then $y \in W \subseteq U_i$, so that $U \cap U_i \neq \emptyset$.

For each $i = 1, \ldots, n$, define $A_i = A \cap \overline{U_i}$. For each such i, there is a $P_i \in \mathcal{P}$ so that $P_i \subseteq U_i$ and every net in U_i with a cluster point in A_i is cofinally in P_i. So we have for each i, $\overline{f}(P_i) \subseteq \overline{f}(U_i) \subseteq \overline{V_i} \subseteq B_i$. Then define $W = W(P_1, \ldots, P_n, B_1, \ldots, B_n)$, which is in $\mathcal{W}$ and contains $\overline{f}$.

Finally, we need to show that $W \subseteq \langle \overline{f}, A, \varepsilon \rangle$. So let $\overline{g} \in W$, and let $x \in A$. Let us start with a $t \in \overline{g}(x)$. Then there is a net (x_α) in $C(f) \cap C(g)$ converging to x such that $(g(x_\alpha))$ converges to t. Now $(f(x_\alpha))$ has a cluster point s, which must be in V_i for some i. So $(f(x_\alpha))$ is cofinally in V_i, which means that (x_α) is cofinally in U_i; and therefore (x_α) is cofinally in P_i. But $\overline{g}(P_i) \subseteq B_i$, so that $(g(x_\alpha))$ is cofinally in B_i, and hence $t \in \overline{B_i}$. Since $s \in V_i \subseteq B_i$ and the diameter of B_i is less than $\frac{\varepsilon}{2}$, we have $|s - t| < \frac{\varepsilon}{2}$. On the other hand, if we start with an $s \in \overline{f}(x)$, then $s \in V_i$ for some i. So there is a net (x_α) in $U_i \cap C(f) \cap C(g)$ which converges to x such that $(f(x_\alpha))$ converges to s. Because (x_α) must be cofinally in P_i, we may assume that each x_α is in P_i. Also since $\overline{g}(P_i) \subseteq B_i$, each $g(x_\alpha)$ is in B_i. Thus, $(g(x_\alpha))$ has a cluster point t in $\overline{g}(x) \cap \overline{B_i}$. As before, we have $|s - t| < \frac{\varepsilon}{2}$. Putting these two cases together gives us $H(\overline{f}(x), \overline{g}(x)) < \frac{\varepsilon}{2}$. This is true for all $x \in A$, so that $p_A(\overline{f}, \overline{g}) \leq \frac{\varepsilon}{2}$. We now see that $W \subseteq \langle \overline{f}, A, \varepsilon \rangle$, showing that $\mathcal{W}$ is indeed a network for $D_k^{\#}(X)$ and establishing our inequality. □

We do not know whether the inequality in Theorem 5.2 is always an equality, but it is if X is locally compact.

Theorem 5.3. *If X is locally compact, then $nw(D_k^{\#}(X)) = pknw(X)$.*

Proof. Because of Theorem 5.2, we need only show that $pknw(X) \leq nw(D_k^{\#}(X))$. Let $\mathcal{N}$ be a network for $D_k^{\#}(X)$. For each $N \in \mathcal{N}$, define

$$N^* = \{x \in X : \overline{g}(x) \cap (0, \infty) \neq \emptyset \text{ for all } \overline{g} \in N\},$$

and let $\mathcal{N}^* = \{\operatorname{Int} N^* : N \in \mathcal{N}\}$.

To show that $\mathcal{N}^*$ is a peripheral k-network for X, let U be a regular open subset of X and let $A \subseteq \overline{U}$ be compact. Choose a compact subset B of X containing A in its interior. Take f to be the characteristic function for U (i.e., $f(x) = 1$ if $x \in U$ and $f(x) = 0$ if $x \in X \setminus U$). Then $\overline{f} \in D_k^{\#}(X)$, so that there is an $N \in \mathcal{N}$ with $\overline{f} \in N \subseteq \langle \overline{f}, B, 1\rangle$. Note that $N^* \subseteq \overline{U}$ because if $x \notin \overline{U}$, then $\overline{f}(x) = \{0\}$, implying that $x \notin N^*$. Because U is regular open, it follows that $\operatorname{Int} N^* \subseteq U$.

Finally, let (x_α) be a net in U that has cluster point x in A. Now (x_α) is cofinally in the interior of B. But for all $x_\alpha \in B$ and for all $\overline{g}$ in N, $\overline{g}(x_\alpha) \cap (0, \infty) \neq \emptyset$. This means that x_α is cofinally in N^*. We can now argue as in the proof of Proposition 5.1 that (x_α) must be cofinally in $\operatorname{Int} N^*$. This finishes the argument that $\mathcal{N}^*$ is a peripheral k-network for X, and shows that $pknw(X) \leq nw(D_k^{\#}(X))$. □

Theorem 5.2 and Corollary 5.1 tell us that a dense subset of $D_k^{\#}(X)$ has no more than $|\mathcal{T}(X)|$ elements. In some cases, this is also true for the set $D^{\#}(X)$ itself. In general, we can use the fact that $|C(X)| \leq 2^{d(X)}$ (see [3]), and hence $|C(X)| \leq 2^{w(X)}$, to obtain an analogous result for $D^{\#}(X)$.

Proposition 5.3. *For every space X, $|D_k^{\#}(X)| \leq 2^{w(X)}$.*

Proof. First note that the cardinality of the family of G_δ-subsets of X is less than or equal to $2^{w(X)}$. For each G_δ-subset G in X, we have $|C(G)| \leq 2^{d(G)} \leq 2^{w(G)} \leq 2^{w(X)}$. Each $\overline{f} \in D^{\#}(X)$ is completely determined by $f|_{C(f)}$, and $C(f)$ is a G_δ-subset of X. So $|D^{\#}(X)| \leq 2^{w(X)} \cdot 2^{w(X)} = 2^{w(X)}$. □

Corollary 5.2. *If X is a second countable space, then $d(D_k^{\#}(X)) \leq |D^{\#}(X)| \leq c$.*

The next proposition gives a strict lower bound for the number of elements in a dense subset of $D_k^{\#}(X)$.

Proposition 5.4. *If $\mathcal{U}$ is a pairwise disjoint family of nonempty open subsets of X such that $\overline{\cup\mathcal{U}}$ is compact, then $|\mathcal{U}| < d(D_k^{\#}(X))$.*

Proof. Let $F \subseteq D_k^{\#}(X)$ with $|F| \leq |\mathcal{U}|$; say $\phi : F \to \mathcal{U}$ is an injection. For each $\overline{f} \in F$, let $x(\overline{f}) \in \phi(\overline{f}) \cap C(f)$, and let $t(\overline{f}) \in [-1, 1]$ be such that $|f(x(\overline{f})) - t(\overline{f})| \geq 1$. Then define $g : X \to \mathbf{R}$ by setting $g(x) = t(\overline{f})$ if $x \in \phi(\overline{f})$ for some $\overline{f} \in F$, and $g(x) = 0$ otherwise. Then g is densely continuous on X and bounded, so that $\overline{g} \in D_k^{\#}(X)$. But by construction, $F \cap \langle \overline{g}, \overline{\cup\mathcal{U}}, 1\rangle = \emptyset$, so that F is not dense in $D_k^{\#}(X)$. □

Corollary 5.3. *If X is a nondiscrete locally compact space, then $d(D_k^{\#}(X))$ is uncountable.*

Putting together these upper and lower bounds for the density of $D_k^{\#}(X)$, and assuming the continuum hypothesis, we get the following result.

Corollary 5.4. *(CH) If X is a nondiscrete locally compact second countable space, then $d(D_k^{\#}(X)) = |D^{\#}(X)| = c$.*

Because a locally compact second countable space is σ-compact, it follows from Theorems 3.1 and 4.2 that the topology on $D_k^{\#}(X)$ is unique for such spaces. This is because any two infinite dimensional completely metrizable locally convex linear topological spaces with the same density are homeomorphic (see [10]).

Theorem 5.4. *(CH) For any two nondiscrete locally compact second countable spaces X and Y, $D_k^{\#}(X)$ and $D_k^{\#}(Y)$ are homeomorphic.*

This leaves a number of questions unanswered. For example, can the assumption of the continuum hypothesis be omitted?

6. Compact Subsets

The Ascoli-type theorem for $D_k(X)$ in [5] can be modified for $D_k^\#(X)$ using the following concepts of densely equicontinuous and densely pointwise bounded.

A subset E of $D^\#(X)$ is *densely equicontinuous* provided that for every x in X and every $\varepsilon > 0$, there exists a finite family $\mathcal{U}$ of open subsets of X such that $\overline{\cup\mathcal{U}}$ is a neighborhood of x and such that for every $\overline{f} \in E$, for every $U \in \mathcal{U}$, and for every $p, q \in U$, the diameter of $\overline{f}(p) \cup \overline{f}(q)$ is less than ε.

A subset E of $D^\#(X)$ is *densely pointwise bounded* provided that there is a dense G_δ-subset G of X such that for every $x \in G$, $\cup\{\overline{f}(x) : \overline{f} \in E\}$ is a bounded subset of $\mathbf{R}$.

The definition of dense equicontinuity given above is similar to but stronger than that given in [5] for subsets of $D(X)$. However, for compact subsets of $D_k^\#(X)$, the concepts are the same, as shown by the following Ascoli-type theorem.

Theorem 6.1. *If X is a locally compact space, then a subset E of $D_k^\#(X)$ is compact if and only if it is closed, densely equicontinuous, and densely pointwise bounded.*

Proof. Let E be closed in $D_k^\#(X)$, densely equicontinuous, and densely pointwise bounded. Since $D_k^\#(X)$ is closed in $D_k(X)$ by Theorem 4.1, we have E closed in $D_k(X)$. So by Theorem 5.7 in [5], E must be compact as a subset of $D_k(X)$, and is hence compact as a subset of $D_k^\#(X)$. On the other hand, if E is compact in $D_k^\#(X)$, and thus compact in $D_k(X)$, Theorem 5.7 in [5] shows that E is densely equicontinuous in a weaker sense than our definition and is densely pointwise bounded. It remains only to show that the proof of dense equicontinuity can be modified to conclude the stronger version.

Given that E is compact in $D_k^\#(X)$, to show that E is densely equicontinuous, let $x_0 \in X$ and let $\varepsilon > 0$. Define finite family $\mathcal{U}$ of open subsets of X as follows. Let A be a compact subset of X containing x_0 in its interior, A^o. By the compactness of E, there exist $\overline{f_1}, \ldots, \overline{f_n} \in E$ such that

$$E \subseteq \langle \overline{f_1}, A, \frac{\varepsilon}{3} \rangle \cup \cdots \cup \langle \overline{f_n}, A, \frac{\varepsilon}{3} \rangle.$$

Each $\overline{f_j}$ is bounded on A, so there exists a number M such that

$$\cup \{ \overline{f_j}(X) : j = 1, \ldots, n \text{ and } x \in A \} \subseteq [-M, M].$$

Let $\mathcal{V} = \{V_1, \ldots, V_m\}$ be a finite open cover of $[-M, M]$ such that each $\overline{V_i}$ has diameter less than $\frac{\varepsilon}{3}$. For each $i = 1, \ldots, m$ and $j = 1, \ldots, n$, let

$$D_i^j = \{ x \in C(f_j) : f_j(x) \in V_i \},$$

for each $x \in D_i^j$, let $W_i^j(x)$ be an open neighborhood of x such that $f_j(W_i^j(x)) \subseteq V_i$, and let

$$W_i^j = \cup \{ W_i^j(x) : x \in D_i^j \}.$$

Finally, define

$$\mathcal{U} = \Big\{ A^o \cap W_{i_1}^1 \cap \cdots \cap W_{i_n}^n \ : \ \text{for each } j \in \{1, \ldots, n\}, \\ i_j \in \{1, \ldots, m\} \Big\}.$$

Now one can check, as in the argument in [5], that $\overline{\cup \mathcal{U}} = A$ and is thus is a neighborhood of x, and that for every $\overline{f} \in E$, for every $U \in \mathcal{U}$, and for every $p, q \in U$, the diameter of $\overline{f}(p) \cup \overline{f}(q)$ is less than ε. □

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