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TOPOLOGIES OF QUASI-UNIFORM CONVERGENCE ON GROUPS OF HOMEOMORPHISMS

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Abstract

In 1997, we proved that a topology of Pervin-type quasi-uniform convergence on a subgroup, G, of the collection of all self-homeomorphisms, $H(X)$, is equivalent to a certain type set-open topology on G. We describe the topologies of two other kinds of quasi-uniform convergence on G using analogies of well-known function space topologies. The paper ends with a discussion of properties of these function spaces.

1. Introduction

In 1965, Murdeshwar and Naimpally [8] defined quasi-uniform convergence as a generalization of uniform convergence and discussed the topology of quasi-uniform convergence on function spaces. The many interesting results that followed, often involving groups of homeomorphisms, unfortunately involved the tedious and cumbersome notation needed when discussing the topology of quasi-uniform convergence.

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Key words: quasi-uniformity, self-homeomorphism, Pervin quasi-uniformity, topology of quasi-uniform convergence, open-open topology, covering quasi-uniformity, lower semi-continuous quasi-uniformity, strong Γ-open-cover topology, $\mathcal{E}$-graph topology

Porter, in 1993 [12], proved that given a topological space, (X, T), the topology of Pervin quasi-uniform convergence on $H(X)$ is equivalent to the open-open topology, T_{oo}, on $H(X)$, which has as its subbasic open sets those that are in the form

$$(U, V) = \{h \in H(X) : h(U) \subset V\}$$

where the sets U and V are open in (X, T). Using the simple notation that the open-open topology affords, the theorems and proofs involving quasi-uniform convergence on groups of homeomorphisms are simpler and more concise.

In 1997, Porter [13] also observed that any topology of Pervin-type quasi-uniform convergence on any subset of $H(X)$ is equivalent to some set-open topology under some conditions on X, once again simplifying notation and proofs.

We shall show that the topologies of quasi-uniform convergence induced on $G \subset H(X)$ by either the covering or lower-semicontinuous type quasi-uniformities on (X, T) may also be classified as function space topologies that are familiar to us.

Throughout this paper we shall assume that (X, T) is a topological space.

2. Preliminaries

Let X be a non-empty set and let $\mathcal{Q}$ be a collection of subsets of $X \times X$ which satisfy the following four conditions:

(1) for all $U \in \mathcal{Q}$, $\triangle = \{(x, x) \in X \times X : x \in X\} \subset U$

(2) for all $U \in \mathcal{Q}$, if $U \subset V$ then $V \in \mathcal{Q}$

(3) for all $U, V \in \mathcal{Q}$, $U \cap V \in \mathcal{Q}$

(4) for all $U \in \mathcal{Q}$, there exists some $W \in \mathcal{Q}$ such that $W \circ W \subset U$ where $W \circ W = \{(p, q) \in X \times X : \exists r \in X \ni (p, r), (r, q) \in W\}$.

Then $\mathcal{Q}$ is a *quasi-uniformity on* X.

A quasi-uniformity, $\mathcal{Q}$, on X induces a topology, $T_{\mathcal{Q}}$, on X, such that for each $x \in X$, the set $\{U(x) : U \in \mathcal{Q}\}$ is a neighborhood system at x, where $U(x) = \{y \in X : (x, y) \in U\}$.

Let $\mathcal{B}$ be a family of subsets of $X \times X$ which satisfies properties (1) and (4) in the definition of quasi-uniformity, and also (3′): for all $B_1, B_2 \in \mathcal{B}$, there exists some $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$. Then $\mathcal{B}$ is a *basis* for a quasi-uniformity, $\mathcal{Q}$, on X. The basis, $\mathcal{B}$, generates the quasi-uniformity $\mathcal{Q} = \{U \subset X \times X : \hat{B} \subset U \text{ for some } \hat{B} \in \mathcal{B}\}$.

A family, $\mathcal{S}$, of subsets of $X \times X$ which satisfies properties (1) and (4) in the definition of quasi-uniformity is a *subbasis* for a quasi-uniformity, $\mathcal{Q}$, on X. This subbasis S generates a *basis*, $\mathcal{B}$, for the quasi-uniformity, $\mathcal{Q}$, where $\mathcal{B}$ is the collection of all finite intersections of elements of $\mathcal{S}$.

Every topological space (X, T) is quasi-uniformizable which means that given a topological space (X, T) there exists a quasi-uniformity, $\mathcal{Q}$, which induces a topology $T_{\mathcal{Q}}$ on X which is equivalent to the original topology, i.e., $T = T_{\mathcal{Q}}$. Krishnan [6] was the first to prove that every topological space is quasi-uniformizable, however, Pervin [11] actually constructed a specific quasi-uniformity which induces an equivalent topology for a given topological space. His construction is as follows: Let (X, T) be a topological space. For $O \in T$ define

$$S_O = (O \times O) \cup ((X \setminus O) \times X).$$

One can show that for $O \in T, S_O \circ S_O = S_O$ and $\triangle \subset S_O$, hence, the collection $\{S_O : O \in T\}$ is a subbasis for a quasi-uniformity, P, on X, called the *Pervin quasi-uniformity.* Note that for $x \in X$, the neighborhoods of x in $T_{\mathcal{Q}}$ are

$$S_O(x) = \begin{cases} X & \text{if } x \notin O \\ O & \text{if } x \in O. \end{cases}$$

Let $\mathcal{Q}$ be a compatible quasi-uniformity for (X,T) and let G be a subgroup of $H(X)$. For $U \in \mathcal{Q}$, define the set

$$W(U) = \{(f,g) \in G \times G : (f(x), g(x)) \in U \text{ for all } x \in X\}.$$

Then the collection $\mathcal{B} = \{W(U) : U \in \mathcal{Q}\}$ is a basis for a quasi-uniformity, $\mathcal{Q}^*$, on G, called *the quasi-uniformity of quasi-uniform convergence w. r. t.* $\mathcal{Q}$. The topology, $T_{\mathcal{Q}^*}$ on G, induced by $\mathcal{Q}^*$ on G, is called *the topology of quasi-uniform convergence w.r.t.* $\mathcal{Q}$ [9].

In the following sections we shall examine the topology of quasi-uniform convergence on subsets of $H(X)$ when the quasi-uniformity, $\mathcal{Q}$, on X is (i) a Pervin-type quasi-uniformity, (ii) a covering quasi-uniformity, or (iii) a lower semi-continuous quasi-uniformity.

3. Pervin-Type Quasi-Uniformities

To generalize the idea of the Pervin quasi-uniformity, let $\mathcal{A}$ be any collection of open sets in (X,T), not necessarily all of T as in Pervin's definition. For each $O \in \mathcal{A}$, define, as before, $S_O = (O \times O) \cup ((X \setminus O) \times X)$. Then, again, we have that for all $O \in \mathcal{A}$, $S_O \circ S_O = S_O$ and $\triangle \subset S_O$, and thus, the collection $\{S_O : O \in \mathcal{A}\}$ is a subbasis for a *Pervin-type quasi-uniformity*, $\mathcal{Q}_{\mathcal{A}}$, on X. The topology induced on X by $\mathcal{Q}_{\mathcal{A}}$ will be denoted by $T_{\mathcal{A}}$.

Let $G \subset H(X)$, then a collection $\mathcal{A} \subset P(X) = \{F : F \subset X\}$ is called a *G-invariant collection of sets* provided that for all $A \in \mathcal{A}$ and for all $g \in G$, $g(A) \in \mathcal{A}$. If a G-invariant collection, $\mathcal{A}$, is also a basis for (X,T), then $\mathcal{A}$ will be called a *G-invariant basis*.

In 1997, Porter [13] proved that the topology, $T_{\mathcal{A}}$, induced on X by the Pervin-type quasi-uniformity $\mathcal{Q}_{\mathcal{A}}$, when $\mathcal{A}$ is a G-invariant basis for (X,T), is equivalent to the original topology, T, on X.

Now suppose that G is a collection of self-homeomorphisms on X and that $\mathcal{A}$ is a G-invariant basis for (X,T). The Pervin-type quasi-uniformity, $\mathcal{Q}_{\mathcal{A}}$, induces a quasi-uniformity, $\mathcal{Q}^*_{\mathcal{A}}$, on G. The topology then induced on G by $\mathcal{Q}^*_{\mathcal{A}}$ will be called *a topology of Pervin-type quasi-uniform convergence* on G and will be denoted by $T_{\mathcal{Q}^*_{\mathcal{A}}}$. Porter [13] proved that each of these topologies of Pervin-type quasi-uniform convergence is, in fact, equivalent to a set-open topology which is defined as follows. Let $\mathcal{A}$ be a collection of subsets of X. For $A \in \mathcal{A}$, and $O \in T$, define the set

$$(A,O) = \{f \in G : f(A) \subset O\}.$$

Set $\mathcal{S}_{\mathcal{A}O} = \{(A,O) : A \in \mathcal{A} \text{ and } O \in T\}$, then if $\mathcal{S}_{\mathcal{A}O}$ is a subbasis for a topology on G, we call this topology a *set-open topology* and denote it by $T_{\mathcal{A}O}$.

Note that if $\mathcal{A}$ is a G-invariant basis, then $S_{\mathcal{A}O}$ is a subbasis for a topology. The finest of this type set-open topology is the *open-open topology* [12] where $\mathcal{A} = T$, and we have:

Theorem 1. [13] *Let (X,T) be a topological space and let G be a subgroup of $H(X)$. Assume $\mathcal{A}$ is a G-invariant basis for (X,T) and let $\mathcal{Q}_{\mathcal{A}}$ be the Pervin-type quasi-uniformity induced on X. Then, the set-open topology, $T_{\mathcal{A}O}$, on G is equivalent to the topology of Pervin-type quasi-uniform convergence, $T_{\mathcal{Q}^*_{\mathcal{A}}}$, on G.*

4. Covering-Type Quasi-Uniformities

Fletcher [2], in 1971 introduced the covering-type quasi-uniformities. The following definitions all appear in the same paper. Let (X,T) be a topological space and let $\mathcal{V}$ be an open cover of X. For each $x \in X$, define the set

$$A^{\mathcal{V}}_x = \bigcap\{V \in \mathcal{V} : x \in V\}.$$

We say $\mathcal{V}$ is a *Q-cover* of X provided that for all $x \in X$, $A^{\mathcal{V}}_x$ is open in X. Let $O \in T$ then $\mathcal{V}$ is a *fundamental cover of X*

about O provided $O \in \mathcal{V}$ and if $W \in \mathcal{V}$ such that $O \cap W \neq \phi$ then $O \subset W$. Note that this implies that $\forall x \in O,\ A_x^{\mathcal{V}} = O$.

Theorem 2. [2] *Let* (X, T) *be a topological space and let* Γ *be a collection of* Q*-covers of* X *such that for each* $O \in T$, Γ *contains a fundamental cover of* X *about* O. *Let* $\mathcal{S}_\Gamma = \{\mathcal{U}_\mathcal{V} : \mathcal{V} \in \Gamma\}$ *where*

$$\mathcal{U}_\mathcal{V} = \bigcup_{x \in X} \{x\} \times A_x^{\mathcal{V}}.$$

Then $\mathcal{S}_\Gamma$ *is a subbasis for a quasi-uniformity,* $\mathcal{Q}_\Gamma$*, for* X*. In addition, the topology,* $T_{\mathcal{Q}_\Gamma}$*, induced by* $\mathcal{Q}_\Gamma$ *on* X *is equivalent to* T*.*

We will call this quasi-uniformity, $\mathcal{Q}_\Gamma$, the Γ-*covering quasi-uniformity for* X. Theorem 2 follows from the following three facts: (i) $\Delta \subset \mathcal{U}_\mathcal{V}$, (ii) $\mathcal{U}_\mathcal{V} \circ \mathcal{U}_\mathcal{V} \subset \mathcal{U}_\mathcal{V}$, and (iii) $\mathcal{U}_\mathcal{V}(x) = A_x^{\mathcal{V}} \in T$.

Fletcher proved that for every Γ-covering quasi-uniformity, $\mathcal{Q}_\Gamma$, on (X, T), $P \subset \mathcal{Q}_\Gamma$ where P is the Pervin quasi-uniformity for X, and, if Γ is the collection of all finite open covers, then $\mathcal{Q}_\Gamma = P$. Thus, it follows that the Pervin quasi-uniformity is the coarsest covering quasi-uniformity for (X, T).

Now, what does the topology of quasi-uniform convergence that is induced on a group of homeomorphisms look like when the underlying quasi-uniformity on X is a Γ-covering quasi-uniformity? The motivation for the definition of the function space topologies which are equivalent to these topologies of quasi-uniform convergence are the cover-close topologies and the open-cover topology which were studied by Irudayanathan [5] and McCoy [7] respectively. The definitions of these topologies are given below for completeness.

Let (X, T) and (Y, T') be topological spaces and let $C(X, Y)$ denote the collection of all continuous functions from X into Y. Let $\Gamma(Y)$ be the set of all open covers of Y. For each $\mathcal{V} \in \Gamma(Y)$ and each $f \in C(X, Y)$, let

$$\mathcal{V}^*(f) = \{g \in C(X, Y) : \forall x \in X, \exists V \in \mathcal{V} \ni (f(x), g(x)) \in V \times V\}.$$

Then $\mathcal{S}_\gamma = \{\mathcal{V}^*(f) : \mathcal{V} \in \Gamma(Y) \text{ and } f \in C(X,Y)\}$ is a subbasis for a topology, T_γ, called the *open-cover topology* [7], on $C(X,Y)$.

Let (X,T) and (Y,T') be topological spaces. Let $\Gamma(Y)$ be a collection of open covers of Y such that $\{St(p,\mathcal{V}) : \mathcal{V} \in \Gamma(Y)\}$ is a neighborhood base at p for each $p \in Y$, where $St(p,\mathcal{V}) = \bigcup_{p \in V \in \mathcal{V}} V$. For each $\mathcal{V} \in \Gamma(Y)$ and each $f \in C(X,Y)$, let

$$W(f,\mathcal{V}) = \{g \in C(X,Y) : \forall x \in X, g(x) \in St(f(x),\mathcal{V})\}.$$

Then $\mathcal{S} = \{W(f,\mathcal{V}) : \mathcal{V} \in \Gamma(Y) \text{ and } f \in C(X,Y)\}$ is a subbasis for a topology, T_γ, called the Γ-*close topology* [5], on $C(X,Y)$.

These topologies are not quite what we need; in fact it is true that the open-cover topology is contained in all the topologies of quasi-uniform convergence which are induced by a Γ-covering type quasi-uniformity, and the Γ-close topology is contained in the topology of quasi-uniform convergence induced by Γ.

However, adaptations of the definitions will work.

Let (X,T) and (Y,T') be topological spaces. Let $\Gamma(Y)$ be a collection of open covers of Y. For each $\mathcal{V} \in \Gamma(Y)$ and each $f \in C(X,Y)$, define

$$\mathcal{V}(f) = \{g \in C(X,Y) : \forall x \in X, f(x) \in V \in \mathcal{V} \Rightarrow g(x) \in V\}.$$

Then $\mathcal{S}_\Gamma = \{\mathcal{V}(f) : \mathcal{V} \in \Gamma(Y) \text{ and } f \in C(X,Y)\}$ is a subbasis for a topology, T_Γ, called the *strong* Γ-*open-cover topology*, on $C(X,Y)$.

It is this topology, with certain conditions on Γ which will describe the topologies of covering quasi-uniform convergence on subsets G of $H(X)$.

Theorem 3. *Let (X,T) be a topological space and let $G \subset H(X)$. Assume Γ is a collection of Q-covers of X such that for each $O \in T$, the collection Γ contains a fundamental cover of X about O. Then the topology of Γ-covering quasi-uniform* *convergence, $T_{\mathcal{Q}^*_\Gamma}$, is equivalent to the strong* Γ*-open-cover topology, T_Γ, on G.*

Proof: Let (X, T), G, and Γ be as defined above. Let $\mathcal{V} \in \Gamma$ and $f \in G$. Note that $\mathcal{V}(f) = \{g \in G : \forall x \in X, \;\; g(x) \in A^{\mathcal{V}}_{f(x)}\}$.

Recall that the basis elements for the topology of quasi-uniform convergence w.r.t. $\mathcal{Q}$, $T_{\mathcal{Q}^*}$, on G, look like

$$W(Q)(f) = \{g \in G : g(x) \in Q(f(x)), \;\; \forall x \in X\},$$

where $f \in G$ and $Q \in \mathcal{Q}$. Since the quasi-uniformity, $\mathcal{Q}$, for (X, T) is the Γ-covering quasi-uniformity, note that

$$\begin{aligned} W(\mathcal{U}_{\mathcal{V}})(f) &= \{g \in G : g(x) \in \mathcal{U}_{\mathcal{V}}(f(x)), \;\; \forall x \in X\} \\ &= \{g \in G : g(x) \in A^{\mathcal{V}}_{f(x)}, \;\; \forall x \in X\}. \end{aligned}$$

This means that $W(\mathcal{U}_{\mathcal{V}})(f)$ consists of all homeomorphisms, $g \in G$, such that if $f(x) \in V \in \mathcal{V}$ then $g(x)$ is also in V. Hence,

$$W(\mathcal{U}_{\mathcal{V}})(f) = \mathcal{V}(f). \qquad \square$$

5. Lower Semi-Continuous Quasi-Uniformities

Let (X, T) be a topological space and let f be a real-valued function on X. Then for $\epsilon > 0$, we define

$$U_{(\epsilon, f)} = \{(x, y) \in X \times X : f(x) - f(y) < \epsilon\}.$$

Let (X, T) be a topological space and let f be a real-valued function on X. We say that f is *lower semi-continuous* on X provided that for all $a \in \mathbf{R}$, $f^{-1}((a, +\infty))$ is open in X.

A collection $\mathcal{E}$ of lower semi-continuous functions is called *admissible* [2] provided that for each $O \in T$ and $x \in O$ there exists $f \in \mathcal{E}$ such that $f(x) = 1$ and $f(X \setminus O) = 0$.

Theorem 4. [4] *Let (X, T) be a topological space and let $\mathcal{E}$ be an admissible collection of lower semi-continuous functions on*

X. Then the collection

$$\mathcal{S} = \{U_{(\epsilon,f)} : f \in \mathcal{E} \text{ and } \epsilon > 0\}$$

is a subbasis for a quasi-uniformity, $\mathcal{Q}_{\mathcal{E}}$, which is compatible with T.

This quasi-uniformity in Theorem 4 is called the $\mathcal{E}$-semi-continuous quasi-uniformity for (X,T) and we denote it by $\mathcal{E}$-SC. The proof of this theorem follows from the following facts: (a) $\Delta \subset U_{(\epsilon,f)}$, (b) $U_{(\frac{\epsilon}{2},f)} \circ U_{(\frac{\epsilon}{2},f)} \subset U_{(\epsilon,f)}$, and (c) $U_{(\epsilon,f)}(x) = f^{-1}(f(x)-\epsilon,+\infty) \in T$.

Again we seek to classify the topologies of quasi-uniform convergence induced on G this time by the $\mathcal{E}$-SC quasi-uniformities on X. To this end let us look at the following well-known topology for function spaces.

Let (X,T) and (Y,T') be topological spaces. For any function $f \in Y^X$, the *graph of* f, denoted by $Gr(f)$, is the set

$$Gr(f) = \{(x, f(x)) \in X \times Y : x \in X\}.$$

Give $X \times Y$ the product topology, T_P. For each open set O in $X \times Y$, define

$$F_O = \{f \in Y^X : Gr(f) \subset O\}.$$

Then the set $\mathcal{B} = \{F_O : O \in T_P\}$ is a basis for a topology, T_{Gr}, on Y^X called the *graph topology* [10].

The graph topology introduced by Naimpally in 1966 is the motivation for the definition of the $\mathcal{E}$-graph topology we define below.

A collection $\mathcal{E}$ of lower semi-continuous functions will be called *G-closed* provided that for each $f \in \mathcal{E}$ and each $h \in G \subset H(X)$, $f \circ h \in \mathcal{E}$.

Now let $\mathcal{E}$ be a G-closed collection of lower semi-continuous functions on X where $G \subset H(X)$. For $f \in \mathcal{E}$, $\epsilon > 0$ and $h \in G$ define

$$(h, U_{(\epsilon,f)}) = \{g \in G : Gr(h^{-1} \circ g) \subset U_{(\epsilon, f\circ h)}\}.$$

Then $\mathcal{S} = \{(h, U_{(\epsilon,f)}) : f \in \mathcal{E},\ \epsilon > 0,\ \text{and } h \in G\}$ is a subbasis for a topology, $T(\mathcal{E})$, on G, which we will call the *$\mathcal{E}$-graph topology.* We claim the following.

Theorem 5. *Let (X, T) be a topological space and let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions on X where $G \subset H(X)$. Then the topology of $\mathcal{E}$-SC quasi-uniform convergence, $T_{\mathcal{Q}^*_{\mathcal{E}}}$, is equivalent to the $\mathcal{E}$-graph topology, $T(\mathcal{E})$, on G.*

Proof. Recall that the basis elements for the topology, $T_{\mathcal{Q}^*}$, of quasi-uniform convergence w.r.t. $\mathcal{Q}$ on $G \subset H(X)$, look like

$$W(Q)(f) = \{g \in G : g(x) \in Q(f(x)),\ \ \forall x \in X\},$$

where $f \in G$ and $Q \in \mathcal{Q}$. Since the quasi-uniformity here, $\mathcal{Q}$, for (X, T) is the $\mathcal{E}$-SC quasi-uniformity, note that

$$\begin{aligned} W(U_{(\epsilon,f)})(h) &= \{g \in G : g(x) \in U_{(\epsilon,f)}(h(x)),\ \ \forall x \in X\} \\ &= \{g \in G : g(x) \in \{y \in X : f \circ h(x) - f(y) < \epsilon\},\ \ \forall x \in X\} \\ &= \{g \in G : f \circ h(x) - f(g(x)) < \epsilon,\ \ \forall x \in X\} \\ &= \{g \in G : f \circ h(x) - f \circ h(h^{-1} \circ g(x)) < \epsilon,\ \ \forall x \in X\}. \end{aligned}$$

This means that $W(U_{(\epsilon,f)})(h)$ consists of all homeomorphisms, $g \in G$, such that $(x, h^{-1} \circ g(x)) \in U_{(\epsilon, f\circ h)}$ $\forall x \in X$, which implies that the graph of $h^{-1} \circ g$ is contained in $U_{(\epsilon, f\circ h)}$. Hence,

$$W(U_{(\epsilon,f)})(h) = (h, U_{(\epsilon,f)}).$$

Therefore the theorem is true. □

6. Some Properties

Let $(X,\mathcal{U})$ and $(Y,\mathcal{Q})$ be quasi-uniform spaces. A function $f : X \to Y$ is *quasi-uniformly continuous* iff for each $Q \in \mathcal{Q}$ there exists some $U \in \mathcal{U}$ such that if $(x,y) \in U$ then $(f(x), f(y)) \in Q$.

In 1997, Porter [13] proved that if G is a subgroup of $H(X)$ with the topology of quasi-uniform convergence induced by the compatible Pervin-type quasi-uniformity on (X,T), then each $g \in G$ is quasi-uniformly continuous. Under suitable hypotheses, Fletcher [3] obtained the same conclusion for G when endowed with one of the two other types of quasi-uniform convergence.

Let Γ be a collection of Q-covers of X such that for each $O \in T$, Γ contains a fundamental cover about O. Let G be a subgroup of $H(X)$. Then Γ is called a *G-invariant* class of Q-covers of X provided that if $\mathcal{V} \in \Gamma$ and $h \in G$ then $\{h(V) : V \in \mathcal{V}\} \in \Gamma$.

Theorem 6. *Let (X,T) be a topological space and let G be a subgroup of $H(X)$. Under any one of the following hypotheses, each $g \in G$ is quasi-uniformly continuous:*

(i) Give G the topology of quasi-uniform convergence induced by a compatible Pervin-type quasi-uniformity on (X,T). [13]

(ii) Let Γ be a G-invariant class of Q-covers of X and give G the topology of Γ-covering quasi-uniform convergence. [3]

(iii) Let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions and give G the topology of $\mathcal{E}$-SC quasi-uniform convergence. [3]

Theorem 7. *Let (X,T) be a topological space and let G be a subgroup of $H(X)$ Under any one of the following hypotheses, if (X,T) is T_i, $i = 0,1,2$, then $(G,\hat{T})$ is T_i, $i = 0,1,2$, respectively:*

(i) Let $\hat{T} = T_{\mathcal{Q}^*_{\mathcal{A}}}$ the topology of quasi-uniform convergence induced by a compatible Pervin-type quasi-uniformity, $\mathcal{Q}^*_{\mathcal{A}}$, on (X, T). [13]

(ii) Let Γ be a G-invariant class of Q-covers of X and let $\hat{T} = T_\Gamma$, the topology of Γ-covering quasi-uniform convergence. [3]

(iii) Let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions and let $\hat{T} = T(\mathcal{E})$, the topology of $\mathcal{E}$-SC quasi-uniform convergence. [3]

Let G be a group with group operation $\cdot$ and topology T^+. Then G is a *paratopological group* (or quasi-topological group) provided that the map $m : (G, T^+) \times (G, T^+) \to (G, T^+)$ with $m(f, g) = f \cdot g$ is continuous.

In 1971, Fletcher [3] showed that for any subgroup, G, of $H(X)$, in which all members are quasi-uniformly continuous with respect to a compatible quasi-uniformity $\mathcal{Q}$, and which has been given the topology of quasi-uniform convergence w.r.t. $\mathcal{Q}$, $(G, T^*_{\mathcal{Q}})$ is a paratopological group. Hence, from Theorem 6 we have:

Corollary 1. *Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Under any one of the following hypotheses, $(G, \hat{T})$ is a paratopological group:*

(i) Let $\hat{T} = T_{\mathcal{Q}^*_{\mathcal{A}}}$ the topology of quasi-uniform convergence induced by a compatible Pervin-type quasi-uniformity, $\mathcal{Q}^*_{\mathcal{A}}$, on (X, T). [13]

(ii) Let Γ be a G-invariant class of Q-covers of X and let $\hat{T} = T_\Gamma$, the topology of Γ-covering quasi-uniform convergence. [3]

(iii) Let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions and let $\hat{T} = T(\mathcal{E})$, the topology of $\mathcal{E}$-SC quasi-uniform convergence. [3]

Let (X,T) and (Y,T') be topological spaces and let F be a collection of functions from X into Y. Suppose $\hat{T}$ is a topology on F, then $\hat{T}$ is *admissible* or *jointly continuous* on F provided the evaluation map, $E:(F,\hat{T})\times(X,T)\longrightarrow(Y,T')$, defined by $E(f,x)=f(x)$ is continuous.

Theorem 8. *Let (X,T) be a topological space and let G be a subgroup of $H(X)$. Under any one of the following hypotheses, $\hat{T}$ is admissible for G.*

(i) Let $\hat{T}=T_{\mathcal{Q}^*_{\mathcal{A}}}$ the topology of quasi-uniform convergence induced by a compatible Pervin-type quasi-uniformity, $\mathcal{Q}^*_{\mathcal{A}}$, on (X,T). [13]

(ii) Let Γ be a G-invariant class of Q-covers of X and let $\hat{T}=T_\Gamma$, the topology of Γ-covering quasi-uniform convergence.

(iii) Let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions and let $\hat{T}=T(\mathcal{E})$, the topology of $\mathcal{E}$-SC quasi-uniform convergence.

Proof: (i) See [13].
(ii) Let O be open in X and let $(g,x)\in E^{-1}(O)$. Thus there exists a cover $\mathcal{V}\in\Gamma$ which is a fundamental Q-cover about O. Then $(g,x)\in\mathcal{V}(g)\times g^{-1}(O)$. If $(h,y)\in\mathcal{V}(g)\times g^{-1}(O)$ then $h(z)\in V\in\mathcal{V}$ whenever $g(z)\in V$. And $y\in g^{-1}(O)$ implies that $g(y)\in O$. But since $\mathcal{V}$ is a fundamental Q-cover about O, it is true that $O=A^{\mathcal{V}}_{g(y)}$. Hence, $h(y)\in O$. Therefore, $\mathcal{V}(g)\times g^{-1}(O)\subseteq E^{-1}(O)$ and we are done.
(iii) Let O be open in X and let $(g,x)\in E^{-1}(O)$. Then $g(x)\in O$ and so there exists a basis element $B=\cap_{i=1}^n U_{(\frac{\epsilon_i}{2},f_i)}(g(x))$ such that $g(x)\in B\subseteq\cap_{i=1}^n U_{(\epsilon_i,f_i)}(g(x))\subseteq O$. Then $g\in\cap_{i=1}^n(g,U_{(\frac{\epsilon_i}{2},f_i)})$ for all $i=1,2,\ldots,n$ and $x\in g^{-1}(B)\subseteq g^{-1}(O)$. Suppose $(h,y)\in\cap_{i=1}^n(g,U_{(\frac{\epsilon_i}{2},f_i)})\times g^{-1}(B)$. Then $h\in(g,U_{(\frac{\epsilon_i}{2},f_i)})$ for all $i=1,2,\ldots,n$ which means that $Gr(g^{-1}\circ h)\subseteq U_{(\frac{\epsilon_i}{2},f_i)}$ for

each $i = 1, 2, \ldots, n$. Therefore, $f \circ g(z) - f \circ h(z) < \dfrac{e_i}{2}$ for any $z \in X$ and any $i \in \{1, 2, \ldots, n\}$. Now $y \in g^{-1}(B)$ so that $g(y) \in B$. Hence, $(g(x), g(y)) \in U_{(\frac{\epsilon_i}{2}, f_i)}$ for all $i = 1, 2, \ldots, n$, which implies that $f_i \circ g(x) - f_i \circ g(y) < \dfrac{\epsilon_i}{2}$ for each i. Thus, $f_i \circ g(x) - f_i \circ h(y) = f_i \circ g(x) - f_i \circ g(y) + f_i \circ g(y) - f_i \circ h(y) < \epsilon_i$ for all $i = 1, 2, \ldots, n$. So $h(y) \in \cap_{i=1}^{n} U_{(\epsilon_i, f_i)}(g(x)) \subseteq O$. Whence, $\cap_{i=1}^{n}(g, U_{(\frac{\epsilon_i}{2}, f_i)}) \times g^{-1}(B) \subseteq E^{-1}(O)$ and thus, $T(\mathcal{E})$ is admissible for G. □

Arens [1] showed that for any collection, F, of continuous functions from (X, T) into (Y, T'), if T_F is an admissible topology for F then T_F contains the compact-open topology, T_{co}, which has as its subbasis all sets of the form

$$(C, O) = \{f \in F : f(C) \subset O\}$$

where C is a compact subset of X and O is open in Y. This gives us our last theorem.

Theorem 9. *Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Under any one of the following hypotheses, $T_{co} \subset \hat{T}$ on G.*

(i) Let $\hat{T} = T_{\mathcal{Q}^*_{\mathcal{A}}}$ the topology of quasi-uniform convergence induced by a compatible Pervin-type quasi-uniformity, $\mathcal{Q}^*_{\mathcal{A}}$, on (X, T). [13]

(ii) Let Γ be a G-invariant class of Q-covers of X and let $\hat{T} = T_\Gamma$, the topology of Γ-covering quasi-uniform convergence.

(iii) Let $\mathcal{E}$ be an admissible G-closed collection of lower semi-continuous functions and let $\hat{T} = T(\mathcal{E})$, the topology of $\mathcal{E}$-SC quasi-uniform convergence.

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