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# COMPACT SEMIRINGS WHICH ARE MULTIPLICATIVELY SIMPLE

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## Abstract

Following the basic ideas of J. Selden, Jr., and building on work of the author, and some results of N. Kimura, K. R. Pearson, and fundamental work of A. D. Wallace, it is shown that a compact semiring which is multiplicatively simple is a disjoint union of compact semirings which are multiplicatively simple and additively the maximal rectangular subbands of Kimura. Two separate decompositions of these basic building blocks are given. Also, a method is given for constructing compact semirings which are multiplicatively simple.

The study of topological semirings (see Definition 1 below) initiated by Selden ([9], [10]) in the '60's, was continued in various directions by Pearson ([4], [5]), Wallace [13], Robbie ([6], [7], [8]) and others. Inevitably due to the focus of the Wallace school on compact semigroups the emphasis was on compact semirings, as many existing tools could be brought to bear. At the same time many mathematicians worked on the algebraic theory of semirings (sometimes with a more restrictive definition). Amongst those were M. Grillet [d.] and H. Weinert. Somewhat earlier, N. Kimura [3] set down the basic facts about rectangular band semigroups which we use again and again in this paper.

*Key words*: Compact simple semigroup, compact semiring, rectangular band, Swelling Lemma, closed semiring congruence

The main motivation for this work was to find an analogue for compact semirings to the Wallace-Rees-Suschkewitsch Theorem for compact simple semigroups. That Theorem of course completely characterises the structure of compact simple semigroups by showing that the internal structure is iseomorphic (ie homeomorphic and homomorphic with the same map) to a certain triple product space with a special operation. After perusal of Selden's, Pearson's and Wallace's papers we decided to investigate those compact semirings in which the multiplication was simple. This built in the previous semigroup result as a foundation. We have managed to get the internal structure result of Theorem 9 where such semirings are shown to be decomposable into blocks of semirings each one of which is additively simple as well, in fact they are each subbands maximal with respect to being rectangular. As well, these blocks form the congruence classes for a closed semiring congruence on the parent semiring and the factor semiring has the structure of a product of a pair of additive bands with left (right) trivial multiplication in each. From the proof of the Theorem it is clear that each block referred to above is a union of $(\cdot)-$ groups which are also $(+)-$ rectangular bands. This is about as close as one might wish for an analogue, but for one thing - getting such internal structure completely characterised externally as in W-R-S. Our results in that direction are Theorems 15 and 16 where we construct compact simple semirings making use of the W-R-S Theorem on the way. Although considerable progress has been made in "closing the circle" we have not quite made it yet!

**Note:** We wish to thank the referee for the several suggestions made for improvement of the paper. These included suggesting that the above sections of an introductory and motivational nature be included along with some indication of the significance of the main results; and also the suggestion that an outline of the proof of Theorem 3 below be included.

**Definition 1.**$(R,+,\cdot)$ *is a compact semiring if,*

*(a) $R$ is a compact Hausdorff space,*

*(b) $+$ and $\cdot$ are two associative binary operations on $R$,*

*(c) the two maps $+$ and $\cdot$ from $R\times R \to R$ are each continuous (where $R \times R$ has the usual product space topology), and*

*(d) the two identities*

$$a\cdot(b+c) = (a\cdot b)+(a\cdot c)$$
$$(b+c)\cdot d = (b\cdot d)+(c\cdot d) \quad \textit{hold on } R.$$

We denote the various parts of the additive and multiplicative semigroups by their usual symbols followed by a $+$ or $\cdot$ in square brackets. For example, $E[\,+\,]$ is the set of additive idempotents and $K[\,\cdot\,]$ is the multiplicative minimal ideal. Of course $E[\,+\,]$ is a multiplicative ideal, due to the two distributive laws, and so $K[\,\cdot\,] \subseteq E[\,+\,]$ . In all work that follows $+$ does not denote an abelian operation unless that is specifically stated. Also, the $\cdot$ will usually be omitted. For easy reference to the necessary material on topological semigroups the reader is referred to Carruth, Hildebrant and Koch, [1].

We now confine our attention to those compact semirings $(R,+,\cdot)$ which have $K[\,\cdot\,] = R$. That is to say, compact semirings which are multiplicatively simple.

**Remark 2.** It is clear that, by using left trivial or right trivial addition, any (compact) topological semigroup may be taken as the multiplication of a (compact) topological semiring. In particular, the multiplication may be any compact simple semigroup. Moreover, by taking the cartesian product of two compact simple semigroups and giving one left trivial addition and the other right trivial addition, the corresponding semiring is a compact semiring which is both $(\cdot)-$ and $(+)-$ simple. We show in Corollary 5 below that every compact simple semiring which is both $(\cdot)-$ and $(+)-$ simple is obtainable in this way.

We begin with a Theorem which we obtained by noting that a result of A.D. Wallace [13] was true under much less stringent hypotheses than he gave.

**Theorem 3.** *Let $(R, +, \cdot)$ be a topological semiring with a multiplicative idempotent $e$, and such that $(R, +)$ is a rectangular band. Then $(R, +, \cdot)$ is iseomorphic (= there is a single map which is a homeomorphism and also a $(\cdot)-$ and a $(+)-$ homomorphism) to the cartesian product of topological semirings $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$, where the addition in $R_1$ is left trivial and the addition in $R_2$ is right trivial. $R_1 = R + e$, $R_2 = e + R$, and the map is $r \longmapsto (r + e,\ e + r)$.*

*Proof.* Given that $(R, +)$ is a rectangular band then the identities $x + y + z = x + z$ and $x + y + x = x$ may (and will) be freely used (Kimura [3]). It is then evident that $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$, where $R_1 = R+e$ and $R_2 = e+R$, are topological subsemirings of $(R, +, \cdot)$ in which the first has left trivial addition and the second has right trivial addition. Taking the cartesian product of these two semirings with the usual product topology and coordinate-wise operations then gives us a topological semiring.

We must now show that our map $r \longmapsto (r + e,\ e + r)$ is a homeomorphism $f$ from $R$ onto $R_1 \times R_2$ which is also a $(\cdot)-$ and $(+)-$ homomorphism. Firstly, as each one of $r \longmapsto r + e$ and $r \longmapsto e + r$ is continuous then by a standard result $r \longmapsto (r + e,\ e + r)$ is also continuous. Next we show that $f$ is one-to-one. Let $(r + e, e + r) = (s + e, e + s)$. Then we must have that $r + e = s + e$ and $e + r = e + s$. Adding $r$ to the right-hand end of each side of the first of these gives $r + e + r = s + e + r$, which in turn gives $r = s + r$. Adding $s$ to the left-hand end of each side of the second equation gives $s + e + r = s + e + s$, which in turn gives $s + r = s$. Thus $r = s$. Now we show that $f$ is onto. Let $(r + e, e + s)$ be an arbitrary element of $R_1 \times R_2$. Then $f(r + s) = (r + s + e, e + r + s) = (r + e, e + s)$.

We so far have a one-to-one, onto, continuous function $f$ from

$R$ to $R_1 \times R_2$ . Thus we have an inverse function by $(r+e, e+r) \longmapsto r$ which may be realised by

$$R_1 \times R_2 \to R \times R \to R$$

where $(r+e, e+r) \longmapsto (r+e, e+r)$ followed by $(r+e, e+r) \longmapsto (r+e)+(e+r) = r+(e+e)+r = r$. As each of these maps is continuous the composition map, which is the inverse of our $f$, is continuous. So $f$ is a homeomorphism of $R$ onto $R_1 \times R_2$.

Now $f(r)f(s) = (r+e, e+r)(s+e, e+s) = ((r+e)(s+e), (e+r)(e+s)) = (rs+re+es+ee, ee+es+re+rs) = (rs+e, e+rs) = f(rs)$.

Also, $f(r)+f(s) = (r+e, e+r)+(s+e, e+s) = (r+e, e+s) = (r+s+e, e+r+s) = f(r+s)$.

So we have established the required iseomorphism. □

The next theorem and its corollary can be obtained as corollaries to the main part of this last theorem since each compact semigroup has at least one idempotent. Indeed Wallace in (Wallace[13]) using the powerful machinery of the minimal ideal structure of a compact semigroup (Wallace[12]) in the special case of a rectangular band semigroup, was able to get the iseomorphism above very quickly. Of course those methods do not apply in case of the hypotheses above so a more plodding approach is required. Anyway we prove the results below separately because that is easy to do, and because the map given, hence the proof, is of a different character to that for the above theorem.

**Theorem 4.** *Let $(R, +, \cdot)$ be a compact semiring in which $(R, +)$ is a rectangular band. Then $(R, +, \cdot)$ is iseomorphic to the cartesian product of compact semirings $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$, where the addition in $R_1$ is left trivial and the addition in $R_2$ is right trivial.*

*Proof.* Choose $e^2 = e \in R$ (possible since $R$ is compact). Then $e+e = e$ as well, because we have $(R, +)$ a band. By the theory of A. D. Wallace [12] , for compact simple semigroups, $R_1 = R+e$ and $R_2 = e+R$ are respectively minimal left and minimal right ideals of $(R,+)$ with corresponding left trivial and right trivial addition . Moreover, by the same theory, $(R_1 \times R_2, +)$ with coordinatewise addition is iseomorphic to $(R,+)$ via the map $(a,b) \longmapsto \theta((a,b)) = a+b$. We define $(a,b) \star (c,d) = (ac,bd)$, coordinatewise multiplication, and claim that $(a,b) \longmapsto \theta((a,b)) = a+b$ is a multiplicative morphism as well.

$$\begin{aligned} \theta((a,b) \star (c,d)) &= \theta((ac,bd)) \\ &= ac+bd \\ &= ac+(ad+bc)+bd \end{aligned}$$

(by Kimura[3], since $(R,+)$ is a rectangular band),

$$\begin{aligned} &= (a+b)(c+d) \\ &= \theta((a,b))\theta((c,d)). \end{aligned}$$

Thus we are done. □

**Corollary 5.** *Let $(R,+,\cdot)$ be a compact semiring which is both $(\cdot)-$ and $(+)-$ simple. Then it is iseomorphic to a semiring obtained as explained in the Remark 2 above.*

*Proof.* In this case the hypotheses of the last theorem are satisfied, because $R = K[\,\cdot\,] \subseteq E[\,+\,] \subseteq R$, which gives $E[\,+\,] = R$, so that $(R,+)$ is a rectangular band. □

**Theorem 6.** *Let $(R,+,\cdot)$ be a compact semiring in which $(R,\cdot)$ is simple. Fix $e' \in E[\,\cdot\,]$. For $x,\ y \in R$, represented uniquely as*

$$x = e_1 a e_2 \text{ and } y = f_1 b f_2, \text{ where}$$
$$e_1, f_1 \in E[\,\cdot\,] \cap Re' \quad \text{and} \quad e_2, f_2 \in e'R \cap E[\,\cdot\,] \quad \text{and,}$$
$$a, b \in G = e'R\,e', \quad \text{then}$$
$$\exists\ g_1 \in E[\,\cdot\,] \cap Re' \quad \text{and} \quad \exists\ g_2 \in e'R \cap E[\,\cdot\,]\ \text{, such that}$$
$$e_1 a e_2 + f_1 b f_2 = g_1(a+b)g_2.$$

*Proof.* $e_1\ a\ e_2 + f_1\ b\ f_2 = g_1\ d\ g_2$ for some $g_1 \in E[\,\cdot\,] \cap Re'$, $g_2 \in e'R \cap E[\,\cdot\,]$ and $d \in G = e'Re'$. Thus $e'\ (e_1\ a\ e_2 + f_1\ b\ f_2)\ e' = e'\ (g_1\ d\ g_2)\ e'$. So $e'\ e_1\ a\ e_2\ e' + e'\ f_1\ b\ f_2\ e' = e'\ g_1\ d\ g_2\ e'$, by the associative and both distributive laws. Now as $e_1, f_1, g_1 \in Re'$, therefore $Rg_1 = Re_1 = Rf_1 = Re' \ni e' = e'^2$. Then $e'g_1 = e'e_1 = e' = e'f_1$. Also, as $e_2,\ f_2,\ g_2 \in e'R$, therefore $g_2R = e_2R = f_2R = e'R \ni e' = e'^2$, and then, $g_2e' = e_2e' = e' = f_2e'$. Thus $e'\ a\ e' + e'\ b\ e' = e'\ d\ e'$, therefore $a + b = d$, since $a,\ b,\ d \in e'\ R\ e'$. □

**Theorem 7.** *Let* $(R, +, \cdot\,)$ *be a compact semiring. Then if* $E[\,\cdot\,] \cap K[\,\cdot\,]$ *is a subsemigroup of* $(R, \cdot)$*, it follows that*

$$((E[\,\cdot\,] \cap K[\,\cdot\,]) + (E[\,\cdot\,] \cap K[\,\cdot\,])) \cap K[\,\cdot\,] \subseteq E[\,\cdot\,] \cap K[\,\cdot\,].$$

*Proof.* Let $e,\ f \in E[\,\cdot\,] \cap K\,[\,\cdot\,]$, such that $e + f \in K[\,\cdot\,]$. Fix $e' \in E[\,\cdot\,] \cap K[\,\cdot\,]$ and let g be the idempotent of $eR \cap Re'$. Then $eR = gR$ and $Rg = Re'$, so $eg = g,\ ge = e,\ ge' = g$, and $e'g = e'$. So $e = ge = ge'e$, and similarly $f = he = he'f$. Thus

$$\begin{aligned} e + f &= g\,e'\,e + h\,e'\,f \\ &= g_1\,(e' + e')\,f_1 \end{aligned}$$

(for some $g_1 \in E[\,\cdot\,] \cap Re'$ and some $f_1 \in e'R \cap E[\,\cdot\,]$, by Theorem 6 above.)

$$\begin{aligned} &= g_1\,e'\,f_1 \\ &= g_1\,f_1 \\ &= t \in E[\,\cdot\,], \text{ by hypothesis.} \end{aligned}$$ □

**Theorem 8.** *Let* $(R, +, \cdot)$ *be a compact semiring in which* $(R, \cdot)$ *is simple, and* $(R, +)$ *is also simple. Choose* $e^2 = e \in R$*. Then denoting* $E[\,\cdot\,] \cap Re$ *by* $R_1, eR \cap E[\,\cdot\,]$ *by* $R_3$*, and* $eRe$ *by* $R_2$*, and considering the product space* $R_1 \times R_2 \times R_3$ *with the usual product topology, we define the following two operations on the product.*

*(1)* $(e_1, g_1, f_1) \circ (e_2, g_2, f_2) = (e_1, g_1 (f_1 e_2) g_2, f_2)$

*(2)* $(e_1, g_1, f_1) + (e_2, g_2, f_2) = (e_1 + e_2, g_1 + g_2, f_1 + f_2)$.

*Then these definitions make sense, obey the two distriburive laws, and we may map our product onto* $R$ *by the map* $(e, g, f) \longmapsto egf$*, which is an iseomorphism between the* $(\cdot)-$ *operations and a homomorphism between the* $(+)-$ *operations. Hence our map is a semiring iseomorphism.*

*Proof.* We get immediately from Wallace[12], that (1) is well defined and is an iseomorphism between the $(\cdot\ )-$ operations. We must show that $e_1+e_2$ is again in $E[\ \cdot\ ]\cap Re$, and that $f_1+f_2$ is again in $eR \cap E[\ \cdot\ ]$.

Considering $Re$, this is a compact subsemiring of $R$, and satisfies the hypotheses of the previous theorem, so $e_1 + e_2$ is again in $E[\ \cdot\ ] \cap Re$, and similarly for the other by considering $eR$. (We note that the ideas involved in the previous theorem and just here are generalizations of work of Pearson[5].) We now demonstrate the left distributive law.

$$\begin{aligned}
&(a, b, c) \circ ((e_1, g_1, f_1) + (e_2, g_2, f_2))\\
&\quad = (a, b, c) \circ (e_1 + e_2, g_1 + g_2, f_1 + f_2)\\
&\quad = (a, b\, c\, (e_1\, g_1 + \cdots + e_2\, g_2), f_1 + f_2)\\
&\quad = (a, b\, c\, (e_1\, g_1 + e_2\, g_2), f_1 + f_2) \text{ (by Kimura [3])}\\
&\quad = (a, b\, c\, e_1\, g_1 + b\, c\, e_2\, g_2, f_1 + f_2)
\end{aligned}$$

$$\begin{aligned}
&(a, b, c) \circ (e_1, g_1, f_1) + (a, b, c) \circ (e_2, g_2, f_2)\\
&\quad = (a, b\, c\, e_1\ g_1, f_1) + (a, b\, c\, e_2\, g_2, f_2)\\
&\quad = (a + a, b\, c\, e_1\, g_1 + b\, c\, e_2\, g_2, f_1 + f_2)\\
&\quad = (a, b\, c\, e_1\, g_1 + b\, c\, e_2\, g_2, f_1 + f_2)
\end{aligned}$$

Similarly for the right distributive law.
It remains to show that our map is an additive homomorphism.

$$\begin{aligned}
(e_1 + e_2)(g_1 + g_2)(f_1 + f_2) &= e_1\, g_1\, f_1 + (....) + e_2\, g_2\, f_2\\
&= e_1\, g_1\, f_1 + e_2\, g_2\, f_2
\end{aligned}$$

again by Kimura[3], since $(R,+)$ is a rectangular band. $\square$

**Theorem 9.** *Let $(R,+,\cdot)$ be a compact semiring in which $(R,\cdot)$ is simple. Then denoting the structure decomposition of $(R,+)$ into maximal rectangular subbands by $(R_\gamma, \gamma \in \Gamma)$* (Kimura [3]),*it follows that this is a partition formed by a closed semiring congruence on R. Moreover, each $(R\gamma,+,\cdot)$ is a compact subsemiring of R, and is a union of maximal $(\cdot)-$ groups and is $(\cdot)-$ and $(+)-$ simple. (As special cases of an $R_\gamma$ we get: (i) $K[+]$ the minimal additive ideal of R, and (ii) the complement of any maximal additive ideal.)*

*Proof.* Following Kimura [3], define the relation $\tau$ by $x\tau y$ if, and only if, $x+y+x=x$ and $y+x+y=y$ . Kimura has shown that this is an additive congruence and the congruence classes form our partition $(R_\gamma, \gamma \in \Gamma)$. We show that $\tau$ is also a $(\cdot)-$ congruence and that it is closed topologically as a subset of $R \times R$ in the usual product topology. Let $(x,y) \in \tau$ and let $q \in R$. Then $qx+qy+qx=qx$ and $qy+qx+qy=qy$, using the left distributive law, so $(qx,qy) \in \tau$. Similarly using the right distributive law, $(xq,yq) \in \tau$. So it is a semiring congruence. To show that $\tau$ is closed, let $(s,t) \in (R \times R)\backslash\tau$. Then either $s+t+s \neq s$ or $t+s+t \neq t$. If the former then $\exists U(s), \exists U(t)$, open sets about $s$ and $t$ respectively, such that

$$(U(s)+U(t)+U(s)) \cap U(s) = \emptyset.$$

Then $W(s,t) = U(s) \times U(t) \subseteq (R \times R)\backslash\tau$, and so $\tau$ is closed. Otherwise similarly consider $t+s+t \neq t$. Thus $\tau$ is closed, so each $R_\gamma$ is closed, hence compact. To complete our proof we need the following lemma:

**Lemma 10.** *Each $\tau-$ class $R_\gamma$ is a union of (pairwise disjoint) maximal $(\cdot)-$ groups.*

*Proof.* Certainly each $R_\gamma$ is contained in a union of maximal $(\cdot)-$ groups as $(R,\cdot)$ is simple by hypothesis. We show that if $e$ is the identity element of the maximal $(\cdot)-$ group $H[\,\cdot\,](e)$ and $e \in R_\gamma$, then $H[\,\cdot\,](e) \subseteq R_\gamma$. This will mean that if $e' \in R_{\gamma'}, \gamma\,' \neq \gamma$, then it cannot be that any element of $H[\,\cdot\,](e')$ is in $R_\gamma$.

Now $H[\,\cdot\,](e) = eRe$ as $(R, \cdot)$ is simple. Also $eRe + eRe \subseteq eRe$ by the two distributive laws, so $(H[\,\cdot\,](e), +, \cdot)$ is a compact subsemiring in which the multiplicative structure is a group. Next we note that Pearson [4] has shown that any such semiring $(R', +, \cdot)$ must have additive structure which is a rectangular band (i.e. be additively simple and consist of additive idempotents). For convenience we also give a proof of that here. We already know that all the elements are additive idempotents since the multiplication is simple, being a single group. To show that $(R', +, \cdot)$ is additively simple we note first that multiplication by any fixed element $a' \in R'$ gives that $a'K'[\,+\,]$ is again an ideal of $(R', +)$ due to the distributive laws and the fact that $a'R = R'$.
Viz. If $q' \in a'K'[\,+\,]$ and $r' \in R'$, $q' + r' = a'x' + a't'$ (where $x' \in K'[\,+\,]$ and $t' = a'^{-1}r'$).
So $q' + r' = a'(x' + t') = a'x''$, where $x'' = x' + t' \in K'[\,+\,]$.
So $q' + r' \in a'K'[\,+\,]$. Similarly, $r' + q' \in a'K'[\,+\,]$.
Thus as $a'K'[\,+\,]$ is an additive ideal of $(R', +, \cdot)$ we must have $K'[\,+\,] \subseteq a'K'[\,+\,]$, since $K'[\,+\,]$ is the minimal (=minimum) ideal of $(R', +, \cdot)$.
Then by the (Non) Swelling Lemma of A. D. Wallace, $K'[\,+\,] = a'K'[\,+\,]$.
Now for any element $y\,' \in R'$ and any element $x' \in K'[\,+\,]$ $(\neq \emptyset)$, $y\,' = (y\,'x'^{-1})x' \in (y\,'x'^{-1})K'[\,+\,] = K'[\,+\,]$. So $R' = K'[\,+\,]$ as stated above.

Since $(H\,\cdot\,](e), +)$ is a rectangular band, and $e \in R_\gamma$, and $R_\gamma$ is a maximal rectangular subband, therefore

$$H[\,\cdot\,](e) \subseteq R_\gamma.$$

So $R_\gamma$ is the precise union of maximal $(\cdot)-$ groups of $(R, \cdot)$. This ends the proof of the lemma. □

Now the product of a pair of maximal $(\cdot)-$ groups is contained

in another, so we need only show that $(E[\,\cdot\,]\cap R_\gamma)\cdot(E[\,\cdot\,]\cap R_\gamma) \subseteq R_\gamma$ , and it then follows that $R_\gamma \cdot R_\gamma \subseteq R_\gamma$ .
Let $e, f \in E[\,\cdot\,] \cap R_\gamma$ . Then

$$ef + f + ef = ef + ff + ef = (e + f + e)f = ef,$$

and

$$f + ef + f = ff + ef + ff = (f + e + f)f = ff = f.$$

Thus $ef\ \tau\ f$ and so $ef \in R_\gamma$ . Finally, as $(R, \cdot)$ is simple and $(R_\gamma, \cdot)$ is a closed subsemigroup of it then $(R_\gamma, \cdot)$ must be compact and simple. So indeed $(R, +, \cdot)$ is a disjoint union of compact semirings each one of which is a Kimura congruence class, and each one has simple multiplication and additive structure a rectangular band.

To prove the special cases mentioned we note the following:

**Lemma 11.** If $(R, +)$ is a semigroup, and $I$ is any ideal of $R$, then for any rectangular band $B$, a subsemigroup of $R$, either $B \subseteq I$, or $B$ does not meet $I$.

*Proof.* If $x \in I \cap B$ and $y$ is any element of $B$, then $y+x+y = y$, which is in $I$, since $I$ is an ideal. □

**Theorem 12. (Fawcett-Koch-Numakura [2])** *If $(S, +)$ is a compact mob* (=topological semigroup) *in which $J$ is a maximal ideal, and if $S\backslash J$ is a union of (additive) groups, then $S\backslash J$ is a subsemigroup of $S$ which is compact and simple.*

**Corollary 13.** *If $(R, +, \cdot)$ is a compact semiring in which $((R, \cdot)$ is simple, then the complement of any maximal additive ideal $J[\,+\,]$, is a subsemigroup of $(R, +)$ and is compact and simple.*

*Proof.* $E[\,+\,] = R$, because $(R, \cdot)$ is simple, and then by Theorem 12 we are done. □

This means that since $K[\,+\,]$ and the complement of any maximal $(+)-$ ideal are rectangular bands of $(R, +)$ when $(R, \cdot)$

is simple, thus they must be maximal rectangular subbands of $(R,+)$. So they are each some $R_\gamma$

This finally concludes the proof of Theorem 9.

A corollary to a result of Pearson [5], gives us the following product theorem for $R/\tau$.

**Theorem 14.** *Choose any element* $\overline{e} \in R/\tau$ . *Then* $R/\tau$ *is iseomorphic to the cartesian product of the semirings*

$$R_1 = (R/\tau)\ \overline{e}\ , \quad R_2 = \overline{e}\ (R/\tau)$$

*via the map* $(a,b) \longmapsto ab$ . *Thus* $R/\tau$ *comes only as the product of a pair of additive bands, with left trivial multiplication in one, and right trivial multiplication in the other.*

*Proof.* Omitted.

We now investigate the construction of compact semirings which are multiplicatively simple.

**Theorem 15.** *If* $(X,+)$ *and* $(Y,+)$ *are compact bands, and if* $(G,+,\cdot)$ *is a compact semiring with* $(G,\cdot)$ *a group, then any continuous function* $\sigma : Y \times X \to G$ *(denoted by juxtaposition where no confusion is likely) which has the properties:*

$$(i) \qquad (y_1 + y_2)\, x = y_1\, x + y_2 x$$

$$(ii) \quad y\,(x_1 + x_2) = y\, x_1 + y\, x_2,$$

*gives rise to a compact semiring which is* $(\cdot)-$ *simple. (We take* $R = X \times G \times Y$ *with the usual topology, and define*

$$(x,g,y)\ \cdot\ (x',g',y\,') = (x, g\ \sigma((y,x'))\ g', y\,')$$
$$(x,g,y) + (x',g',y') = (x+x', g+g', y+y')\ .$$

*Proof.* That $(R,\cdot)$ is a compact semigroup follows from Wallace[12]. $(R,+)$ is merely the product of three bands, and so is a band semigroup. We need only check the distributive laws.

$$\begin{aligned}
&(x,g,y) \cdot ((x_1,g_1,y_1) + (x_2,g_2,y_2))\\
&= (x,g,y)\cdot(x_1+x_2,\ g_1+g_2,\ y_1+y_2)\\
&= (x, g\,\sigma((y,x_1+x_2))\ (g_1+g_2),\ y_1+y_2)\\
&= (x, g\,(\sigma((y,x_1)) + \sigma((y,x_2)))\ (g_1+g_2),\ y_1+y_2) \text{ (by (i))}\\
&= (x, g\,\sigma((y,x_1))\,g_1 + \ldots + g\,\sigma((y,x_2))\,g_2, y_1+y_2)\\
&= (x, g\,\sigma((y,x_1))\,g_1 + g\,\sigma((y,x_2))\,g_2,\ y_1+y_2)
\end{aligned}$$

(Kimura [15] since as shown above, $(G,+)$ is a rectangular band).

$$\begin{aligned}
&(x,g,y) \cdot (x_1,g_1,y_1) + (x,g,y)\cdot(x_2,g_2,y_2)\\
&= (x\ ,\ g\ \sigma((y\ ,\ x_1))\ g_1\ ,\ y_1) + (x\ ,\ g\ \sigma((y\ ,\ x_2))\ g_2\ ,\ y_2)\\
&= (x+x\ ,\ g\ \sigma((y\ ,\ x_1))\ g_1 + g\ \sigma((y\ ,\ x_2))\ g_2\ ,\ y_1+y_2)\\
&= (x\ ,\ g\ \sigma((y\ ,\ x_1))\ g_1 + g\ \sigma((y\ ,\ x_2))\ g_2\ ,\ y_1+y_2)
\end{aligned}$$

(because $(X,+)$ is a band).

Thus we have left distribution. The right distribution follows similarly from (ii) above and the fact that $(Y,+)$ is a band. □

**Theorem 16.** *For any compact semiring $(R,+,\cdot)$ we may define, on the minimal multiplicative ideal $K[\ \cdot\ ]$, an addition $\oplus$ , and multiplications $\sigma_g$ for each $g$ in a fixed $H[\ \cdot\ ](e) \subseteq E[\ +\ ]$, such that $(K[\ \cdot\ ],\oplus,\sigma_g)$ is a compact semiring with simple multiplication. When $g=e$, we obtain the original multiplication, and if $K[\ \cdot\ ] \subseteq E[\ \cdot\ ]$, and $(K[\ \cdot\ ],+,\cdot)$ is a subsemiring of $R$, then $\oplus$ is the same as $+$.*

*Proof.* For any fixed $e^2 = e \in K[\ \cdot\ ]$, $H[\ \cdot\ ](e) = eK[\ \cdot\ ]e$ is a compact semiring under $+$ and $\cdot$ , and is a $(\cdot)-$ group. Also, $E[\cdot]\cap K[\cdot]e$ and $eK[\cdot]\cap E[\cdot]$ are compact semirings under $+$ and $\cdot$ (see Theorem 7 above). More importantly, these last two sets are $(+)-$ bands, since they are included in $K[\cdot] \subseteq E[+]$.

Since we are in a semiring, if we select $g \in K[\cdot]e$, then the map $(y, x) \longmapsto \sigma_g((y, x)) = ygx$ from $(eK[\cdot] \cap E[\cdot]) \times (E[\cdot] \cap K[\cdot]e) \to eK[\cdot]e$ is continuous, and has the properties needed to apply Theorem 15 above.

We know from Wallace [12], that the space

$$(E[\cdot] \cap K[\cdot]e) \times eK[\cdot]e \times (eK[\cdot] \cap E[\cdot])$$

with multiplication $(a, b, c) \cdot (a', b', c') = (a, b\ \sigma_g((c, a'))\ b', c')$, is a compact simple semigroup. Moreover, when $g = e$, it is iseomorphic to $(K[\cdot], \cdot)$ by the map $(a, b, c) \longmapsto abc$ . Thus this map is always a homeomorphism of our product to $K[\cdot]$ . Now by applying Theorem 15 above, we may take componentwise addition for $\oplus$, and then, via the homeomorphism, $(K[\cdot], \oplus, \sigma_g)$ is a compact semiring with simple multiplication. When $g = e$, $\sigma_g$ corresponds to $(\cdot)$ . If $K[\cdot] \subseteq E[\cdot]$, and $K[\cdot]$ is a subsemiring of $R$, then Pearson [5], has shown that our addition corresponds to $(+)$ . So the theorem is proved. □

We do not know whether compact semirings with simple multiplication are more complicated than this, but, due to our inability to produce a semifactorization theorem corresponding to our construction, it is suspected that they are more complicated. It is not known to the author, for compact connected semirings in the plane, whether or not $K[\cdot]$ has to be a subsemiring, but once again it is conjectured that it need not be.

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