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ON INJECTIVE TOPOLOGICAL SPACES AND BISPACES

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Dedicated to George Strecker on the occasion of his 60th birthday as an expression of respect and esteem.

Abstract

Dana Scott introduced and characterized Injective Topological Spaces as "exactly the continuous lattices". Our intention has been to obtain an analogous characterization of Injective Bitopological Spaces.

1. Introduction

Injective T_0 topological spaces were defined by Dana Scott in [6] (Definition 1.1), where he gave two characterizations of such spaces which we now recall:

External Characterization ([6] 1.6 Corollary; [1] 3.4 Lemma; [4] Chapter VII, 4.7 Corollary):

A T_0–topological space is injective if and only if it is a retract of a power of the Sierpiński dyad.

Internal Characterization ([6] 2.12 Theorem; [1] Chapter II, 3.8 Theorem; [4] Chapter VII, 4.8 Proposition):

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The injective T_0 topological spaces are precisely the spaces (X,T) such that $(X,\leq_T)$ is a continuous lattice, and the associated Scott topology coincides with T, where $\leq_T$ denotes the specialization order.

The purpose of this paper is to characterize injective bispaces both externally and internally. The results are given in Section 3, and Section 4, respectively.

Firstly, we define, in a natural way, the concept of an injective T_0–bispace.

Definition 1.1. ([2]) A bitopological space (X,P,Q) is a T_0–**bispace** if $P \vee Q$ is a T_0–topology.

Definition 1.2. (I,L,R) will be called an **injective bispace** if it has the following extension property with respect to embeddings:

> If $f:(X,P,Q)\to(I,L,R)$, and $e:(X,P,Q)\to(X',P',Q')$ is an embedding, then there is F, called an extension of f, such that $F:(X',P',Q')\to(I,L,R)$ and $F\circ e=f$.

As an important example of an injective T_0–bispace we have the space $(\mathbb{Q},U,L)$ introduced in [2], where $\mathbb{Q}=\{\alpha,\ \beta,\ \gamma,\ \delta\}$ and

$$\begin{aligned} U &= \{\phi,\ \{\alpha,\ \beta\},\ \{\alpha,\ \beta,\ \gamma,\ \delta\}\} \\ L &= \{\phi,\ \{\beta,\ \gamma\},\ \{\alpha,\ \beta,\ \gamma,\ \delta\}\}. \end{aligned}$$

In fact (see also [2]), every T_0–bispace (X,P,Q) is homeomorphic to a subspace of the canonical product

$$(\mathbb{Q},U,L)^{C(X,\mathbb{Q})}.$$

We now recall some relevant definitions and notation. As usual, $f:(X,T)\to(X',T')$ will always denote a continuous map;

$f : (X, P, Q) \to (X', P', Q')$ will denote a map such that

$$f : (X, P) \to (X', P') \text{ and } f : (X, Q) \to (X', Q').$$

$e : (X, P, Q) \to (X', P', Q)$ is an embedding of (X, P, Q) into (X', P', Q') if $e : (X, P) \to (X', P')$ and $e : (X, Q) \to (X', Q')$ are embeddings. We denote the set of maps $f : (X, T) \to (X', T')$ by $C(X, X')$, we shall also denote the set of maps $f : (X, P, Q) \to (X', P', Q')$ by $C(X, X')$; the context making it clear whether it is spaces that are involved or alternatively, bispaces.

In what follows, $\mathbb{R}$ will denote the real line, $\mathbb{I}$ the unit interval $[0, 1]$ and $\mathbb{D}$ the two point space $\{0, 1\}$. The topology of upper semicontinuity on $\mathbb{R}$, $\mathbb{I}$, $\mathbb{D}$, with basic open sets of the form $(\leftarrow, a)$, will be denoted by u. Thus $(\mathbb{D}, u)$ represents the space $\mathbb{D}$ with topology u consisting of ϕ, $\{0\}$, $\{0, 1\}$.

Topologically, we could say that a point x is "above" y, if every open neighbourhood of x contains y. This gives rise to a partial order relation denoted by $y \leq_T x$ (so that $y \leq_T x$ if and only if $x \in cl_T y$). We have $x \leq_T x$; $x \leq_T y$ and $y \leq_T z \Rightarrow x \leq_T z$. For T_0 topologies it is also true that: $x \leq_T y$ and $y \leq_T x \Rightarrow x = y$.

In $(\mathbb{R}, u)$, the order relation induced by u is precisely the usual order on $\mathbb{R}$. This order relation, sometimes its dual, is referred to as the **specialization order** induced by T on the set X and will be written $\leq_T$, or, simply $\leq$ when the context makes it clear that the topology involved is T.

2. Injective T_0 Topological Spaces

Because our characterization of injective T_0–bispaces is given in terms of injective T_0–spaces, we have thought that it would be appropriate to present a characterization of this class of spaces which, although essentially known, does not quite appear, in the literature, in the form given below.

Related to the concept of Monotone Convergence Space ([1], Chapter II, 3.9 Definition) we have the following concept of a monotone net.

Definition 2.1. Let D be a directed set and $x : D \to X$ a net with values in a topological space (X, T). The net will be said to be **monotonic decreasing** if, for all d, d' in D, we have $d \leq d' \Rightarrow x_{d'} \leq_T x_d$.

Proposition 2.2. *Let (X, T) be an injective T_0-space. Then $(X, \leq_T)$ is a complete partially ordered set in which monotone decreasing nets converge to their infima and every x_0 is the infimum of the monotone decreasing net $x : \mathcal{N}_{x_0} \to X$ given by $x_V = \sup V$, where $V \in \mathcal{N}_{x_0}$.*

Proof. We shall only give a sketch of the proof. As is well known, the canonical map $e : (X, T) \to (\mathbb{D}, u)^{C(X,\mathbb{D})}$ provides a topological and order embedding ($x \leq_T y \Leftrightarrow e(x) \leq_P e(y)$, where P refers to the product topology). It is also well known and easily verified that $\leq_P$ coincides with the product of the $\leq_u$ on the factors (the usual order on $D = \{0, 1\}$). Since (X, T) is injective, there is $r : (\mathbb{D}, u)^{C(X,\mathbb{D})} \to (X, T)$ such that $r \circ e = 1_X$. The retraction map, being continuous, is necessarily monotone:

$$\alpha \leq_P \beta \Rightarrow r(\alpha) \leq_T r(\beta).$$

The mapping r will yield the properties given in the statement of the proposition.

Firstly, given $A \subseteq X$, by order completeness of the product, there are α, β such that $\beta = \inf e[A]$, $\alpha = \sup e[A]$. It is readily verified that $r(\beta) = \inf A$, $r(\alpha) = \sup A$.
For the second property, consider a monotone decreasing net $x : D \to (X, \leq_T\ )$. Then $e \circ x$ is a monotone decreasing net in the product and, as such, converges to its infimum β, say. Then $r(\beta) = \inf\{x_d \,|\, d \in D\}$ and the net x converges to $r(\beta)$.

Finally, given $x_0 \in X$, it is clear that $x : V \mapsto x_V = \sup V$, $V \in \mathcal{N}_x$, yields a monotone decreasing net on the set $\mathcal{N}_x$ directed by reverse inclusion. We show that $\inf x_V = x$. It is clear that $x \leq x_V$ for all $V \in \mathcal{N}_x$, hence $x \leq \inf x_V$. Now, given $V \in \mathcal{N}_x$, let V_0 be a neighbourhood of $e(x)$ of the form $\sqcap G_i$, where all $G_i = \mathbb{D}$, except for one i, say i_0, such that $x \in e^{\leftarrow}[V_0] \subseteq V$. It is clear that there is a point α_{V_0} in V_0 such that $\beta \leq \alpha_{V_0}$ for all β in V_0. In particular, $e(z) \leq \alpha_{V_0}$ for all z in $e^{\leftarrow}[V_0]$. Thus $z \leq r(\alpha_{V_0})$ for all z in $e^{\leftarrow}[V_0]$. Hence $x_{V_0} \leq r(\alpha_{V_0})$. It is clear that, as W_0 ranges through the (sub-)basic neighbourhoods of $e(x)$, we have $\lim\limits_{W_0} \alpha_{W_0} = \inf\limits_{W_0} \alpha_{W_0} = e(x)$. By continuity of r, we get $\lim\limits_{W_0} r(\alpha_{W_0}) = r(e(x)) = x$. Hence, there is one such W_0 for which $r(\alpha_{W_0}) \in V$, so that $x_{W_0} \in V$. Hence $\inf x_V \in V$, as required. $\square$

We shall express the property "$x = \inf\limits_{V \in \mathcal{N}_x} \sup V$", by saying that the neighbourhoods of points have "**small spread**".

Proposition 2.3. *Let (X, T) be a T_0-space for which $(X, \leq_T)$ is a complete partially ordered set in which monotone decreasing nets converge to their infima and points have small spreads. Then (X, T) is an injective space.*

Proof. We shall show that a T_0-space (X, T) with the given properties is a retract of its canonical product $(\mathbb{D}, u)^{C(X,\mathbb{D})}$, hence injective ([6], 1.4 Proposition; [4] Chapter VII, 4.7 Corollary; [1] Chapter II, 3.2 Lemma). Following D. Scott, we shall use a "$\overline{lim}$ approximation" expressed by the formula

$$r : (\mathbb{D}, u)^{C(X,\mathbb{D})} \to X, \quad \text{where } r(\alpha) = \inf_{V \in \mathcal{N}_\alpha} \sup e^{\leftarrow}[V]$$

(see Section 2 on Continuous Lattices in [6]; also [1], Chapter II, Exercise 3.14), where e is the canonical embedding $e : X \to \mathbb{D}^{C(X,\mathbb{D})}$. We shall use the convention that $\sup \phi = 0$, where 0 is the smallest element of $(X, \leq_T)$.

We show that r is continuous at α: observe that if $W \subseteq V$, where $V, W \in \mathcal{N}_\alpha$, then $e^{\leftarrow}[W] \subseteq e^{\leftarrow}[V]$, so that $\sup(e^{\leftarrow}[W]) \leq \sup(e^{\leftarrow}[V])$. Thus $x : \mathcal{N}_\alpha \to (X, T)$ is a monotone decreasing net when $\mathcal{N}_\alpha$ is directed by reverse inclusion and $x_V = \sup(e^{\leftarrow}[V])$. By the assumption concerning monotone decreasing nets, the net converges to x_0, say, where $x_0 = \inf_{V \in \mathcal{N}_\alpha} x_V$. Define $r(\alpha)$ to be x_0. Consider V_0, an open set which contains $r(\alpha) = x_0$. Since x converges to x_0, there is x_U in V_0, for some $U \in V_\alpha$. Consider β in U. Then $r(\beta) \leq x_U$, by definition of r. Now V_0 is $\leq_T$–decreasing, hence $r(\beta) \in V_0$. Thus, r is continuous.

Finally, in terms of r, the condition that neighbourhoods have small spreads is expressed by $r(e(x)) = x$. Hence X is a retract of its canonical product, as required. ◻

3. Injective T_0-Bispaces

It may be useful to consider some examples.

Example 3.1. ([2]) As mentioned in the Introduction, let $\mathbb{Q}$ denote the Quad: the set $\{\alpha, \beta, \gamma, \delta\}$ with four points, and with topologies U, L specified as follows:

$$U = \{\phi, \{\alpha, \beta\},\ \{\alpha, \beta, \gamma, \delta\}\}$$

$$L = \{\phi, \{\beta, \gamma\},\ \{\alpha, \beta, \gamma, \delta\}\}.$$

It is readily verified that every T_0–bispace (X, P, Q) is homeomorphic to a subspace of the canonical product

$$(\mathbb{Q}, U, L)^{C(X,\mathbb{Q})},$$

and that $(\mathbb{Q}, U, L)$ is an injective bispace.

Example 3.2. The Triad $(\mathbb{T}, U, L)$ is the subspace of the Quad which consists of the three points (α, β, γ). This space is not injective, indeed, the identity map on $\mathbb{T}$ cannot be extended as a bicontinuous map from $(\mathbb{Q}, U, L)$ to $(\mathbb{T}, U, L)$.

It is natural to try to determine the relationship between injective spaces and injective bispaces. We address some of the immediate questions that arise. If (X, T) is injective, does it follow that (X, T, T) is an injective bispace? The following example answers the question negatively.

Example 3.3. Let $1_{\mathbb{D}} : (\mathbb{D}, u, u) \to (\mathbb{D}, u, u)$. Consider $e : (\mathbb{D}, u, u) \to (\mathbb{T}, U, L)$, where $e(0) = \beta$, $e(1) = \delta$. Then, there does not exist $F : (\mathbb{T}, U, L) \to (\mathbb{D}, u, u)$, such that $F \circ e = 1_{\mathbb{D}}$.

However, in the category of **strongly–$\boldsymbol{T_0}$ bispaces** — spaces (X, P, Q) for which $P \wedge Q$ is T_0, we have the following.

Proposition 3.4. *If (X, T) is an injective T_0–space, then (X, T, T) is an injective bispace in the category of strongly T_0 bispaces.*

Proof. Suppose $(A, L, R) \xrightarrow{e} (A', L', R')$ is an embedding, and $f : (A, L, R) \to (X, T, T)$. Then $e : (A, L \wedge R) \to (A', L' \wedge R')$ is also an embedding and $f : (A, L \wedge R) \to (X, T)$ is continuous. Hence there is $F : (A', L' \wedge R') \to (X, T)$ such that $F \circ e = f$. Now $1_{A'} : (A', L') \to (A', L' \wedge R')$ and $1_{A'} : (A', R') \to (A', L' \wedge R')$ are both continuous, hence, so is $F : (A', L', R') \to (X, T, T)$. The proof is complete. $\square$

Proposition 3.5. *If (X, P, Q) is injective, then so is $(X, P \vee Q)$.*

Proof. Given an embedding $e : (A, T) \to (A', T')$ and $f : (A, T) \to (X, P \vee Q)$, then $e : (A, T, T) \to (A', T', T')$ is an embedding and $f : (A, T, T) \to (X, P, Q)$. Hence, there is $G : (A', T', T') \to (X, P, Q)$ such that $G \circ f = g$. Observe that $G : (A', T') \to (X, P \vee Q)$ is continuous. The proof is complete. $\square$

However, as may be expected, $(X, P \vee Q)$ may be injective whilst (X, P, Q) is not, as the following example illustrates.

Example 3.6. Consider $(\mathbb{T}', U, L)$ the subspace of $(\mathbb{Q}, U, L)$, where $\mathbb{T}' = \{\delta, \alpha, \beta\}$. The identity map from $(\mathbb{T}', U, L)$ to itself cannot be extended to $(\mathbb{Q}, U, L)$. Thus $(\mathbb{T}', U, L)$ is not an injective bispace.
Note that $(\mathbb{T}', U \vee L)$ is an injective space (the 3–chain with u–topology).

Proposition 3.7. *If (X, P, Q) is an injective bispace, then (X, P) and (X, Q) are injective spaces.*

Proof. Consider an embedding $(Y, L) \stackrel{e}{\to} (Y', L')$ and $f : (Y, L) \to (X, P)$. Letting D and D' denote the discrete topologies on Y and Y', respectively, we have an embedding $e : (Y, L, D) \to (Y', L', D')$ and $f : (Y, L, D) \to (X, P, Q)$. Hence, there is $F : (Y', L', D') \to (X, P, Q)$ such that $F \circ e = f$. Thus, $G : (Y', L') \to (X, P)$ is such that $F \circ e = f$, showing that (X, P) is injective. By symmetry, (X, Q) is also injective. Note that neither (X, P) nor (X, Q) need be T_0 topological spaces when $P \vee Q$ is a T_0 topology. □

The following simple example shows that injectivity for bispaces requires more than the injectivity of the component topologies.

Example 3.8. $(\mathbb{D}, u)$ and $(\mathbb{D}, \ell)$ are both injective T_0-spaces, where $\mathbb{D} = \{0, 1\}$ and $u = \{\phi, \{0\}, \{0, 1\}\}$, $\ell = \{\phi, \{1\}, \{0, 1\}\}$. However, $(\mathbb{D}, u, l)$ is not an injective bispace:

Let $(\mathbb{T}^*, U, L)$ denote the subspace of the Quad, determined by $\mathbb{T}^* = \{\gamma, \delta, \alpha\}$. Let $e : (D, u, l) \to (\mathbb{T}^*, U, L)$ be given by $e(0) = \alpha$, $e(1) = \gamma$. Then e is an embedding and there is no map F, $F : (\mathbb{T}^*, U, L) \to (D, u, l)$ such that $F \circ e = 1_{\mathbb{D}}$. Thus (D, u, l) is not an injective T_0–bispace.

It is now appropriate to give the external characterization of injective T_0 bispaces, where $(\mathbb{Q}, U, L)$ plays the role of $(\mathbb{D}, u)$ for injective T_0 topological spaces.

Proposition 3.9. *(X, P, Q) is an injective bispace if and only if it is a retract of a product of copies of $(\mathbb{Q}, U, L)$.*

Proof. The argument is categorical, but will be sketched here for the sake of completeness. Assume (X, P, Q) is injective, then there is

$$r : (\mathbb{Q}, U, L)^{C(X,\mathbb{Q})} \longrightarrow (X, P, Q)$$

such that $r \circ e_X = 1_X$, where e_X is the canonical embedding of (X, P, Q) in the product space $(\mathbb{Q}, U, L)^{C(X,\mathbb{Q})}$.

Conversely, suppose $(A, L, R) \xrightarrow{e} (A', L', R')$ is an embedding and $g : (A, L, R) \to (X, P, Q)$. Then there is a naturally induced map g^* such that $g^* : (\mathbb{Q}, U, L)^{C(A,\mathbb{Q})} \longrightarrow (\mathbb{Q}, U, L)^{C(X,\mathbb{Q})}$ and $g^* \circ e_A = e_X \circ g$. Observe that $(\mathbb{Q}, U, L)^{C(A,\mathbb{Q})}$ is injective, so there is $H : (A', L', R') \to (\mathbb{Q}, U, L)^{C(A,\mathbb{Q})}$ such that $H \circ e = e_A$. It is easy to check that if (X, P, Q) is a retract of a power of $(\mathbb{Q}, U, L)$, then there is $r_X : (\mathbb{Q}, U, L)^{C(X,\mathbb{Q})} \longrightarrow (X, P, Q)$ such that $r_X \circ e_X = 1_X$. Then, $G = r_X \circ g^* \circ H : (A', L', R') \longrightarrow (X, P, Q)$ is such that $G \circ e = g$, as required. $\square$

The result above is simply a consequence of the observation that $(\mathbb{Q}, \mathbb{U}, \mathbb{L})$ is an injective cogenerator for T_0 bispaces and the fact that the injective hull of the singleton $\{(\mathbb{Q}, U, L)\}$ in $2\underline{Top}_0$, the category of T_0–bispaces, consists of retracts of products of $(\mathbb{Q}, U, L)$ (see, for example [5], Proposition 4).

4. Injective T_0–Bispaces, an Internal Characterization

It would be of interest to formulate criteria that would enable one to decide, fairly directly, whether or not a given bitoplogical space is injective. This is what we shall attempt to do in this section.

Because a T_0 bispace (X, P, Q) is one where distinct points are separated either by a P–open set or by a Q–open set, neither P nor Q need be T_0– topologies. We denote the T_0–reflection of (X, P) by $([X]_P, P_0)$, and of (X, Q) by $([X]_Q, Q_0)$: recall that the points of $[X]_P$ are equivalence classes $[x]_P$ which consist of all x' such that $cl_P x' = cl_P\, x$; similarly, the elements of $[X]_Q$ are equivalence classes $[y]_Q$, consisting of all y' in X such that $cl_Q\, y' = cl_Q\, y$. Moreover (X, T) is an injective in the category of topological spaces $\underline{Top}$ if, and only if, $([X]_T, T_0)$ is an injective in the category of T_0 topological spaces $\underline{Top}_0$, as is easily verified.

It was observed in Proposition 3.7 that if (X, P, Q) is an injective T_0– bispace, then (X, P) and (X, Q) are injective spaces, hence $([X]_P, P_0)$, $([X]_Q, Q_0)$ are injective T_0 topological spaces. It was pointed out in Example 3.8 that injectivity of (X, P) and of (X, Q) in $\underline{Top}_0$ does not imply injectivity of (X, P, Q) in $2\underline{Top}_0$, the category of T_0–bispaces.

The internal characterization given below is formulated in terms of an additional requirement which we have decided to call **intertwinement.**

Definition 4.1. A bispace (X, P, Q) is **intertwining** if we have $[x]_P \bigcap [x']_Q \neq\ \phi$ for all x, x' in X.

Examples 4.2. 1. $(\mathbb{D}, u, u)$ is not intertwining; $(\mathbb{D}, u, \ell)$ is not intertwining

2. $(\mathbb{D}, u, i)$ is intertwining, where i denotes the indiscrete topology on $\mathbb{D}$.

3. $(\mathbb{Q}, U, L)$ is intertwining.

Observation: R. Börger (oral communication) has observed that, when (X, P, Q) is intertwining and $P \vee Q$ is T_0, then $[x]_p \cap [x']_Q$ is always a singleton set.

We can now formulate the internal characterization promised above: (X, P, Q) is an injective T_0–bispace if both (X, P) and (X, Q) are injective in $\underline{Top}$ and (X, P, Q) is intertwining, equivalently:

Proposition 4.3. (X, P, Q) *is an injective T_0–bispace if and only if it is an intertwining bispace and both* $([X]_P, P_0)$, $([X]_Q, Q_0)$ *are injective T_0–spaces.*

Proof. Assume (X, P, Q) is injective. It has been shown that (X, P) and (X, Q) are both injective spaces, hence both $([X]_P, P_0)$ and $([X]_Q, Q_0)$ are injective T_0 topological spaces. It remains to show that (X, P, Q) is intertwining. Consider the embedding of (X, P, Q) into $(X \times X,\ P \times I,\ I \times Q)$ by the diagonal map $\triangle$, where $\triangle(x) = (x, x)$ and I denotes the indiscrete topology on X.

Observe that $(X \times X,\ P \times I,\ I \times Q)$ is a T_0–bispace. By injectivity of (X, P, Q), there is a bicontinuous

$$F : (X \times X,\ P \times I,\ I \times Q) \to (X, P, Q)$$

such that $F \circ \triangle = 1_X$, i.e. $F(x, x) = x$. Fix x, x' in X. We show that $F(x, x') \in [x]_P \bigcap [x']_Q$. To prove that $F(x, x') \in [x]_P$, let V be a P–neighbourhood of x. Since $F(x, x) = x$, there is a P–neighbourhood of x, W, such that $W \times X$ is mapped to V by F. In particular $F(x, x') \in V$. Since V is arbitrary, we conclude that $x \in cl_P\ F(x, x')$. Consider now a P–neighbourhood U of $F(x, x)$. By continuity of F, there exists a P–neighbourhood of x, W, such that $W \times X$ is mapped to U by F. Thus $F(x, x) \in U$. But $x = F(x, x)$, so $x \in U$. Since U is arbitrary, we have $F(x, x') \in cl_P x$. Hence $F(x, x') \in [x]_P$. Similarly, $F(x, x') \in [x']_Q$. Thus (X, P, Q) is an intertwined bispace, as required.

Conversely, assume that (X, P) and (X, Q) are injective topological spaces and that (X, P, Q) is intertwined. As might have been expected from above, the intertwining property allows one to define a continuous map

$$F : (X \times X,\ P \times I,\ I \times Q) \to (X, P, Q) :$$

For each pair (x, y), let $F(x, y) = z$, where $z \in [x]_P \bigcap [y]_Q$. Note that z need not be unique if (X, P, Q) is not a T_0–bispace, but it is unique when (X, P, Q) is a T_0–bispace. Thus $F(x, x) = x$. To prove that

$F : (X \times X,\ P \times I) \to (X, P)$ is continuous, let x, y be given. Let V be a P–open set containing $z = F(x, y)$. Since $z \in [x]_P$, we have $x \in V$. For any x' in V, we have $z' = F(x', y) \in [x']_P \bigcap [y]_Q$, hence $z' \in V$. Thus $F : (X \times X,\ P \times I) \to (X, P)$ is continuous. Similarly, $F : (X \times X,\ I \times Q) \to (X, Q)$ is continuous, as required.

We conclude the proof by showing that (X, P, Q) is an injective bispace: Consider an embedding $e : (Y, L, R) \to (Y', L', R')$, and $g : (Y, L, R) \to (X, P, Q)$, then $e : (Y, L) \to (Y', L')$ is an embedding and $g : (Y, L) \to (X, P)$, similarly, $e : (Y, R) \to (Y', R')$ is an embedding $g : (Y, R) \to (X, Q)$. By injectivity of (X, P) and (X, Q), there are maps $G_1 : (Y', L') \to (X, P)$ and $G_2 : (Y', R') \to (X, Q)$ such that $G_1 \circ e = g$, $G_2 \circ e = g$. Now observe that

$G_1 \times G_2 : (Y', L', R') \to (X \times X,\ P \times I,\ I \times Q)$ is bicontinuous. With F defined above, we have

$$F \circ (G_1 \times G_2) : (Y', L', R') \to (X, P, Q).$$

It remains to verify that $F \circ (G_1 \times G_2) \circ e = g$.

$$\begin{aligned}(F \circ (G_1 \times G_2) \circ e)(x) &= F(G_1(e(x)), G_2(e(x))) \\ &= F(g(x), g(x)) \\ &= g(x).\end{aligned} \qquad \square$$

We shall give two applications of the above, bearing in mind Example 4.2.

Example 4.4. 1. $(\mathbb{D}, u, u)$ is not injective in $2\underline{Top}_0$, since it is not intertwined.

2. $(\mathbb{Q}, U, L)$ is injective in $2\underline{Top}_0$, since $([\mathbb{Q}]_U, U_0) = (D, u)$, $([\mathbb{Q}]_L, L_0) = (D, u)$, and $\overline{(\mathbb{Q}, U, L)}$ is intertwined.

It is, perhaps, worthwile to point out, once again, the extent to which the separation of points by open sets influences injectivity. This will be done by means of the following Corollary to Proposition 4.3.

Proposition 4.5. *The injective T_0–bispaces (X, P, Q) for which* ***both*** *P and Q are T_0 topologies cannot have more than one point.*

Thus, we see that, once again, $(\mathbb{D}, u, u)$ is not an injective T_0 bispace. However, it is an injective bispace in the category of bispaces (X, P, Q) for which $P \wedge Q$ is a T_0–topology, as observed in Proposition 3.4. These observations lead to the formulation of the following Problem:

Problem 4.6. Characterize the injective objects in the category of **doubly $\boldsymbol{T_0}$ bispaces** ((X, P, Q) such that both P and Q are T_0) and in the category of **strongly $\boldsymbol{T_0}$ bispaces** ((X, P, Q) such that $P \wedge Q$ is T_0).

References

[1] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *A Compendium of Continuous Lattices*, Springer Verlag, 1980.

[2] E. Giuli and S. Salbany, $2T_0$ *spaces and closure operators*, Sem. Fach. Math. und Inf. Univ. Hagen **29** (1988) 11–40.

[3] H. Herrlich, *Topological Structures and Injectivity*, Rendiconti del Circolo Matematico di Palermo, Serie II, **12** (1986), 87–92.

[4] P. Johnstone, *Stone Spaces*, Cambridge University Press, **3**, 1983.

[5] S. Salbany, *Injective Objects and Morphisms*, in Categorical Topology and its Relation to Analysis, Algebra and Combinatorics, pages 394–409. Editors Jiří Adámek and Saunders MacLane, 1990, World Scientific Publishing Co. Pty. Ltd, ISBN 9971-50-9660.

[6] D.S. Scott, *Continuous Lattices*, in Toposes, Algebraic Geometry and Logic, Springer Lecture Notes in Mathematics **274**, (1972) 97–136.

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