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SPANS OF CERTAIN SIMPLE CLOSED CURVES AND RELATED SPACES

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Abstract

In this paper we define a class of planar simple closed curves. We refer to members of this class as upward concave symmetric simple closed curves. We calculate the spans of each simple closed curve in this class. Also, we determine the spans of various surfaces and solids related to members of this class. We show that if Y is a plane sepatating continuum and X, one of these simple closed curves, is contained in a bounded component of $R^2 - Y$, then the span of Y is larger than the span of X.

1. Introduction

The concept of the span of a metric span was introduced in [L1]. The span can be thought of as a continuous type analogue of the diameter. Modified versions of the span have been introduced (cf [L2] and [L3]). Most of the interest has been in the various spans of continua, that is metric spaces which are compact and connected. In general, it is difficult to calculate the span of a given continuum.

Questions have been asked about how these various spans are related for a given continuum. For instance, the following questions have been asked (see [CIL],[L3]):

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(a) For a continuum X, is it true that the span of X is less than or equal to twice the surjective span of X?

(b) For a continuum X, is it true that the semispan of X is less than or equal to twice the surjective semispan of X?

(c) Is it true that the surjective span of X is less than or equal to twice the surjective semispan of X?

(d) For a simple closed curve X, is it true that the span of X is equal to the semispan of X?

For results related to (a), (b), and (c) see ([L2], [W1]). For results related to (d) see ([T1], [W2]). In this paper we calculate the spans for continua in various classes of continua. Also, we note that the questions asked above are answered in the affirmative for the continua considered in this paper.

Other questions concern the relationships of the spans of related continua. The following question by H. Cook, has generated much interest.

> If S_1 and S_2 are two simple closed curves in the plane and S_2 is contained in the bounded component of $R^2 - S_1$, then is the span of S_1 larger than the span of S_2? ([CIL]).

This question is answered in the affirmative for the class of simple closed curves considered in this paper. For other patial answers to this question and related results see ([T1], [T2], [W2], [W3], [W4], [W5], [W6]).

2. Preliminaries

If X is a non-empty metric space, we define the *span of* X, $\sigma(X)$, to be the least upper bound of the set of real numbers α which satisfy the following condition: there exists a connected space C and continuous mappings $g, f : C \to X$ such that

$$g(C) = f(C) \tag{σ}$$

and $\alpha \leq$dist$[g(c), f(c)]$ for $c \in C$.

The definition does not require X to be connected, but to simplify our discussion we will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, except we change conditions (σ) to the following:

$$g(C) = f(C) = X, \tag{σ^*}$$

$$g(C) \subseteq f(C), \tag{σ_0}$$

$$g(C) \subseteq f(C) = X, \tag{σ_0^*}$$

Equivalently (see [L1], p.209), the span $\sigma(X)$ is the least upper bound of numbers α for which there exist connected subsets C_α of the product $X \times X$ such that

$$p_1(C_\alpha) = p_2(C_\alpha) \tag{σ}'$$

and $\alpha \leq$dist(x, y) for $(x, y) \in C_\alpha$, where p_1 and p_2 denote the projections of $X \times X$ onto X, i.e., $p_1(x, y) = x$ and $p_2(x, y) = y$ for $x, y \in X$. Again, we will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, execpt we change conditions $(\sigma)'$ to the following (see L3):

$$p_1(C_\alpha) = p_2(C_\alpha) = X, \tag{σ^*}'$$

$$p_1(C_\alpha) \subseteq p_2(C_\alpha), \tag{σ_0}'$$

$$p_1(C_\alpha) \subseteq p_2(C_\alpha) = X. \tag{σ_0^*}'$$

We note that for a compact space X, C in the first set of definitions and C_α in the second set can be considered to be closed. The following inequalities follow immediately from the definitions.

$$0 \le \sigma^*(X) \le \sigma(X) \le \sigma_0(X) \le \text{diam } X,$$
$$0 \le \sigma^*(X) \le \sigma_0^*(X) \le \sigma_0(X) \le \text{diam} X.$$

It can easily be shown that, if J is an arc then $\sigma(J) = \sigma_0(J) = \sigma^*(J) = \sigma_0^*(J) = 0$. A simple consequence of this is that when X is a simple closed curve, $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$.

It has been shown by Lelek (see [L4] and [L5]) that in the above definitions, the inequalites

$$\alpha \le \text{dist}(x, y) \textit{ and } \alpha \le \text{dist}(g(c), f(c))$$

can be actually be replaced by (*)

$$\alpha = \text{dist}(x, y) \textit{ and } \alpha = \text{dist}(g(c), f(c))$$

respectively, in the case of compact spaces.

We utilize the following theorem from [L1], section 7.

Theorem L *If Y is a closed subset of the Hilbert cube I^ω and $f : Y \to S$ is an essential mapping of Y into the circumference S, then*

$$\inf_{s \in S}(f^{-1}(s), f^{-1}(-s)) \le \sigma(Y).$$

To simplify our exposition, we define and use the following notation. We let p_x denote either $p_x : R^3 \to R$ where $p_x(x, y, z) = x$ or $p_x : R^2 \to R$ where $p_x(x, y) = x$. Similarly, we let p_y denote either $p_y : R^3 \to R$ where $p_y(x, y, z) = y$ or

$p_y : R^2 \to R$ where $p_y(x, y) = y$. Let W be a subset of either R^2 or R^3. Let

$$R_W = \{w \in W \mid p_x(w) \geq 0\},$$

$$L_W = \{w \in W \mid p_x(w) \leq 0\},$$

$$T_W = \{w \in W \mid p_y(w) \geq 0\}, \text{ and}$$

$$B_W = \{w \in W \mid p_y(w) \leq 0\}.$$

Let W be a subset of R^2 and let $w \in W$. We use C_w to denote either the circle in R^3 generated by w when W is rotated about the x-axis or the circle generated by w when W is rotated about the y-axis. We let D_w represent the disc corresponding to C_w. Let J be an arc in the plane such that J and the y-axis intersect in a single point, w, and for each $(x, y) \in J$ either $y \geq 0$ or $y \leq 0$. In the surface, which is generated by rotating J about the x-axis. We let Js denote the following: Js is the copy of J which is obtained when J is rotated (in a rigid manner) and the point w on J is rotated to the point s on C_w. We define Js in the comparable way when the rotation is about the y-axis. By $cl(W)$ we mean the closure of W in the space under consideration. We let θ represent the origin either in R^2 or in R^3.

3. Main Results

Let f be a concave upward function where $f : [0, p] \to [0, q]$ and $f(0) = q$ and $f(p) = 0$. Let $A_1 = G$ where G is the graph of f in $R \times R$. Let A_2 be the reflection of A_1 through the y-axis. Let A_3 be the reflection of A_2 through the x-axis. Let A_4 be the reflection of A_1 through the x-axis. Let $X = A_1 \cup A_2 \cup A_3 \cup A_4$. We refer to the simple closed curve X, defined in this manner, as a concave upward symmetric simple closed curve. Also, let $P = (p,0)$ and $Q = (0,q)$ or $P = (p,0,0)$ and $Q = (0,q,0)$ when X is considered as a subset of R^3. Let (X,d) be a metric space. Let $x \in X$ and A be a closed set in X. We denote $\min\{d(x,y) \mid y \in A\}$ by $d(x,A)$.

Theorem 1. *Let X be a concave upward symmetric simple closed curve (as defined above). Then* $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = \min\{d(-P,A_1), d(-Q,A_1)\}$.

Proof. We can assume without a loss of generality that $\min\{d(-P,A_1), d(-Q,A_1)\} = d(-P,A_1)$. If this is not the case, then rotate X 90^o counterclockwise and the proof will be similar. Consider the set in $X \times X$ given by

$$\begin{aligned} C \;=\; & (\{-P\} \times (A_4 \cup A_1)) \cup ((A_3 \cup A_4) \times \{Q\}) \\ & \cup(\{P\} \times (A_2 \cup A_3)) \cup ((A_1 \cup A_2) \times \{-Q\}). \end{aligned}$$

Clearly, $p_1(C) = p_2(C) = X$ and C is connected. Also, for each $(x,y) \in C$, $d(x,y) \geq d(-P,A_1)$. When $x = -P$ and $y = S \in A_1$ such that $d(-P,S) = d(-P,A_1)$, then $d(x,y) = d(-P,S) = d(-P,A_1)$. So, $\sigma(X) \geq d(-P,A_1)$. Now, we will show that $\sigma_0(X) \leq d(-P,A_1)$. Note that $S \neq P$, since the line through P and Q is neither tangent to the circle centered at $-P$ of radius $2p$ nor does it have positive slope. Let $S_1 = \overline{PS}$, $S_2 = \overline{-PS}$, $S_3 = \overline{-P(-S)}$, and $S_4 = \overline{-SP}$. Let R_1 be the ray in quadrant I emanating from S which is parallel to the y-axis or on the y-axis in the case when $S = Q$. Let R_2 be the ray in quadrant II emanting from $-P$ which is parallel to the y-axis. Let R_3 be the ray in quadrant III emanting from $-S$ which is parallel to the y-axis or on the y-axis in the case where $-S = -Q$. Let R_4 be the ray in quadrant IV emanating from P which is parallel to the y-axis. Note that X must be contained in the portion of the plane bounded by $R_1 \cup S_1 \cup R_4$ and $R_2 \cup S_3 \cup R_3$ which contains the origin.

Let L be the line through the origin which is perpendicular to S_2. The line L is not the y-axis since $S \neq P$. Let $p : X \to L$ be the map from X to L which takes each point of X and projects it perpendicularly onto L. Suppose $f, g : C \to X$ are continuous functions from a connected set C into X where $f(C) = X$. Now, consider $p \circ f$, $p \circ g : C \to L$. Consider the ordering on L given by $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ for each pair of points (x_1, y_1) and (x_2, y_2) on L. Let $A = \{t \in C \mid p \circ f(t) \leq p \circ g(t)\}$ and $B = \{t \in C \mid p \circ f(t) \geq p \circ g(t)\}$. Since, $C = A \cup B$, C is a connected set, and A and B are both closed, it must be that $A \cap B \neq \phi$. So, there exists $t^{'} \in C$ such that $p \circ f(t^{'}) = p \circ g(t^{'})$ so, $f(t^{'})$ and $g(t^{'})$ must both be on a line segment S^* which is perpendicular to L where the endpoints of S^* are either on S_1 and S_3, R_1 and R_2, or R_3 and R_4. Clearly, $d(f(t^{'}), g(t^{'})) \leq \mathrm{diam}(S^*) \leq \mathrm{diam}(S_2) = d(-P, S)$. Consequently, $\sigma_0(X) \leq d(-P, S) = d(-P, A_1)$. Since, $d(-P, A_1) \leq \sigma(X) \leq \sigma_0(X) \leq d(-P, A_1)$, we see that $\sigma(X) = \sigma_0(X) = d(-P, A_1)$. Also, since X is a simple closed curve

$$\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X). \qquad \square$$

Hence, question (d) has been answered in the affirmative for simple closed curves in this class.

Theorem 2. *Let X be a concave upward symmetric simple closed curve. Suppose Y is a continuum such that Y is contained in $B \cup X$ where B is the bounded component of $R^2 - X$. Then $\alpha(Y) \leq \alpha(X)$ where $\alpha = \sigma$, σ_0, σ^*, σ_0^*.*

Proof. Similar to the last part of the proof of Theorem 1. $\square$

Theorem 3. *Let X be a concave upward symmetric simple closed curve. Suppose Y is any plane separating continuum such that X is contained in cl(B), where B is a bounded component of $R^2 - Y$. Then $\sigma(Y) \geq \sigma(X)$.*

Proof Again we assume without loss of generality that $\min\ \{d(-P, A_1), d(-Q, A_1)\} = d(-P, A_1)$. Let $0 < \varepsilon <$

(diam A_1)/4. Let $\theta \in (0, \frac{\pi}{8})$ such that the diameter of the portion of X contined in the wedge of angle 2θ formed by these pairs of rays, $\overrightarrow{\theta e^{i(\pi/2-\theta)}}$ and $\overrightarrow{\theta e^{i(\pi/2+\theta)}}$, $\overrightarrow{\theta e^{i(\pi-\theta)}}$ and $\overrightarrow{\theta e^{i(\pi+\theta)}}$, and $\overrightarrow{\theta e^{i(3\pi/2-\theta)}}$ and $\overrightarrow{\theta e^{i(3\pi/2+\theta)}}$ is less than $\varepsilon/2$.

Let $q : Y \to S^1$ be given by $q(re^{i\gamma}) = e^{i\gamma}$. Since Y is a plane separating continuum and θ is in a bounded component of $R^2 - Y$, q is an essential map. Let U be the unbounded component of $R^2 - X$. We partition cl(U) into six sets as follows:

$$\begin{aligned}
A &= \{re^{i\alpha} \in cl(U) \mid 0 \leq \alpha \leq \pi/2 - \theta \text{ or } 3\pi/2 + \theta \leq \alpha \leq 2\pi\},\\
A' &= \{re^{i\alpha} \in cl(U) \mid \pi - \theta \leq \alpha \leq \pi + \theta\},\\
B &= \{re^{i\alpha} \in cl(U) \mid \pi/2 + \theta \leq \alpha \leq \pi - \theta\},\\
B' &= \{re^{i\alpha} \in cl(U) \mid 3\pi/2 - \theta \leq \alpha \leq 3\pi/2 + \theta\},\\
C &= \{re^{i\alpha} \in cl(U) \mid \pi + \theta \leq \alpha \leq 3\pi/2 - \theta\}, \text{ and}\\
C' &= \{re^{i\alpha} \in cl(U) \mid \pi/2 - \theta \leq \alpha \leq \pi/2 + \theta\}.
\end{aligned}$$

If $x \in A'$ and $y \in A$ then $d(x,y) \geq d(-P, A_1 \cup A_4) - \varepsilon = d(-P, A_1) - \varepsilon$. If $x \in B'$ and $y \in B$ then $d(x,y) \geq d(-Q, A_2) - \varepsilon = d(-Q, A_1) - \varepsilon \geq d(-P, A_1) - \varepsilon$. If $x \in C'$ and $y \in C$ then $d(x,y) \geq d(Q, A_3) - \varepsilon = d(-Q, A_1) - \varepsilon \geq d(-P, A_1) - \varepsilon$. In each of the three cases $d(x,y) \geq d(-P, A_1) - \varepsilon$.

Let $r : S^1 \to S^1$ be a one-to-one continuous function on S^1 such that:

$$\begin{aligned}
r(r^{i0}) &= e^{i0},\\
r(e^{i(\pi/2-\theta)}) &= e^{i\pi/6},\\
r(e^{i(\pi/2+\theta)}) &= e^{i\pi/2},\\
r(e^{i(\pi-\theta)}) &= e^{i5\pi/6},\\
r(e^{i(\pi+\theta)}) &= e^{i7\pi/6},\\
r(e^{i(3\pi/2-\theta)}) &= e^{i3\pi/2}, \text{ and}\\
r(e^{i(3\pi/2+\theta)}) &= e^{i11\pi/6}.
\end{aligned}$$

Consider the function $r \circ q : Y \to S^1$. It is an essential map from Y onto S^1 such that

$$\inf_{s\in S^1} d\left((r\circ q)^{-1}(s),(r\circ q^{-1})(-s)\right)\geq d(-P,A_1)-\varepsilon.$$

Consequently, by Theorem L $\sigma(Y)\geq d(-P,A_1)-\varepsilon$. Since ε was arbitrary, we see that $\sigma(Y)\geq d(-P,A_1)=\sigma(X)$. □

Let Z_x be the surface generated by rotating X about the x-axis in 3-space. Let W_x be the corresponding solid. We have the following results.

Theorem 4. *Let Z_x be as defined above. Then $\sigma(Z_x)=\sigma_0(Z_x)=2q$.*

Proof. Let $C=\{(s,-s)\mid s\in C_Q\}$. The set C is connected and $p_1(C)=p_2(C)$. Hence, $\sigma(Z_x)\geq 2q$. Suppose $D\subset Z_x\times Z_x$, where D is connected and $p_1(D)\subseteq p_2(D)$. Define $p_x^{'}:R^3\to R^3$ by $p_x^{'}(x,y,z)=(x,0,0)$. Consider $(p_x^{'}\circ p_1)(D)$ and $(p_x^{'}\circ p_2)(D)$. They are connected and $(p_x^{'}\circ p_1)(D)\subseteq(p_x^{'}\circ p_2)(D)\subseteq[-p,p]\times\{0\}\times\{0\}$. Hence, there is a $d\in D$ such that $(p_x^{'}\circ p_1)(d)=(p_x^{'}\circ p_2)(d)$. So, $p_1(d)$ and $p_2(d)$ are both on the circle in Z_x generated by a point $(x^{'},y^{'},0)\in X$ where $(p_x^{'}\circ p_1)(d)=(p_x^{'}\circ p_2)(d)=(x^{'},0,0)$. Consequently, $d(p_1(d),p_2(d))\leq 2|y^{'}|\leq 2q$. So, $\sigma_0(Z_x)\leq 2q$. Hence, $\sigma(Z_x)=\sigma_0(Z_x)=2q$. □

Theorem 5. *Let W_x be as defined above. Then $\sigma(W_x)=\sigma_0(W_x)=2q$.*

Proof. The proof is similar to the proof of Theorem 4. □

Let Z_y be the surface generated by rotating X about the y-axis. Let W_y be the corresponding solid.

Theorem 6. *Let Z_y be as defined above. Then $\sigma(Z_y)=\sigma_0(Z_y)=2p$.*

Proof. The proof is similar to the proof of Theorem 4. □

Theorem 7. *Let W_y be as defined above. Then* $\sigma(W_y) = \sigma_0(W_y) = 2p$.

Proof. The proof is similar to the proof of Theorem 4. □

Theorem 8. *Let W_x be as defined above. If $d(-P, A_1) \leq d(-Q, A_1)$ then $\sigma^*(W_x) = \sigma_0^*(W_x) = diam X/2$.*

Proof. We consider two cases.
case 1: $p \geq q$.
Let $D = (\{-P\} \times R_{W_x}) \cup (L_{W_x} \times \{P\}) \cup (A_3 \times \{Q\}) \cup (\{-Q\} \times A_2) \cup (R_{W_x} \times \{-P\}) \cup (\{P\} \times L_{W_x})$. The set D is connected and $p_1(D) = p_2(D) = W_x$. For each $(x, y) \in D$, $d(x, y) \geq d(\theta, P) = p$. So, $\sigma^*(W_x) \geq p$. Since for every $x \in W_x$, $d(\theta, x) \leq p$, $\sigma_0^*(W_x) \leq p$. So, $\sigma^*(W_x) = \sigma_0^*(W_x) = p = \mathrm{diam} X/2$.

case 2: $q > p$.
Let $D = (\{Q\} \times B_{W_x}) \cup (T_{W_x} \times \{-Q\}) \cup \{(s, -s) \mid s \in C_Q\} \cup (B_{W_x} \times \{Q\}) \cup (\{-Q\} \times T_{W_x})$. For each $(x, y) \in D$, $d(x, y) \geq q$ and $p_1(D) = p_2(D) = W_x$. So, $\sigma^*(W_x) \geq q$. Also, for each $x \in W_x$, $d(\theta, x) \leq q$, and $\sigma_0^*(W_x) \leq q$. Hence, $\sigma^*(W_x) = \sigma_0^*(W_x) = q = \mathrm{diam} X/2$. In both cases we have shown that $\sigma^*(W_x) = \sigma_0^*(W_x) = \mathrm{diam} X/2$. □

Theorem 9. *Let W_y be as defined above. If $d(-P, A_1) \leq d(-Q, A_1)$ then $\sigma^*(W_y) = \sigma_0^*(W_y) = \min\{\frac{diam X}{2}, d(-P, A_1)\}$.*

Proof. We consider two cases.
case 1: $q \leq p$.
Let $D = (\{-P\} \times R_{W_y}) \cup (L_{W_y} \times \{P\}) \cup \{(s, -s) \mid s \in C_P\} \cup (R_{W_y} \times \{-P\}) \cup (\{P\} \times L_{W_y})$. The set D is connected $p_1(D) = p_2(D) = W_y$ and for all $(x, y) \in D$, $d(x, y) \geq p$. So, $\sigma^*(W_y) \geq p$. Also, $\sigma_0^*(W_y) \leq p$ since $d(\theta, x) \leq p$ for all $x \in W_y$. So, $\sigma^*(W_y) = \sigma_0^*(W_y) = p = \mathrm{diam} X/2$.

case 2: $q > p$.
Let $D = (\{Q\} \times B_{W_y}) \cup (\{-Q\} \times T_{W_x}) \cup (A_2 \times \{P\}) \cup (\{-P\} \times A_1) \cup (B_{W_y} \times \{Q\}) \cup (T_{W_y} \times \{-Q\})$. The set D is connected, $p_1(D) =$

$p_2(D) = W_y$. For all $(x, y) \in D, d(x, y) \geq \min\{q, d(-P, A_1)\}$. If $q \leq d(-P, A_1)$ then by an argument similar to the one in case 1 we can see that $\sigma^*(W_y) = \sigma_0^*(W_y) = \text{diam}X/2$. Now consider the case where $d(-P, A_1) < q$. We know that $\sigma^*(W_y) \geq d(-P, A_1)$. We need to show that $\sigma_0^*(W_y) \leq d(-P, A_1)$. Let $f, g : C \to W_y$ be continuous functions from C into W_y, where C is connected, $f(C) = W_y$, and $d(f(c), g(c)) = \sigma_0^*(W_y)$ for all $c \in C$. Such functions exist by (*). Let $p_y^{'} : R^3 \to \{0\} \times R \times \{0\}$ be given by $\mathrm{p}_y^{'}(x, y, z) = (0, y, 0)$. Let $r : \{0\} \times [-q, q] \times \{0\} \to A_1 \cup A_4$ be the function given by$r((0, y, 0)) = (x, y, 0)$ where $(x, y, 0)$ is the corresponding element of $A_1 \cup A_4$. Clearly, this is a continuous, one-to-one function. Let $l : \{0\} \times [-q, q] \times \{0\} \to A_2 \cup A_3$ be the function given by $l((0, y, 0)) = (x, y, 0)$ where $(x, y, 0)$ is the corresponding element of $A_2 \cup A_3$. Again this is a continuous one-to-one function. So,

$$r \circ p_y^{'} \circ f : C \to A_1 \cup A_4 \text{ and } l \circ p_y^{'} \circ g : C \to A_2 \cup A_3$$

are continuous functions and $r \circ p_y^{'} \circ f(C) = A_1 \cup A_4$. Consider $m : C \to R$ given by $m(c) = m(r \circ p_y^{'} \circ f(c), l \circ p_y^{'} \circ g(c)) =$ slope between the points $r \circ p_y^{'} \circ f(c)$ and $l \circ p_y^{'} \circ g(c)$. This is a well defined continuous function as long as the slope between the points is defined. The line segment between $r \circ p_y^{'} \circ f(c)$ and $l \circ p_y^{'} \circ g(c)$ is never degenerate because for each $c \in C$ neither $f(c) = g(c) = Q$ nor $f(c) = g(c) = -Q$ can occur since $\sigma_0^*(W_y) \neq 0$. Also, when $f(c) = Q$, $g(c) \neq -Q$ since clearly $\sigma_0^*(W_y) \neq 2q = \text{diam}(W_y)$. Similary, when $f(c) = -Q$, $g(c) \neq Q$. Since f is onto, there is a $c^{'}$ in C such that $f(c^{'}) = Q$ and a $c^{''} \in C$ such that $f(c^{''}) = -Q$. Observe that when $f(c^{'}) = Q$, $m(c^{'}) = m(r \circ p_y^{'} \circ f(c^{'}), l \circ p_y^{'} \circ g(c^{'})) > m(-P, S)$ (where $S \in A_2$ such that $d(-P, S) = d(-P, A_2)$) since the line of slope $m(-P, S)$ through Q does not intersect W_y at any other point and $m(x, Q) > m(-P, S)$ for all $x \in A_2 \cup A_3 - \{Q\}$. Also, observe that when $f(c^{''}) = -Q$,

$$m(c^{''}) = m(r \circ p_y^{'} \circ f(c^{''}), l \circ p_y^{'} \circ g(c^{''})) < 0.$$

Since m is continuous and C is connected, $[m(c''),\ m(c')] \subset m(C)$. So, there is a c^* in C such that $m(c^*) = m(-P, S)$. Let $r \circ p_y' \circ f(c^*) = (x_1, y_1, 0) = p_1$ and $l \circ p_y' \circ g(c^*) = (x_2, y_2, 0) = p_2$. In the proof of Theorem 1 we showed that $d(p_1, p_2) \leq d(-P, S)$. Clearly, $f(c^*) \in D_{p_1}$, $g(c^*) \in D_{p_2}$, and $d(f(c^*), g(c^*)) \leq d(-P, S)$. Hence, $\sigma_0^*(W_y) \leq d(-P, S)$. So, for case 2, we can conclude that $\sigma^*(W_y) = \sigma_0^*(W_y) = \min\{\text{diam}X/2, d(-P, A_1)\}$. □

Theorem 10. *Let W_x be as defined above. If $d(-Q, A_1) < d(-P, A_1)$ then $\sigma^*(W_x) = \sigma_0^*(W_x) = \min\{diamX/2, d(-Q, A_1)\}$.*

Proof. If we rotate X 90° clockwise and then rotate it about the y-axis, we get a space W which is isometric to W_x. We can apply to W an argument similar to the one used in the proof of Theorem 9. The conclusion, which is comparable to the conclusion in Theorem 9, is

$$\sigma^*(W_x) = \sigma_0^*(W_x) = \min\{\text{diam}X/2, d(-Q, A_1)\}. \qquad \square$$

Theorem 11. *Let W_y be as defined above. If $d(Q, A_1) < d(-P, A_1)$ then $\sigma^*(W_y) = \sigma_0^*(W_y) = diamX/2$.*

Proof. If we rotate X 90° clockwise and then rotate it about the x-axis, we get a space W which is isometric to W_y. We can apply to W an argument which is similar to the one used in the proof of Theorem 8. The conclusion, which is comparable to conclusion in Theorem 8, is $\sigma^*(W_y) = \sigma_0^*(W_y) = \text{diam}X/2$. □

Theorem 12. *Let Z_y be as defined above. Then $\sigma^*(Z_y) = \sigma_0^*(Z_y) = d(-P, A_1)$.*

Proof. Let $D = \bigcup_{s \in C_P} (\{s\} \times (A_1 \cup A_4)_{-s}) \cup (\bigcup_{s \in C_P} (A_1 \cup A_4)_{-s} \times \{s\})$ where $(A_1 \cup A_4)_{-s}$ is as defined in section 2 and $-s$ is the element of C_P which is antipodal to s. For each $(x, y) \in D$, $d(x, y) \geq d(-P, A_1 \cup A_4) = d(-P, A_1) = d(-P, S)$. Clearly, D is connected and $p_1(D) = p_2(D) = Z_y$. Hence $\sigma^*(Z_y) \geq d(-P, A_1)$. Now we need to show that $\sigma_0^*(Z_y) \leq d(-P, A_1)$.

Suppose $f, g : C \to Z_y$ are continuous functions from a connected set C into Z_y such that $f(C) = Z_y$ and $d(f(c), g(c)) = \sigma_0^*(Z_y)$ for each $c \in C$. We know that such functions exist by (*).

It cannot be the case that there is a $c \in C$ such that $f(c) = Q$ and $g(c) = -Q$. If $q > p$ this is clear since $\operatorname{diam}(Z_y) = 2q$ and Q and $-Q$ are the only two points in Z_y such that $d(x, y) = 2q$. Suppose $q \leq p$. It is clear from the construction of X and Z_y that we can choose $\varepsilon > 0$ small enough, such that if $x \in B(Q, \varepsilon)$ and $y \in B(-Q, \varepsilon)$ then $d(x, y) < 2q$ when either $x \neq Q$ or $y \neq -Q$. Let $U = f^{-1}(B(Q, \varepsilon)) \cap g^{-1}(B(-Q, \varepsilon))$ where ε is as described above. So for each $c' \in U$, $f(c') = Q$ and $g(c') = -Q$. Hence $U = f^{-1}(Q) \cap g^{-1}(-Q)$ is both open and closed. This contradicts the connectedness of C. So there is no $c \in C$ such that $f(c) = Q$ and $g(c) = -Q$. Similarly, there is no $c \in C$ such that $f(c) = -Q$ and $g(c) = Q$. Now, consider $l \circ p_y' \circ g, r \circ p_y' \circ f : C \to X$. Because of the above observation, $m : C \to R$ given by $m(c) = m(l \circ p_y' \circ g(c), r \circ p_y' \circ f(c))$ is well defined and continuous, since the line segment $\overline{l \circ p_y' \circ g(c)\ r \circ p_y' \circ f(c)}$ is never vertical nor degenerate. Note that for $c \in C$ where $f(c) = Q$, $p_y' \circ f(c) = (0, q, 0)$, $r \circ p_y' \circ f(c) = Q$ and $m(c) > m(-P, S)$, since the line through Q in R^2 of slope $m(-P, S)$ does not intersect X and X is 'below' this line. Also, for $c \in C$ where $f(c) = -Q$, $m(c) < 0$. Since $f(C) = Z_y$ there is a $c' \in C$ such that $m(c') > m(-P, S)$ and there is a $c'' \in C$ such that $m(c'') < 0$. Since m is continuous and C is connected, $[m(c''), m(c')] \subset m[C]$. Hence, there is a $c''' \in C$ such that $m(c''') = m(-P, S)$. As in the proof of Theorem 9, we see that $d(f(c'''), g(c''')) \leq d(-P, S)$. So, $\sigma_0^*(Z_y) \leq d(-P, S)$ and $\sigma^*(Z_y) = \sigma_0^*(Z_y) = d(-P, A_1)$. □

Theorem 13. *Let Z_x be as defined above. Then $\sigma^*(Z_x) = \sigma_0^*(Z_x) = d(-Q, A_1)$.*

Proof. Let $D = \bigcup_{s \in C_Q} (\{s\} \times (A_1 \cup A_2)_{-s}) \cup (\bigcup_{s \in C_Q} (A_1 \cup A_2)_{-s} \times \{s\})$. The set D is connected, $p_1(D) = p_2(D)$, and for each

$(x, y) \in D$, $d(x, y) \geq d(-Q, A_1 \cup A_2) = d(-Q, A_1)$. So, $\sigma^*(Z_x) \geq d(-Q, A_1)$. If we rotate X in the plane by $90°$ clockwise and then rotate it about the y-axis, we get a space W which is isometric to Z_x. We can apply to W the argument used in Theorem 12 to show that $\sigma_0^*(Z_x) = \sigma_0^*(W) \leq d(-Q, A_1)$. Hence, $\sigma^*(Z_x) = \sigma_0^*(Z_x) = d(-Q, A_1)$. □

For the spaces Z_x, W_x, Z_y and W_y we observe that questions (a), (b), and (c) are all answered in the affirmative. For Z_x , we showed that

$$\sigma(Z_x) = \sigma_0(Z_x) = 2q$$

and

$$\sigma^*(Z_x) = \sigma_0^*(Z_x) = d(-Q, A_1).$$

So,

$$\frac{\sigma_0^*(Z_x)}{\sigma_0(Z_x)} = \frac{\sigma^*(Z_x)}{\sigma(Z_x)} = \frac{d(-Q, A_1)}{2q}$$

and

$$\frac{1}{2} < \frac{d(-Q, A_1)}{2q} < 1,$$

since $q < d(-Q, A_1) < 2q$. For Z_y we showed that $\sigma(Z_y) = \sigma_0(Z_y) = 2p$ and $\sigma^*(Z_y) = \sigma_0^*(Z_y) = d(-P, A_1)$. So,

$$\frac{\sigma_0^*(Z_y)}{\sigma_0(Z_y)} = \frac{\sigma^*(Z_y)}{\sigma(Z_y)} = \frac{d(-P, A_1)}{2p}$$

and

$$\frac{1}{2} < \frac{d(-P, A_1)}{2p} < 1,$$

since $p < d(-P, A_1) < 2p$. We showed that $\sigma(W_x) = \sigma_0(W_x) = 2q$ and that $\sigma(W_y) = \sigma_0(W_y) = 2p$. When $d(-P, A_1) \leq d(-Q, A_1)$ we showed that

$$\sigma^*(W_x) = \sigma_0^*(W_x) = \frac{\mathrm{diam} X}{2}$$

and that

$$\sigma^*(W_y) = \sigma_0^*(W_y) = \min\{\frac{\mathrm{diam} X}{2}, d(-P, A_1)\}.$$

So, in this case

$$\frac{\sigma_0^*(W_x)}{\sigma_0(W_x)} = \frac{\sigma^*(W_x)}{\sigma(W_x)} = \frac{(\mathrm{diam}X)/2}{2q} = \frac{\mathrm{diam}X}{4q} = \begin{cases} \frac{1}{2} & , q \geq p \\ \frac{p}{2q} > \frac{1}{2} & , p > q \end{cases} .$$

Also, in this case

$$\frac{\sigma_0^*(W_y)}{\sigma_0(W_y)} = \frac{\sigma^*(W_y)}{\sigma(W_y)} = \frac{(\min\{\frac{\mathrm{diam}X}{2}, d(-P, A_1)\})}{2p}.$$

If $(\mathrm{diam}X)/2 \leq d(-P, A_1)$ then

$$\frac{\sigma_0^*(W_y)}{\sigma_0(W_y)} = \frac{\sigma^*(W_y)}{\sigma(W_y)} = \frac{\mathrm{diam}X}{4p} = \begin{cases} \frac{1}{2} & , p \geq q \\ \frac{q}{2p} > \frac{1}{2} & , q > p \end{cases} .$$

If $d(-P, A_1) < (\mathrm{diam}X)/2$ then

$$\frac{\sigma_0^*(W_y)}{\sigma_0(W_y)} = \frac{\sigma^*(W_y)}{\sigma(W_y)} = \frac{d(-P, A_1)}{2p}$$

and

$$\frac{1}{2} < \frac{d(-P, A_1)}{2p} < 1,$$

since $p < d(-P, A_1) < 2p$. Similarly when $d(-Q, A_1) < d(-P, A_1)$ we can see that

$$\frac{\sigma_0^*(W_y)}{\sigma_0(W_y)} = \frac{\sigma^*(W_y)}{\sigma(W_y)} \geq \frac{1}{2}$$

and that

$$\frac{\sigma_0^*(W_x)}{\sigma_0(W_x)} = \frac{\sigma^*(W_y)}{\sigma(W_y)} \geq \frac{1}{2}.$$

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