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ORBIT SPACES OF SUBGROUP COMPLEXES, MORSE THEORY, AND A NEW PROOF OF A CONJECTURE OF WEBB

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In [3] K.S. Brown introduced the complex $\mathcal{S}_p(G)$ of non-trivial p-subgroups in the finite group G. This is a simplicial complex whose simplices are chains of non-trivial p-subgroups of G ordered by inclusion. Since G acts on these chains by conjugation, $\mathcal{S}_p(G)$ is a G-complex.

In his fundamental paper [6], D. Quillen related this complex to other complexes, e.g., the poset of elementary abelian p-groups in G. He provided evidence that it should be considered a generalization of the building associated to a finite Chevalley group over a field of characteristic p.

In [10] P. J. Webb conjectured that the orbit space $\tilde{\mathcal{S}}_p(G) := \mathcal{S}_p(G)/G$ is contractible. He was able to prove that it is acyclic "mod p". J. Thévenaz was able to settle some cases of the conjecture in [8], and recently P. Symonds showed in [7] that Webb's Conjecture holds in general.

One might ask if there is something particular about finite groups behind Webb's conjecture, and in this note it will be argued that this is not the case. We will see that analogues of Webb's Conjecture should hold for the orbit space of a subgroup complex whenever an analogue of Sylow's theorem is true for the subgroups used to build the complex. In particular we will find some orbit spaces of subgroup complexes associated to linear algebraic groups to be contractible.

Let us make this more precise. Fix a linear algebraic group $\mathbf{G}$ and consider the following simplicial complexes. Let $\mathcal{T}(\mathbf{G})$ be the simplicial complex whose vertices are the non-trivial tori in $\mathbf{G}$ and where a set of tori forms a simplex if they form a chain with respect to inclusion. Let $\mathcal{U}(\mathbf{G})$ be the simplicial complex obtained the same way from the set of non-trivial connected unipotent subgroups of $\mathbf{G}$, and let $\mathcal{P}(\mathbf{G})$ be the simplicial complex based on the ordered set of parabolic strict subgroups of $\mathbf{G}$.

The group $\mathbf{G}$ acts on these complexes by conjugation in an obvious way. Let $\tilde{\mathcal{T}}(\mathbf{G})$, $\tilde{\mathcal{U}}(\mathbf{G})$, and $\tilde{\mathcal{P}}(\mathbf{G})$ denote the orbit spaces of $\mathcal{T}(\mathbf{G}), \mathcal{U}(\mathbf{G})$, and $\mathcal{P}(\mathbf{G})$ respectively. These are CW-complexes whose cells look like simplices. However, a cell in these complexes is not determined by its set of vertices.

We will prove

Theorem 1. The complexes $\tilde{\mathcal{S}}_p(G)$, $\tilde{\mathcal{T}}(\mathbf{G})$, and $\tilde{\mathcal{U}}(\mathbf{G})$ are contractible. The complex $\tilde{\mathcal{P}}(\mathbf{G})$ is empty if $\mathbf{G}$ is solvable otherwise it is contractible.

Moreover, our argument will show that these complexes are contractible for very similar reasons. The argument is inspired by Morse theory for piecewise Euclidean complexes in the sense of [1].

In Section 1 we will give a quick treatment of Webb's Conjecture that will provide the background for the main technical result of Section 2 which is used to prove Theorem 1 in Section 3.

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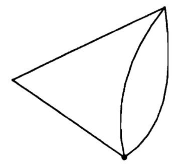
1. Webb's Conjecture

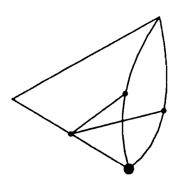
First we recall some Morse theoretic facts from [1, Section 2] in a simplified form that fits to our situation.



A ΔCW-complex is a piecewise Euclidean CW-complex where all cells are equilateral simplices. That is a CW-complex whose closed cells are given the structure of equilateral Euclidean simplices such that characteristic functions are injective and restrict to characteristic functions of faces, possibly precomposed by an isometry of the face. This is nothing but a simplicial complex except that one allows for different simplices to intersect in more than one face. In [4] A. Haefliger introduced the similar concept of a simplicial cell complex. Note, however, that we do not allow for identifications of faces of one simplicial cell by requiring that the characteristic functions be injective. Hence, for example, a rose or the dunce hat are *not* ΔCW-complexes.

Remark 2. The face relation induces a partial order on the set of simplices of a ΔCW-complex X. We formally add a minimal element—representing the empty simplex. Then for each element of the poset, the subposet of its faces is isomorphic to the poset of subsets of a finite set. Note that any partially ordered set of this kind gives rise to a ΔCW-complex being its "geometric realization".

The notion of the star of a vertex carries over to ΔCW-complexes in an obvious way. We denote the open star of the vertex x by $\mathrm{St}_X(x)$. The notion of the link of a vertex, however, has to be slightly modified. We want to think of points in the link of x as "local directions starting at x".



So we construct the link $\mathrm{Lk}_X(x)$ of a vertex inductively on the skeleta as follows: Take the midpoints of edges starting at x to be the vertices of $\mathrm{Lk}_X(x)$. For each m-simplex in $\mathrm{St}_X(x)$ of higher dimension, the vertices of $\mathrm{Lk}_X(x)$ it contains span a $(m-1)$-simplex which we glue in along its boundary. The usual concept of the link of x as the subcomplex of all simplices in X that are faces of simplices in $\mathrm{St}_X(x)$ but do not contain x leads to the notion of the embedded link $\mathrm{eLk}_X(x)$. Note that there is a natural map $\mathrm{Lk}_X(x) \to \mathrm{eLk}_X(x)$. It sends a vertex in $\mathrm{Lk}_X(x)$, i.e., a midpoint of an edge starting at x to the opposite endpoint of this edge. To simplices of higher dimensions, the map extends linearly.

Observation 3. *X is obtained from $X \setminus \mathrm{St}_X(x)$ by first gluing in the mapping cone of $\mathrm{Lk}_X(x) \to \mathrm{eLk}_X(x)$ and then coning off $\mathrm{Lk}_X(x)$.*

A height function on a ΔCW-complex X is a function that assigns integers (heights) to the vertices of X such that vertices which are joined by an edge have different heights. By linear interpolation, a height function induces a Morse function in the sense of [1, Definition 2.2]. For a given height $t \in \mathbb{Z}$, let $X^{\geq t}$ be the subcomplex of X formed by all simplices above t. Here we say that a simplex lies above t if all its vertices are of height $\geq t$. We denote by $X^{>t}$ the subcomplex formed by all simplices strictly above t. That means that all vertices of the simplex are of height $> t$.

Given a height function on the ΔCW-complex X and a vertex $x \in X$ of height t, we define the ascending star $\mathrm{St}^{\uparrow}_X(x)$ to be the open star of x in $X^{\geq x}$. We define the ascending link $\mathrm{Lk}^{\uparrow}_X(x)$ and the embedded ascending link $\mathrm{eLk}^{\uparrow}_X(x)$ analogously.

Of course, there are analogously defined sublevel complexes $X^{\leq t}$ and $X^{<t}$, descending stars, descending links, and descending embedded links.

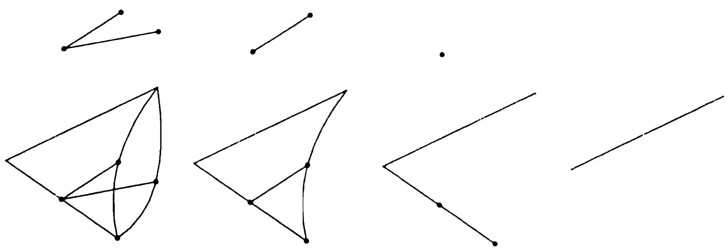
For our purposes, the following observation is an appropriate version of [1, Lemma 2.5].

Morse Lemma 4. *$X^{\geq t}$ is homotopy equivalent to $X^{>t} = X^{\geq t+1}$ with the ascending links of the vertices of height t coned* *off. In particular, if the ascending links are contractible then* $X^{\geq t}$ *and* $X^{>t} = X^{\geq t+1}$ *are homotopy equivalent.*

Proof: Since there are no "horizontal" edges joining vertices of the same height, ascending stars of vertices of height t are disjoint and we can consider them separately. Applying Observation 3 to each of the vertices, the lemma follows.□

Concerning finite ΔCW-complexes with height function, one can prove a slightly stronger version that deals with simple homotopy types.

Observation 5. *Suppose* $\mathrm{Lk}_X(x)$ *can be collapsed to a single point. So the cone over the link can be collapsed to a single edge joining* x *to* $X \backslash \mathrm{St}_X(x)$*. This edge can be collapsed since* x *is a free face.*



Therefore X *can be collapsed to* $X \backslash \mathrm{St}_X(x)$. Hence X and $X \backslash \mathrm{St}_X(x)$ *have the same simple homotopy type.*

Using the same argument as above we infer from this

Lemma 6. *Suppose that* X *has only finitely many vertices of height* t *and that all ascending links of vertices of height* t *are collapsible. Then* $X^{\geq t}$ *can be collapsed to* $X^{\geq t+1}$.

We now present a "stand alone proof" of Webb's Conjecture. First we pass to a different complex introduced in [5] that is also used in P. Symond's proof of Webb's Conjecture. This complex $\mathcal{R}_p(G)$ is the simplicial complex where a m-simplex is a chain $P_0 \lhd P_1 \lhd \cdots \lhd P_m$ of p-subgroups in G such that all P_i are normal in P_m. Note that $\mathcal{R}_p(G)$ is a subcomplex of $\mathcal{S}_p(G)$

containing all of its vertices. In [9] the inclusion $\mathcal{R}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is proven to be a G-equivariant homotopy equivalence.

Let us denote the orbit space by $\tilde{\mathcal{R}}_p(G) := \mathcal{R}_p(G)/G$ which is a ΔCW-complex. Vertices of this complex are conjugacy classes of p-subgroups in G. Hence the map $\tilde{P} \mapsto |P|$ assigning to each p-group the number of elements induces a height function on $\tilde{\mathcal{R}}_p(G)$.

The following theorem implies Webb's Conjecture.

Theorem 7. *The complex $\tilde{\mathcal{R}}_p(G)$ can be collapsed to a point.*

Proof: We argue indirectly by "descent infini". So fix a prime number p and suppose G was a minimal counter example. So G is a finite group such that $\tilde{\mathcal{R}}_p(H)$ is collapsible for every group H with fewer elements but not so for $H = G$.

The ascending link of a vertex $P \in \mathcal{R}_p(G)$ of non-maximal height is the simplicial complex of chains $P_0 \lhd \cdots \lhd P_m$ such that $P \lhd P_i \trianglelefteq P_m$ for all i. Since all these p-subgroups are contained in the normaliser $N_G(P)$ of P, the ascending link is isomorphic to the simplicial complex $\mathcal{R}_p(N_G(P)/P)$.

Modding out the action of the normaliser which is the stabiliser of the vertex P, we see that the ascending link of $\tilde{P}$ in $\tilde{\mathcal{R}}_p(G)$ is $\tilde{\mathcal{R}}_p(N_G(P)/P)$. This complex can be collapsed to a point since otherwise $N_G(P)/P$ would be a smaller counterexample.

There is a unique vertex of maximal height corresponding to the Sylow subgroups in G that form a conjugacy class. Hence applying lemma 6 along our way down to minimal p-subgroups we see that $\tilde{\mathcal{R}}_p(G)$ can be collapsed to a point. So G is not a counter example. $\square$

Remark 8. Note that you cannot run the argument of this proof using embedded ascending links instead of ascending links in $\tilde{\mathcal{R}}_G(p)$. Inside $\mathcal{R}_G(p)$, however, both notions coincide. Therefore it might be of interest to locate the difference.

The embedded ascending link of the vertex $P \in \mathcal{R}_p(G)$ is isomorphic to the complex $\mathcal{R}_p(N_G(P)/P)$, and modding out the action of the normaliser $N_G(P)$ yields the complex $\tilde{\mathcal{R}}_p(N_G(P)/P)$. From this, however, we may not conclude that this is the embedded ascending link of $\tilde{P}$ in $\tilde{\mathcal{R}}_p(G)$.

The point is that the action of G may identify endpoints of edges starting at P even when it does not identify the midpoints of these edges. This, however, is not true for the action of $N_G(P)$. Hence both actions on the ascending link of P in $\mathcal{R}_G(p)$ have different quotients. Modding out the action of G yields the embedded ascending link of $\tilde{P}$ in $\tilde{\mathcal{R}}_G(p)$ whereas modding out the action of $N_G(P)$ yields its ascending link in $\tilde{\mathcal{R}}_G(P)$. Hence, to make the induction work, we have to consider ascending links instead of embedded ascending links.

2. Groups acting on directed graphs

In the preceding argument we were given a height function on the complex $\tilde{\mathcal{R}}_p(G)$ for free. The aim of this section is to extend the method to contexts where such a height function is less obvious. This is only in part motivated by the applications to linear algebraic groups we aim at since, in that context, one could use the dimension of subgroups as a height function. Our approach, however, leads to a unified treatment of the examples.

The idea we present here is to use directly the structure of the poset of subgroups under consideration to construct a height function. Since we do not restrict ourselves to finite posets, we cannot guarantee that this height will take integer values. Instead, we have to allow for ordinal numbers. This, however, presents only minor technical difficulties.

A generalized height function on a ΔCW-complex X is a map that assigns an ordinal number to each vertex in X such that vertices joined by an edge have different heights. Everything said for height functions also applies to generalized height functions, but there is a small difference due to the existence of limit ordinals. Hence, from the contractibility of descending links, we may infer that $X^{\leq\eta}$ is homotopy equivalent to

$X^{<\eta}$, but we may not be able to describe the latter complex as $X^{\leq \eta-1}$ since an ordinal $\eta - 1$ might not exist. Nevertheless, we have

Lemma 9. *Let X be a ΔCW-complex with a generalized height function such that all descending links are contractible. Then the inclusion $X^{\leq 0} \hookrightarrow X$ is a homotopy equivalence.*

Proof: Since $X = X^{\leq \omega}$ for some ordinal ω we only need to prove that $X^{\leq 0} \hookrightarrow X^{\leq \eta}$ is a homotopy equivalence for all η. This is done by transfinite induction. So assume $X^{\leq 0} \hookrightarrow X^{\leq \omega}$ is a homotopy equivalence for every $\omega < \eta$. The complex $X^{\leq \eta}$ is homotopy equivalent to $X^{<\eta}$ by contractibility of descending links. $X^{<\eta}$, in turn, is the ascending union of the complexes $X^{\leq \omega}$ for $\omega < \eta$ with all inclusion maps being homotopy equivalences. Hence

$$X^{\leq 0} \hookrightarrow X^{<\eta} \hookrightarrow X^{\leq \eta}$$

is a homotopy equivalence being a weak homotopy equivalence of CW-complexes. □

A generalized height function on X induces a total order on the vertices of each simplex $\sigma \in X$. A <u>top face</u> τ of σ is a face of σ above all vertices of σ not in τ. <u>Bottom faces</u> are defined mutatis mutandis. In particular each simplex has a top vertex and a bottom vertex. Note that for every bottom face of σ there a <u>complementary</u> top face of σ containing the other vertices of σ.

We now extend the notion of descending links to simplices. We proceed inductively. For 0-simplices the notion is already defined. So let σ be a $(m+1)$-simplex with top vertex x. We define its <u>descending link</u> to be

$$\mathrm{Lk}^{\downarrow}_{X}(\sigma) := \mathrm{Lk}^{\downarrow}_{\mathrm{Lk}^{\downarrow}_{X}(x)}(\sigma')$$

where σ' denotes the simplex in $\mathrm{Lk}^{\downarrow}_{X}(x)$ represented by σ. The generalized height function on $\mathrm{Lk}^{\downarrow}_{X}(x)$ is induced by the generalized height function on X via the map

$$\mathrm{Lk}^{\downarrow}_{X}(x) \subseteq \mathrm{Lk}_{X}(x) \to \mathrm{eLk}_{X}(x) \subseteq X.$$

Observation 10. *Let the group G act on a ΔCW-complex X with G-invariant generalized height function, let σ be a simplex in X, and let $\tilde{\sigma}$ be its image in $\tilde{X} := X/G$. The stabiliser G_σ acts on the descending link $\mathrm{Lk}_X^{\downarrow}(\sigma)$ and there is a canonical isomorphism*

$$\mathrm{Lk}_X^{\downarrow}(\sigma)/G_\sigma = \mathrm{Lk}_{\tilde{X}}^{\downarrow}(\tilde{\sigma})$$

We now describe a source of ΔCW-complexes with generalized height functions. Let Γ be a directed graph. We call a vertex $x \in \Gamma$ minimal if there is no edge pointing away from x. A directed path in Γ is a sequence of edges $e_1, e_2, \dots$ such that the terminal vertex of e_i is the initial vertex of e_{i+1}. For two vertices $v, w \in \Gamma$, we write $v \leftarrow\!\!-\!- w$ if there is an edge pointing from w to v, $v \prec w$ if v is reachable from w, i.e., there is a directed path from w to v, and $v \preceq w$ if $v \prec w$ or $v = w$. Γ is well founded if it contains no directed paths of infinite length. This implies in particular that Γ contains no directed circles whence $\preceq$ is a partial order relation on Γ. Even more is true: Every non empty subset of Γ contains minimal elements—otherwise there was an infinite descending sequence corresponding to an infinite directed path. Hence every chain (totally ordered subset) in Γ is well ordered whence it determines an ordinal number. The maximum number that occurs this way is the depth of Γ denoted by $\mathrm{dp}(\Gamma)$.

Let $X(\Gamma)$ be the flag complex over Γ. This is the simplicial complex with vertex set Γ whose simplices are those finite subsets of Γ whose vertices are pairwise joined by an edge in Γ. For every vertex $x \in \Gamma$ let $\Gamma^{\prec x}$ be the subgraph of Γ spanned by all vertices reachable from x and $\mathrm{ht}(x) := \mathrm{dp}(\Gamma^{\prec x})$. This is a generalized height function on Γ which maps minimal vertices to 0. Moreover, let $\Gamma_{\downarrow}^{x}$ be the subgraph spanned by the endpoints of edges starting at x.

For every simplex $\sigma \in X(\Gamma)$, let $\Gamma_\downarrow^\sigma := \cap_{x\in\sigma}\Gamma_\downarrow^x$ be the descending subgraph. Note that there is a canonical isomorphism

$$X(\Gamma_\downarrow^\sigma) = \mathrm{Lk}_{X(\Gamma)}^\downarrow(\sigma).$$

Let G be a group. A directed G-graph is a directed graph Γ on which G acts by automorphisms of directed graphs. If, additionally, Γ is well founded, then $X(\Gamma)$ is a simplicial G-complex with a G-invariant generalized height function. Hence the corresponding orbit space $\tilde{X}(\Gamma) := X(\Gamma)/G$ is a ΔCW-complex with generalized height function.

Remark 11. In the applications of the next section, the relation $\leftarrow\!\!--$ is transitive. In this case, for each simplex $\sigma \in X(\Gamma)$ with bottom vertex x, the equality $\Gamma_\downarrow^\sigma = \Gamma_\downarrow^x$ holds.

We call a simplex minimal if it contains a minimal vertex. Our main result is

Theorem 12. *Let Γ be a well founded directed G-graph. Then the following are equivalent:*

1. *For every non-minimal $\sigma \in X(\Gamma)$ the stabiliser G_σ acts transitively on the set of minimal elements of $\Gamma_\downarrow^\sigma$.*
2. *For every non-minimal $\sigma \in X(\Gamma)$ the orbit space* $\mathrm{Lk}_{X(\Gamma)}^\downarrow(\sigma)/G_\sigma$ of the descending link is connected.
3. *For every non-minimal $\sigma \in X(\Gamma)$ the orbit space $Lk_{X(\Gamma)}^\downarrow(\sigma)/G_\sigma$ of the descending link is contractible.*

Corollary 13. *If these conditions hold and G acts transitively on the set of minimal vertices of Γ then $\tilde{X}(\Gamma)$ is contractible. This follows immediately by adding a new vertex and edges pointing from this vertex to the old vertices.*

Proof of Theorem 12. Let $\tilde{\sigma}$ denote the image of σ in $\tilde{X}(\Gamma)$. We already observed

$$\mathrm{Lk}_{X(\Gamma)}^\downarrow(\sigma)/G_\sigma = \mathrm{Lk}_{\tilde{X}(\Gamma)}^\downarrow(\tilde{\sigma})$$

and

$$\mathrm{Lk}_{X(\Gamma)}^\downarrow(\sigma) = X(\Gamma_\downarrow^\sigma)$$

whence $\mathrm{Lk}^{\downarrow}_{\tilde{X}(\Gamma)}(\tilde{\sigma}) = X(\Gamma^{\sigma}_{\downarrow})/G_{\sigma}$ is a ΔCW-complex with generalized height function.

For any simplex $\tau \in X(\Gamma)$, consider the following conditions:

a. G_{τ} acts transitively on the set of minimal elements of $\Gamma^{\tau}_{\downarrow}$.

b. $\mathrm{Lk}^{\downarrow}_{X(\Gamma)}(\tau)/G_{\tau}$ is connected.

c. $\mathrm{Lk}^{\downarrow}_{X(\Gamma)}(\tau)/G_{\tau}$ is contractible.

Suppose conditions 1, 2, and 3 do not hold simultaneously. Then there is a non-minimal simplex σ such that not all of a, b, and c hold for $\tau = \sigma$. We fix such a simplex σ with minimal $\mathrm{dp}(\Gamma^{\sigma}_{\downarrow})$. From the minimality, it follows that c holds for every simplex $\tau \in X(\Gamma)$ with $\mathrm{dp}(\Gamma^{\tau}_{\downarrow}) < \mathrm{dp}(\Gamma^{\sigma}_{\downarrow})$. This holds in particular for each $\tau \neq \sigma$ that has σ as a top face. Hence all descending links in $X(\Gamma^{\sigma}_{\downarrow})/G_{\sigma}$ are contractible. Lemma 9 therefore implies that $X(\Gamma^{\sigma}_{\downarrow})/G_{\sigma}$ is homotopy equivalent to $(X(\Gamma^{\sigma}_{\downarrow})/G_{\sigma})^{\leq 0}$, which is a discrete set of points. Hence, for $\tau = \sigma$, a, b, and c fail whence 1, 2, and 3 fail simultaneously. $\square$

Remark 14. The equivalence 1$\Longleftrightarrow$3 of Theorem 12 essentially says that Webb's Conjecture for a given finite group is nothing but Sylow's Theorem applied to the group and all its subfactors whereas Sylow's Theorem for a given finite group follows from Webb's Conjecture applied to the group and all its subfactors. So at least philosophically, one should expect an analogue of Webb's conjecture to be true in those situations where an analogue of Sylow's Theorem holds.

3. Applications to algebraic groups

To complete the proof of Theorem 1, let us first consider the complex $\tilde{\mathcal{P}}(\mathbf{G})$. Here the corresponding graph is the set of all parabolic strict subgroups of the linear algebraic group $\mathbf{G}$ where directed edges correspond to inclusion and point towards the smaller subgroup. The minimal elements in this graph are the Borel subgroups of $\mathbf{G}$ provided $\mathbf{G}$ is not solvable—if it is then the graph is empty.

According to [2, Theorem 11.1] all Borel subgroups of $\mathbf{G}$ are conjugate. Hence $\mathbf{G}$ acts transitively on the minimal elements. So let us fix a simplex σ whose bottom vertex is the parabolic subgroup $P < \mathbf{G}$. This group is contained in all of the other groups belonging to σ. Hence it normalizes these groups, whence P is contained in the stabiliser $\mathbf{G}_\sigma$. From this, it follows that the Borel subgroups of $\mathbf{G}$ contained in P—these are the minimal vertices of the descending link of σ—are Borel subgroups of $\mathbf{G}_\sigma$ whence they are conjugate with respect to $\mathbf{G}_\sigma$. Hence Corollary 13 applies, and we recognize $\tilde{\mathcal{P}}(\mathbf{G})$ as contractible.

The complex $\tilde{\mathcal{T}}(\mathbf{G})$ arises from the graph $\mathcal{T}(\mathbf{G})$ of all non trivial tori in $\mathbf{G}$ where edges are given by inclusion. But now an edge points towards the bigger subgroup. So maximal tori are minimal elements in the complex $\mathcal{T}(\mathbf{G})$. By [2, Corollary 11.3 (1)], these are all conjugate whence $\mathbf{G}$ acts transitively on the minimal elements. Fix a simplex $\sigma = (T_0 > \cdots > T_m)$ with bottom vertex $T_0 \leq \mathbf{G}$. Since all tori are abelian, every T_i is normal in each maximal torus T of $\mathbf{G}$ containing T_0 whence T is a maximal torus in the stabiliser $\mathbf{G}_\sigma$, which in turn acts transitively on the set of these maximal tori above T_0 by [2, Corollary 11.3 (1)]. Hence Corollary 13 shows $\tilde{\mathcal{T}}(\mathbf{G})$ to be contractible. Observe that the same reasoning applies to $\tilde{\mathcal{U}}(\mathbf{G})$. We only have to use [2, Corollary 11.3 (2)] instead.

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