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## VARIATIONS OF HOMOGENEITY

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**ABSTRACT.** Results concerning various concepts related to homogeneity are recalled and discussed. After bihomogeneity and  $n$ -homogeneity, special attention is paid to homogeneity with respect to classes of mappings. Numerous open questions are either recalled or stated, and directions for further study in the area are indicated.

A topological space is said to be *homogeneous* provided that for every two points of the space there is a homeomorphism of the space onto itself which maps one of the points to the other. The notion of homogeneity was introduced [102] in 1920 by Waław Sierpiński, and shortly thereafter some related concepts were defined. The aim of this article is to present a survey of known results pertinent to these concepts and indicate some directions for further study in the area.

The paper consists of five sections. After a short discussion concerning homogeneity itself, results on bihomogeneity are reviewed. The third section is devoted to  $n$ -homogeneity, where  $n$  is a positive integer. Generalized homogeneity, i.e., homogeneity with respect to various classes of mappings, is

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considered in the fourth section. In the last section a common generalization of the two concepts of previous sections is introduced as  $\mathfrak{M} - n$ -homogeneity, where  $\mathfrak{M}$  is a class of mappings and  $n$  is a positive integer. Open problems and questions are recalled, and directions for further investigations are pointed out.

## 1. HOMOGENEITY

The simplest example of a homogeneous topological space is a *discrete* space, i.e. a space any subset of which is assumed to be open. Another simple example is an *indiscrete* space, i.e. a space with the only two open subsets, viz. the empty set and the whole space. A very nice common generalization of these two examples was given in [48] by J. Ginsburg who has proved the following characterization of finite homogeneous topological spaces. Given a cardinal number  $k$ , let  $D(k)$  and  $I(k)$  denote a set of cardinality  $k$  equipped with the discrete and with the indiscrete topology, respectively.

**Theorem 1.1.** *A finite topological space  $X$  is homogeneous if and only if there are natural numbers  $m$  and  $n$  such that  $X$  is homeomorphic to the product  $D(m) \times I(n)$ .*

One implication of the above theorem is a consequence of the following well known result, from which a number of examples of homogeneous spaces can be obtained.

**Theorem 1.2.** *The product of homogeneous spaces is a homogeneous space.*

**Proof:** Let  $X_1$  and  $X_2$  be homogeneous spaces, and let  $p = (p_1, p_2)$ ,  $q = (q_1, q_2) \in X_1 \times X_2$ . Thus  $p_i, q_i \in X_i$  for  $i \in \{1, 2\}$ . Since both spaces  $X_i$  are homogeneous, there are homeomorphisms  $h_1 : X_1 \rightarrow X_1$  with  $h_1(p_1) = q_1$ . Then  $h = h_1 \times h_2 : X_1 \times X_2 \rightarrow X_1 \times X_2$  is a homeomorphism with  $h(p) = q$ , as needed.

Remark that the above argument can be extended to an arbitrary (finite as well as infinite) number of factors. Therefore

the Cantor set, which is known to be homeomorphic to the countable product of the two point discrete space,  $\{0, 1\}^{\aleph_0}$ , see e.g. [43, p. 84 and Example 3.1.28, p. 131], is homogeneous. This example is also a particular case of another more general (and well known) result. Recall that a *topological group* means a space  $G$  equipped with a topology and with a group structure such that the two group operations, viz. the multiplication,  $xy$ , and taking the inverse element,  $x^{-1}$ , are continuous with respect to the topology in the sense that if the sequences  $\{x_n\}$  and  $\{y_n\}$  are converging to points  $x$  and  $y$ , respectively, then the sequences  $\{x_n y_n\}$  and  $\{x_n^{-1}\}$  are converging to points  $xy$  and  $x^{-1}$ , correspondingly.

It is known that each topological group is a homogeneous space. An even stronger result is true, see Theorem 2.1 below. Thus the real line  $\mathbb{R}$  as well as the unit circle  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$  are examples of homogeneous spaces.

In 1920 B. Knaster and K. Kuratowski [60] asked whether a nondegenerate plane homogeneous *continuum* (i.e., a compact connected metric space) is necessarily a *simple closed curve* (i.e. a space homeomorphic to  $S^1$ ). A partial affirmative answer was given in 1924 by S. Mazurkiewicz, [79], as the following theorem.

**Theorem 1.3.** *The unit circle  $S^1$  is the only locally connected homogeneous plane continuum.*

This result has been generalized by R. H. Bing in 1960 who proved [9] that a simple closed curve is the only homogeneous bounded plane continuum that contains an arc. For generalizations to higher dimensional cases see [90] and [91].

A negative answer to the question was shown in 1948 by R. H. Bing [5], who showed that the pseudo-arc, constructed in 1922 by B. Knaster, [59], and rediscovered by E. E. Moise in [83], is homogeneous. Shortly thereafter Moise presented his own proof of this result, [84]. See W. Lewis' expository article [76] for various properties of the pseudo-arc.

The reader is referred to the 8-th chapter of the author's article [27] for information on the numerous related results concerning homogeneity of continua; see also [26, Section 2, p. 80]. F. Burton Jones' contribution to continuum theory and, in particular, to homogeneity of continua is presented in M. E. Rudin's article [97]. See also W. Lewis' informative article [77] for classifications of homogeneous continua. A characterization of homogeneous locally connected continua which are not  $n$ -manifolds for  $n \leq 2$  is given in [66, Theorem 2, p. 85]. See also [64] and [65]. Properties of Menger manifolds and results on their homogeneity are collected in a comprehensive article [39]. Compare also [56].

## 2. BIHOMOGENEITY

The concept of homogeneity has been modified in many ways. One of them is bihomogeneity. A topological space  $X$  is said to be *bihomogeneous* provided that for every two points  $p, q \in X$  there is a homeomorphism  $h : X \rightarrow X$  of the space onto itself such that  $h(p) = q$  and simultaneously  $h(q) = p$ .

**Theorem 2.1.** *Each topological group is a bihomogeneous space.*

**Proof:** Given a topological group  $G$ , fix two points  $p, q \in G$  and define a function  $h : G \rightarrow G$  by  $h(x) = px^{-1}q$  for each  $x \in G$ . Then  $h$  is one-to-one. Indeed, for any  $x_1, x_2 \in G$  we have

$$\begin{aligned} (1) \quad h(x_1) = h(x_2) &\iff px_1^{-1}q = px_2^{-1}q \\ (2) &\iff p^{-1}px_1^{-1}qq^{-1} = p^{-1}px_2^{-1}qq^{-1} \\ (3) &\iff x_1^{-1} = x_2^{-1} \iff x_1 = x_2. \end{aligned}$$

Then  $h^{-1} : G \rightarrow G$  is a function determined by the formula  $h^{-1}(x) = qx^{-1}p$ . Since  $G$  is a topological group, both  $h$  and  $h^{-1}$  are continuous. Thus  $h$  is a homeomorphism. Finally,  $h(p) = pp^{-1}q = q$  and  $h(q) = pq^{-1}q = p$ , as needed.

Around 1921 B. Knaster asked the question of whether every homogeneous space is bihomogeneous, and shortly after

that K. Kuratowski described an example of a 1-dimensional, non-locally compact, homogeneous subset of the plane, which is not bihomogeneous, [73]. In the same paper Kuratowski gave a partial affirmative answer to Knaster's question for *totally disconnected* spaces (i.e., spaces with one-point quasi-components), and for subspaces of closed intervals of reals. In 1930 D. van Dantzig asked whether homogeneity implies bihomogeneity for continua, [41]. A locally compact homogeneous metric space which is not bihomogeneous was found by H. Cook in 1986, [40]. His space is of dimension 2. It is still not known if there is a 1-dimensional (locally) compact metric example.

Finally Knaster and van Dantzig's questions have been answered in the negative in 1990 by Krystyna Kuperberg [69] who constructed a locally connected homogeneous and not bihomogeneous continuum of dimension 7. She asks what is the lowest dimension of a homogeneous but not bihomogeneous locally compact metric space (continuum), [69, Problem 3, p. 142]. Another example of a homogeneous but not bihomogeneous continuum (that is neither locally connected nor of a finite dimension) is constructed by P. Minc in [82]. Outlines of both these examples are nicely described in the introduction of [70]. K. Kawamura in [55] noticed that, using some methods from [39] and [38], the Minc example can be modified so that its dimension can be lowered to 2. Kawamura's ideas applied to the construction of [69] gives a locally connected homogeneous and not bihomogeneous continuum of dimension 4, see [70]. Let us also recall that G. Kuperberg constructed in [67], for each pair of integers  $m \geq 1$  and  $n \geq 2$ , a homogeneous, non-bihomogeneous locally connected continuum whose every point has a neighborhood homeomorphic to the product of two Menger compacta of dimension  $m$  and  $n$ , respectively.

### 3. $n$ -HOMOGENEITY, LOCAL HOMOGENEITY, $1/n$ -HOMOGENEITY

A natural generalization of homogeneity is the concept of an  $n$ -homogeneous space. Let  $n$  be a positive integer. A topological space  $X$  is said to be  $n$ -homogeneous provided that for every pair  $A, B$  of  $n$ -element subsets of  $X$  there is a homeomorphism of  $X$  onto itself which maps  $A$  onto  $B$ . Thus a space is homogeneous if (and only if) it is 1-homogeneous. Note the following statement.

**Statement 3.1.** *Each 2-homogeneous space is homogeneous.*

**Proof:** Let  $p, q$  be distinct points on a 2-homogeneous space  $X$ , let  $r$  be another point of  $X$ , and  $h : X \rightarrow X$  be a homeomorphism such that  $h(\{p, r\}) = \{q, r\}$ . Then either  $h(p) = q$  and  $h(r) = r$ , and we are done in this case, or  $h(p) = r$  and  $h(r) = q$ . Then  $h^2 = h \circ h : X \rightarrow X$  is a homeomorphism such that  $h^2(p) = h(r) = q$ . The argument is complete.

The concept of  $n$ -homogeneity was defined in 1930 by van Dantzig [41], and studied by C. E. Burgess in [10] and by G. S. Ungar in [104], among others. In 1972 R. B. Bennett introduced in [3] the concept of a countable dense homogeneous space. A space  $X$  is said to be *countable dense homogeneous* provided that for every pair  $A, B$  of countable dense subsets of  $X$  there is a homeomorphism of  $X$  onto itself which maps  $A$  onto  $B$ . Connected manifolds without boundary are the simplest and the most natural examples of spaces which satisfy all of these homogeneity conditions. A nice result linking the considered concepts was shown in 1978 by G. S. Ungar [105]: for continua distinct from a simple closed curve countable dense homogeneity is equivalent to  $n$ -homogeneity for each natural  $n$ .

To discuss some other results concerning homogeneity it is needed to recall some theorems related to dimension theory. The famous Menger-Nöbeling embedding theorem says that every metric separable  $n$ -dimensional space  $X$  (where  $n$

is a nonnegative integer) is topologically contained in the cube  $[0, 1]^{2n+1}$ , that is, there exists a homeomorphism  $h : X \rightarrow h(X) \subset [0, 1]^{2n+1}$  (see [80] and [86]; see also [74, §45, VII, Theorem 1, p. 116], where a stronger related result of W. Hurewicz is quoted). The exponent  $2n + 1$  cannot be lessened in the Menger-Nöbeling theorem since for each nonnegative integer  $n$  there exists a metric separable  $n$ -dimensional space which is not homeomorphic with any subset of the cube  $[0, 1]^{2n}$ , [46]. For  $n = 1$  the two Kuratowski primitive skew graphs (see e.g. [74, §51, VII, Theorem 1 and Figure 11, p. 305, and footnote (1) on p. 306]) illustrate this result.

We describe now a subset of the Euclidean  $n$ -space  $\mathbb{R}^n$  which is a generalization of the Cantor ternary set as well as of the Sierpiński universal plane curve (see e.g. [74, §51, I, Example 5, p. 275, and Fig. 8, p. 276]). Given a collection  $\mathcal{C}$  of subsets of a space, we denote by  $|\mathcal{C}|$  the union of all elements of  $\mathcal{C}$ . Let  $\mathcal{C}$  be a collection of  $n$ -dimensional cubes lying in  $\mathbb{R}^n$ . Such cubes are closed bounded subsets of  $\mathbb{R}^n$ , with their boundaries being finite unions of faces. Each face is itself isometric with a  $k$ -dimensional cube, where  $k \in \{0, \dots, n\}$ . Denote by  $\mathcal{C}^{(k)}$  the collection of all  $k$ -dimensional faces of cubes from  $\mathcal{C}$ , and by  $\text{par } \mathcal{C}$  the collection of all  $n$ -dimensional cubes which we get by the partitioning of each cube from  $\mathcal{C}$  into  $3^n$  congruent cubes. Take the collection  $\mathcal{C}_0 = \{[0, 1]^n\}$  consisting of the only  $n$ -cube  $[0, 1]^n$ , and define inductively  $\mathcal{C}_i$  by the formula

$$\mathcal{C}_i = \{C \in \text{par } \mathcal{C}_{i-1} : C \cap |\mathcal{C}_{i-1}^{(k)}| \neq \emptyset\} \quad \text{for } i \in \mathbb{N}$$

and fixed  $k \in \{0, \dots, n\}$ .

The *Menger  $k$ -dimensional universal compactum*  $M_k^n$  in  $[0, 1]^n$  is defined by  $M_k^n = \bigcap \{|\mathcal{C}_i| : i \in \{0, 1, 2, \dots\}\}$ . It can be shown that for each  $n \in \mathbb{N}$  the set  $M_0^n$  is homeomorphic to the Cantor set, and that  $M_n^n = [0, 1]^n$ . The set  $M_1^2$  is the Sierpiński universal plane curve mentioned above, and  $M_1^3$  is called the *Menger universal curve*. We denote it by  $\mathbb{M}$  for shortness, i.e.,  $\mathbb{M} = M_1^3$ . It was proved by R. D. Anderson in [2, Theorem XII, p. 13] that every 1-dimensional continuum, with no local



cut points and no nonempty open subsets embeddable in the plane, is homeomorphic to  $\mathbb{M}$ . The reader is referred to [1], [2] and [78] for characterizations any many various properties of this important continuum. See also [66, Theorem 3, p. 86], where another characterization of  $\mathbb{M}$  is presented.

It is known from [80] that  $\dim M_k^n = k$  and that  $M_k^n$  is universal with respect to containing homeomorphic copies of every  $k$ -dimensional compactum which can be embedded in  $\mathbb{R}^n$ . The compacta  $M_k^n$  were characterized in 1989 by M. Bestvina, [4].

In 1958 R. D. Anderson proved in [1] that the Menger universal curve  $\mathbb{M}$  is  $n$ -homogeneous for every natural  $n$ . Even a stronger result is shown therein. Recall that a space  $X$  is said to be *strongly  $n$ -homogeneous* provided that for any two ordered sequences  $A = \{a_1, \dots, a_n\}$  and  $B = \{b_1, \dots, b_n\}$  there is a homeomorphism  $h : X \rightarrow X$  carrying  $A$  onto  $B$ , i.e., such that  $h(a_i) = b_i$  for each  $i \in \{1, \dots, n\}$ . It is proved that  $\mathbb{M}$  is strongly  $n$ -homogeneous for every natural  $n$ , see [1, Theorems III and IV, p. 322]. Another important result is that the circle  $S^1$  and the Menger universal curve  $\mathbb{M}$  are the only locally connected 1-dimensional homogeneous continua, see [2, Theorem XIII, p. 14]. Concerning universality of other Menger compacta note that the Cantor set  $M_0^1 = M_0^n$  is homogeneous as a topological group (see Theorem 2.1 above), while the Sierpiński universal plane curve  $M_1^2$  is not (it has exactly two orbits of points: the union  $U$  of boundaries of complementary domains, called the rational part of the curve, and  $M_1^2 \setminus U$ , called its irrational part; see [61]). Extending Anderson's result [1] on the homogeneity of the Menger curve  $\mathbb{M} = M_1^3$ , M. Bestvina proved in 1989 (in [3], announced in 1984) that the continua  $M_n^{2n+1}$  are homogeneous for each  $n \in \{0, 1, 2, \dots\}$ , whence it follows that all  $M_n^m$  are for  $m \geq 2n + 1$ . Finally so called intermediate Menger compacta  $M_n^m$  for each  $n \in \{1, 2, \dots\}$  and  $m \in \{1, 2, \dots, 2n\}$  are not homogeneous, as it was shown in 1987 by W. Lewis, [75].

Using another result of Anderson [2] concerning the homogeneity of curves, R. B. Bennett [3] showed that  $\mathbb{M}$  is countable dense homogeneous. Looking for higher dimensional countable dense homogeneous continua which are not manifolds, R. B. Bennett asked if the property of being countable dense homogeneous is preserved by taking Cartesian products.

Answering a question asked in 1955 by C. E. Burgess [10], G. S. Ungar proved in [104] the following result (for a generalization see [109]).

**Theorem 3.2.** *Each 2-homogeneous metric continuum is locally connected.*

Further, investigating the  $n$ -homogeneous spaces, Ungar asked if there exists a homogeneous locally connected metric continuum which is not 2-homogeneous.

Both Bennett's and Ungar's questions were answered in 1980 by K. Kuperberg, W. Kuperberg and W. R. R. Transue, who proved in [71] that the product of the circle  $S^1$  and the Menger universal curve  $\mathbb{M}$  is not countable dense homogeneous, or even 2-homogeneous. Namely, it is shown in [71] that every homeomorphism  $h$  of  $S^1 \times \mathbb{M}$  onto itself preserves the circular fibers, i.e., for every point  $a \in \mathbb{M}$  there exists a point  $b \in \mathbb{M}$  such that  $h(S^1 \times \{a\}) = S^1 \times \{b\}$ . Thus if  $p, q \in S^1 \times \{a\}$  and  $r \in S^1 \times \{b\}$ , then there is no homeomorphism of  $S^1 \times \mathbb{M}$  onto itself that maps  $\{p, q\}$  onto  $\{p, r\}$ . Later, in 1984, this result has been extended by J. Kennedy Phelps in [58, Corollary 4, p. 97] by showing the following theorem (for its generalization see Section 4 of [72, p. 294]).

**Theorem 3.3.** *The Cartesian product of  $\mathbb{M}$  with an arbitrary continuum is not 2-homogeneous.*

The next theorem is the key argument she used in the proof of the above result.

**Theorem 3.4.** *Each 2-homogeneous primitively stable space is representable.*

Recall that a space is said to be *primitively stable* provided that it admits an auto-homeomorphism different from the identity which is the identity on a nonempty open subset. A space  $X$  is said to be *representable* (called also *strongly locally homogeneous*) provided that for every point  $x \in X$  and every neighborhood  $U$  of  $x$  there is a neighborhood  $V$  of  $x$  with  $V \subset U$  such that for every point  $y \in V$  there is a homeomorphism  $h : X \rightarrow X$  such that  $h(x) = y$  and  $h|(X \setminus U)$  is the identity.

In [47] Dennis J. Garity showed by constructing suitable examples that adding the additional requirement of local  $n$ -connectivity is not enough to get a converse to Ungar's theorem (i.e., Theorem 3.2). For each positive integer  $n$ , a homogeneous metric continuum of dimension  $n + 1$  that is locally  $(n - 1)$ -connected is constructed which is not 2-homogeneous. The examples are produced by taking products of the universal Menger  $n$ -dimensional space with  $S^1$ . Other examples are obtained in [47] by taking products of Menger spaces.

Another generalization of homogeneity is the concept of a locally homogeneous space. A space  $X$  is said to be *locally homogeneous* (originally called micro-homogeneous in [41], see also [68]) provided that for every pair of points  $p, q \in X$  there are neighborhoods  $U$  and  $V$  of  $p$  and  $q$  respectively and a homeomorphism  $h : U \rightarrow V$  with  $h(p) = q$ . Let us mention the following two results related to the Cartesian products that are due to H. Patkowska, [87].

**Theorem 3.5.** *If the Cartesian product of finitely many locally connected 1-dimensional continua is 2-homogeneous, then each factor is homeomorphic to the circle  $S^1$ .*

**Theorem 3.6.** *The Cartesian product of finitely many 1- or 2-dimensional ANR's is locally homogeneous if and only if each factor is a manifold.*

Let us merely mention, without any particular discussion, one more line of investigations related to  $n$ -homogeneity, and connected with a special construction of higher dimensional continua. See [103] and references therein for details.

Given a topological space  $X$ , let  $H(X)$  stand for the group of autohomeomorphisms of  $X$ . If a point  $p \in X$  is fixed, then the set  $A = \{h(p) : h \in H(X)\}$  is called an *orbit* of  $p$ . It is obvious that orbits of points of  $X$  either are mutually disjoint or coincide, and that their union is the whole  $X$ . We say that the *action on  $X$  of the group  $H(X)$  has exactly  $n$  orbits* provided that there are  $n$  subsets  $A_1, \dots, A_n$  of  $X$  such that  $X = A_1 \cup \dots \cup A_n$  and, for any  $x \in A_i$  and  $y \in A_j$ , there is a homeomorphism  $h \in H(X)$  with  $h(x) = y$  if and only if  $i = j$ .

Let  $n$  be a positive integer. A space  $X$  is called *1/n-homogeneous* provided that the action on  $X$  of the group  $H(X)$  has exactly  $n$  orbits. As it was mentioned above, J. Krasinkiewicz proved in [61] that the Sierpiński universal plane curve  $M_1^2$  is 1/2-homogeneous. Using Whyburn's characterization of the curve [110, Corollary, p. 323] (as a plane locally connected 1-dimensional continuum such that the boundary of each complementary domain of the continuum is a simple closed curve and no two of these complementary domain boundaries intersect; see [93, Corollary 18, p. 36] for an extension of this result) one can list all 1/2-homogeneous planar locally connected continua, and using Anderson's characterization of the Menger universal curve  $\mathbb{M}$  [1] and [2] (quoted above) all compact 1/2-homogeneous locally connected curves can also be classified. In 1989 H. Patkowska classified [88] all 1/2-homogeneous compact ANR-spaces of dimension at most 2, and also gave a full classification of 1/2-homogeneous polyhedra.

Recall that a *dendrite* means a locally connected continuum containing no simple closed curve. For a given nonempty set  $S \subset \{3, 4, \dots, \omega\}$  denote by  $D_S$  any dendrite  $X$  such that each ramification point of  $X$  is of order belonging to  $S$ , and, for each arc  $A$  contained in  $X$  and for every  $m \in S$  there is in  $A$  a point  $p$  of order  $m$  in  $X$ . It is shown in [36, Theorem 6.2, p. 229] that the dendrite  $D_S$  is topologically unique.  $D_S$  is called the *standard universal dendrite of orders in  $S$* . If  $S = \{m\}$  for some  $m \in \{3, 4, \dots, \omega\}$ , then the mentioned

dendrite is denoted by  $D_m$  and is called the *standard universal dendrite of order  $m$* . In particular,  $D_\omega$  is called the *standard universal dendrite*. For its construction and the proof on the universality see T. Ważewski dissertation [108, Chapter K, p. 137] (compare also [81, Chapter X, §6, p. 318] and [20, p. 168]).

The following result is known, see [23, Theorem 3.16 and Corollary 3.20, p. 466].

**Theorem 3.7.** *Each standard universal dendrite  $D_S$  of orders in  $S \subset \{3, 4, \dots, \omega\}$  satisfies the conditions:*

- (3.7.1)  $D_S$  is  $1/n$ -homogeneous, where  $n = 2 + \text{card } S$ ;
- (3.7.2) for each orbit  $A$  of  $D_S$  and for each arc  $J$  in  $D_S$  the intersection  $A \cap J$  is a dense subset of  $J$ ;
- (3.7.3) each orbit of  $D_S$  is a dense subset of  $D_S$ .

*In particular, for any  $m \in \{3, 4, \dots, \omega\}$  the standard universal dendrite  $D_m$  of order  $m$  is  $1/3$ -homogeneous.*

Conditions (3.7.1) and (3.7.2) above characterize the dendrites  $D_S$  in the sense of the next theorem, which is a reformulation of Theorem 6.2 of [36, p. 229]. See also [23, Theorem 3.22, p. 467].

**Theorem 3.8.** *Let a dendrite  $X$  satisfy the following conditions:*

- (3.8.1)  $X$  is  $1/n$ -homogeneous for some integer  $n \geq 3$ ;
- (3.8.2) for each orbit  $A$  of  $X$  and for each arc  $J$  in  $X$  the intersection  $A \cap J$  is a dense subset of  $J$ .

*Then  $X$  is homeomorphic to  $D_S$  for some  $S \subset \{3, 4, \dots, \omega\}$  and  $\text{card } S = n - 2$ . In particular, if  $n = 3$ , then  $X$  is homeomorphic to the standard universal dendrite  $D_m$  of order  $m$  for some  $m \in \{3, 4, \dots, \omega\}$ .*

#### 4. SPACES HOMOGENEOUS WITH RESPECT TO OTHER CLASSES OF MAPPINGS (GENERALIZED HOMOGENEITY)

Generalizing the concept of a homogeneous space, David P. Bellamy in a conversation with the author in 1976, at the University of Wrocław, Wrocław (Poland), replaced the homeomorphism (which is used in the definition of homogeneity) by an arbitrary (continuous) surjective mapping. In this way the concept of homogeneity with respect to continuity has been introduced. Later the concept was extended to a more general setting. A topological space is said to be *homogeneous with respect to a class  $\mathfrak{M}$  of mappings* (shortly  *$\mathfrak{M}$ -homogeneous*) provided that for every two points of the space there is a surjective mapping of the space onto itself belonging to  $\mathfrak{M}$  which maps one of the points to the other. If the class  $\mathfrak{M}$  in the above definition is taken to be the class of homeomorphisms  $\mathfrak{H}$ , we obtain the familiar concept of homogeneity. If  $\mathfrak{M}$  is taken to be the class of continuous surjections  $\mathfrak{C}$ , Bellamy's concept of homogeneity with respect to continuity (shortly called  *$\mathfrak{C}$ -homogeneity*) is obtained.

A general problem, which can be considered as a research program rather than a particular question, is to verify what results concerning homogeneity can be strengthened so that the usual concept of homogeneity is replaced by homogeneity with respect to a wider (thus less restrictive than homeomorphisms) class of mappings.

The following classes of mappings are of a special interest. A surjective mapping  $f : X \rightarrow Y$  between topological spaces is said to be:

- *open* provided that the images of open sets under  $f$  are open;
- *monotone* provided that for each point  $y \in Y$  the set  $f^{-1}(y)$  is connected;
- *light* provided that for each point  $y \in Y$  the set  $f^{-1}(y)$  has one-point components (note that if the point-inverses are compact, this condition is equivalent to the property that they are zero-dimensional);

- *confluent* provided that for each subcontinuum  $Q$  of  $Y$  each component of  $f^{-1}(Q)$  is mapped onto  $Q$  under  $f$ ;
- *semi-confluent* provided that for each subcontinuum  $Q$  of  $Y$  and for every two components  $C_1$  and  $C_2$  of  $f^{-1}(Q)$  either  $f(C_1) \subset f(C_2)$  or  $f(C_2) \subset f(C_1)$ ;
- *weakly confluent* provided that for each subcontinuum  $Q$  of  $Y$  some component of  $f^{-1}(Q)$  is mapped onto  $Q$  under  $f$ .

We briefly summarize results achieved in the area of generalized homogeneity. Let us start with a general assertion. Recall that a class  $\mathfrak{M}$  of mappings is said to have the *composition property* provided that the composition of any two mappings from  $\mathfrak{M}$  belongs to  $\mathfrak{M}$  as well.

**Assertion 4.1.** *Let a class  $\mathfrak{M}$  of mappings have the composition property, and let some two spaces  $X$  and  $Y$  be given. If there exist in  $\mathfrak{M}$  surjections  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ , then  $X$  is  $\mathfrak{M}$ -homogeneous if and only if  $Y$  is.*

**Proof:** Indeed, let  $X$  be  $\mathfrak{M}$ -homogeneous. Take points  $p, q \in Y$ ,  $a = g(p) \in X$ ,  $b \in f^{-1}(q) \subset X$ , and let  $h : X \rightarrow X$  be a surjection in  $\mathfrak{M}$  such that  $h(a) = b$ . Then the composition  $fhg : Y \rightarrow Y$  is in  $\mathfrak{M}$  and  $fhg(p) = q$ , as needed.

The simplest cases are ones of finite and of countable spaces. Since each surjective mapping from a finite topological space onto itself is a homeomorphism, [28, Proposition 1.1, p. 197], the Ginsburg characterization of finite topological spaces (Theorem 1.1. above) has been extended in [28, Theorem 1.2, p. 197] as follows.

**Theorem 4.2.** *For a finite topological space  $X$  the following conditions are equivalent.*

- (4.2.1)  $X$  is homogeneous;
- (4.2.2)  $X$  is open-homogeneous;
- (4.2.3)  $X$  is continuous-homogeneous;
- (4.2.4) there are natural numbers  $m$  and  $n$  such that  $X$  is homeomorphic to the product  $D(m) \times I(n)$ .

The next quoted results concern countable spaces (see [28, Theorems 2.1 and 2.3, p. 201, and Theorem 2.15, p. 209]).

**Theorem 4.3.** *If a countable topological space  $X$  is metrizable (or, equivalently, is a regular  $T_1$ -space satisfying the first or the second countability axiom), then the following conditions are equivalent.*

- (4.3.1)  $X$  is homogeneous;
- (4.3.2)  $X$  is open-homogeneous;
- (4.3.3)  $X$  is homeomorphic either to a discrete space ( $\mathbb{Z}$  of integers) or to a space which is dense in itself ( $\mathbb{Q}$  of rationals).

**Theorem 4.4.** *For a countable regular  $T_1$ -space the following conditions are equivalent:*

- (4.4.1)  $X$  is continuous-homogeneous;
- (4.4.2) there is a mapping from  $X$  onto a countable discrete space;
- (4.4.3) there is a countable open covering of  $X$  whose elements are mutually disjoint;
- (4.4.4)  $X$  is noncompact.

However, no characterization is known of countable topological spaces which are either homogeneous or open-homogeneous in the general case (i.e., not only for metric spaces but for the nonmetric setting as well), see [28, Remark 2.7, p. 204].

Let us turn our attention to continuum theory and recall some basic facts on continuous-homogeneity of continua. For shortness, denote by  $\mathfrak{C}$  the class of all (continuous) mappings between spaces. Then  $\mathfrak{C}$ -homogeneity means homogeneity with respect to the class of mappings. The following result is known (see [13, Assertion 1, p. 272/292]; for  $\mathfrak{C}$ -homogeneity compare [62, Theorem 1, p. 347]).

**Theorem 4.5.** *Each locally connected metric continuum is  $\mathfrak{C}$ -bihomogeneous (and thus it is  $\mathfrak{C}$ -homogeneous).*

**Proof:** Let  $X$  be a metric continuum, and let  $d$  be a metric on  $X$ . For any two distinct points  $p, q \in X$  define  $f_1 : X \rightarrow [0, 1]$  by  $f_1(x) = d(x, p)/[d(x, p) + d(x, q)]$ . Then  $f_1 \in \mathfrak{C}$ ,  $f_1(p) = 0$



and  $f_1(q) = 1$ . Further, since the continuum  $X$  is locally connected, there is a surjective mapping  $g : [1/3, 2/3] \rightarrow X$ , and since  $X$  is arcwise connected, there are arcs  $A$  and  $B$  in  $X$  from  $q$  to  $g(1/3)$  and from  $g(2/3)$  to  $p$  respectively. Define a mapping  $f_2 : [0, 1] \rightarrow X$  by  $f_2(0) = q$ ,  $f_2(1/3) = g(1/3)$ ,  $f_2(2/3) = g(2/3)$ ,  $f_2(1) = p$ ,  $f_2|_{[0, 1/3]} : [0, 1/3] \rightarrow A$ ,  $f_2|_{[1/3, 2/3]} = g$ ,  $f_2|_{[2/3, 1]} : [2/3, 1] \rightarrow B$ , where the partial mappings  $f_2|_{[0, 1/3]}$  and  $f_2|_{[2/3, 1]}$  are arbitrary surjections. Then the composition  $f_2 f_1 : X \rightarrow X$  is a surjection that takes  $p$  to  $q$  and  $q$  to  $p$ , as needed.

Let  $\mathfrak{H}$  stand for the class of homeomorphisms. Then  $\mathfrak{H}$ -homogeneity (i.e., homogeneity) of locally connected continua does not imply their  $\mathfrak{H}$ -bihomogeneity, see [69] and [67]. On the other hand, the implication holds for the class  $\mathfrak{C}$  of all mappings by Theorem 4.5. Thus the following question is natural.

**Question 4.6.** For what classes  $\mathfrak{M}$  of mappings of locally connected metric continua, with  $\mathfrak{H} \subset \mathfrak{M} \subset \mathfrak{C}$  (as monotone, open, confluent, etc.)  $\mathfrak{M}$ -homogeneity implies  $\mathfrak{M}$ -bihomogeneity?

It is shown in [62, Proposition 3, p. 346] that the cone over any topological space is  $\mathfrak{C}$ -homogeneous. In particular the cone over the Cantor set, i.e., the *Cantor fan* is  $\mathfrak{C}$ -homogeneous. More general, each uniformly pathwise connected continuum containing an open subset with uncountably many components is  $\mathfrak{C}$ -homogeneous. These results were applied to show  $\mathfrak{C}$ -homogeneity of some hyperspaces. Namely for each (metric) continuum  $X$  the hyperspace  $2^X$  of its nonempty compact subsets is  $\mathfrak{C}$ -homogeneous, and if  $X$  is either locally connected or contains an open subset with uncountably many components, then also the hyperspace  $C(X)$  of all its nonempty subcontinua is  $\mathfrak{C}$ -homogeneous, see [37, Theorems 1 and 2, p. 341]. It is not known if there is a metric continuum  $X$  whose hyperspace  $C(X)$  is not  $\mathfrak{C}$ -homogeneous, [37, Question 2, p. 341].

In 1959 R. H. Bing proved that each homogeneous nondegenerate arc-like continuum is a pseudo-arc [8] obtaining in

this way a characterization of the pseudo-arc as the only homogeneous nondegenerate arc-like continuum. Using a result of I. Rosenholtz [96] Bing's result has been extended to open mappings in [11] (see also [12] for some generalizations of this result). Finally in 1986 it was shown that each confluent-homogeneous nondegenerate arc-like continuum is the pseudo-arc [33, Corollary 3.6, p. 33], which lead to the following result (see [33, Theorem 3.9, p. 34]).

**Theorem 4.7.** *Let a nondegenerate continuum  $X$  be arc-like. Then the following conditions are equivalent:*

- (4.7.1)  $X$  is homogeneous;
- (4.7.2)  $X$  is open-homogeneous;
- (4.7.3)  $X$  is monotone-homogeneous;
- (4.7.4)  $X$  is confluent-homogeneous;
- (4.7.5)  $X$  is the pseudo-arc.

A natural class of mappings larger than that of confluent ones (but not as large as the class of all continuous mappings) is the class of weakly confluent mappings. However, if we assume that the range space is an arc-like continuum, then each mapping is weakly confluent [94, Theorem 4, p. 236], so the classes of all mappings and of weakly confluent ones coincide if considered on arc-like continua. Note that weakly confluent-homogeneity (and therefore  $\mathfrak{C}$ -homogeneity) cannot be joined to properties listed in Theorem 4.7, because the arc, as a locally connected continuum, is  $\mathfrak{C}$ -homogeneous according to Theorem 4.5. There are also other arc-like continua which are  $\mathfrak{C}$ -homogeneous, as e.g. the simplest Knaster indecomposable continuum [74, Example 1, Fig. 4, p. 204 and 205] or the irreducible continuum (also due to Knaster) described in [74, Example 5, p. 191]; see [33, Remark 3.10, p. 34] for details. The problem of characterizing all nondegenerate arc-like  $\mathfrak{C}$ -homogeneous continua remains open, [33, Problem 3.11, p. 35]. An important step in this direction has been made by J. R. Prajs who, investigating in [89] arc components in  $\mathfrak{C}$ -homogeneous Hausdorff continua  $X$  proved that (1) if  $X$  is

the countable union of arcwise connected continua, then  $X$  is arcwise connected; (2) if  $X$  is nondegenerate and metric, the number of its arc components is countable and it contains no simple triod, then it is either an arc or a simple closed curve; and, in particular, (3) an arc is the only nondegenerate  $\mathfrak{C}$ -homogeneous arc-like metric continuum with countably many arc components.

A class  $\mathfrak{M}$  of mappings is named in [33, p. 30] *admissible* provided that it contains the class  $\mathfrak{H}$  of homeomorphisms and if for each mapping in  $\mathfrak{M}$  its composite with a homeomorphism is in  $\mathfrak{M}$ , too. Recall that a point  $p$  of a continuum  $X$  is called an *end point* of  $X$  if for each two subcontinua of  $X$  both containing  $p$ , one of them is contained in the other, [7, p. 660 and 661]. A point  $p$  of an arc-like (metric) continuum  $X$  is called a *pseudo-end point* of  $X$  provided that for each neighborhood  $U$  of  $p$  and for each positive number  $\varepsilon$  there is an  $\varepsilon$ -chain covering  $X$ , one of whose end links lies in  $U$ , see [8, p. 346] and [33, p. 35]. A mapping  $f$  between (arc-like) continua  $X$  and  $Y$  is said to *preserve ends (pseudo-ends)* provided that for each end point (pseudo-end point)  $p$  of  $X$  its image  $f(p)$  is an end point (pseudo-end point, respectively) of  $Y$ .

Another characterization of the pseudo-arc using generalized homogeneity runs as follows [33, Corollary 4.8, p. 38].

**Theorem 4.8.** *Let an admissible class  $\mathfrak{M}$  of mappings between arc-like continua preserve ends and pseudo-ends. A non-degenerate arc-like continuum is the pseudo-arc if and only if it is  $\mathfrak{M}$ -homogeneous.*

We say that a Hausdorff continuum  $X$  has the *property of Kelley at a point*  $p \in X$  if for any subcontinuum  $K$  of  $X$  containing  $p$  and for any open neighborhood  $\mathcal{U}$  of  $K$  in  $C(X)$  there is a neighborhood  $U$  of  $p$  in  $X$  such that if  $q \in U$ , then there is a continuum  $L \in C(X)$  with  $q \in L \in \mathcal{U}$ . A continuum  $X$  has the *property of Kelley* if it has the property of Kelley at each of its points. The property, introduced for metric continua by J. L. Kelley as property 3.2 in [57, p. 26], has been used there to

study hyperspaces, in particular their contractibility (see e.g. Chapter 16 of [85], where references for further results in this area are given). Now the property, which has been recognized as an important tool in investigation of various properties of continua, is interesting by its own right, and has numerous applications to continuum theory. Many of them are not related to hyperspaces. Its pointed version (also for the metric case) has been defined in [107, p. 292]. It is shown in [107, Theorem 2.5, p. 293] that metric homogeneous continua have the property of Kelley. Metrizable is indispensable in the result, because there is a nonmetrizable homogeneous continuum without the property of Kelley, [35]. Wardle's result has been sharpened in [15, Statement, p. 380] to the following.

**Theorem 4.9.** *Each open-homogeneous metric continuum has the property of Kelley.*

This result cannot be extended to confluent-homogeneous continua. Answering the author's question, H. Kato has constructed in [53] two examples of metric continua (one contractible and 2-dimensional, and the other 1-dimensional) which are confluent-homogeneous and which do not have the property of Kelley. Recently A. Illanes has shown that even monotone-homogeneous does not imply the property of Kelley. Namely there is a *dendroid*, i.e., an arcwise connected and hereditarily unicoherent continuum, which is monotone-homogeneous and which does not have the property of Kelley, [52].

Another of Kato's examples exhibits a continuum which is homogeneous with respect to the class of open and monotone mappings and which is not homogeneous, [53, p. 58]. The construction employs D. C. Wilson's result [111] that for each locally connected continuum  $X$  there is an open monotone mapping from the Menger universal curve  $\mathbb{M}$  onto  $X$  with all fibers homeomorphic to  $\mathbb{M}$ .

The property of Kelley can also be used to characterize solenoids, see [51] and [63]. Namely the following theorem is known, see [14, Theorem, p. 173].

**Theorem 4.10.** *Let a continuum  $X$  be circle-like. Then the following conditions are equivalent:*

- (4.10.1)  *$X$  is homogeneous, and  $X$  contains an arc;*
- (4.10.2)  *$X$  is homogeneous, and each nondegenerate proper subcontinuum of  $X$  is an arc;*
- (4.10.3)  *$X$  is open-homogeneous, and each nondegenerate proper subcontinuum of  $X$  is an arc;*
- (4.10.4)  *$X$  has the property of Kelley, and each point  $x \in X$  belongs to an arc with end points different from  $x$ ;*
- (4.10.5)  *$X$  is a solenoid.*

The following question remains open, [14, Question 2, p. 173].

**Question 4.11.** Let a circle-like open-homogeneous continuum that is not  $S^1$  contain an arc. Is it then a solenoid?

In connection with circle-like continua recall that in 1951 R. H. Bing constructed a hereditarily indecomposable circle-like continuum called the *pseudo-circle* (see [6, p. 48] for the definition) and asked about its homogeneity. The pseudo-circle was shown to be unique, [44] and not homogeneous, [45] and [95]. Thus natural questions arise (see [16, Problem 7, p. 5] and [24, Problems 1 and 2, p. 10]), which are (as far as I know) still open.

**Question 4.12.** Is the pseudo-circle a) open-homogeneous, b) confluent-homogeneous?

Answering a question of H. Kato [53, p. 62] the author has shown in [19, Theorem, p. 409] that no dendroid is open-homogeneous.

H. Kato has shown in [53, Example 2.4, p. 59] (compare also [54, Proposition 2.4, p. 223]) that the standard universal dendrite  $D_3$  of order 3 is monotone-homogeneous. This result has been generalized by the author to all standard universal dendrites  $D_m$  for each  $m \in \{3, 4, \dots, \omega\}$  in [20, Theorem 7.1, p. 186], and next to all dendrites  $D_S$  in [25, Theorem 3.3, p. 292],

where an uncountable family of monotone-homogeneous dendrites is constructed, [25, Corollary 3.8, p. 293]. The strongest result in this direction says that if a dendrite  $X$  has the set of its ramification points  $R(X)$  dense in  $X$ , then  $X$  is monotone-homogeneous, [30, Proposition 15, p. 364]. The converse is not true and, moreover, it can be seen that the condition  $\text{cl}R(X) = X$  is far from being necessary for a dendrite  $X$  to be monotone-homogeneous. Namely a monotone-homogeneous dendrite  $L_0$  is known having the set  $R(L_0)$  of its ramification points discrete (thus nowhere dense in  $L_0$ ). For its construction and the proof of its monotone homogeneity see [20, Example 6.9, p. 182] and also [30, p. 365]. It is shown in [30, Proposition 20, p. 366] that if a dendrite contains a homeomorphic copy of  $L_0$ , then it is monotone-homogeneous. The converse is not known, and the following two questions are still open (see [20, Question 7.2, p. 186], [21, Problems 5, p. 771] and [30, Question 21, p. 366]).

**Question 4.13.** Does every monotone-homogeneous dendrite contain a homeomorphic copy of the dendrite  $L_0$  (equivalently, does it admit any monotone mapping onto  $D_3$ )?

**Question 4.14.** What is an internal (structural) characterization of monotone-homogeneous dendrites?

The reader is referred to Section 3 of [31] for a summary of known results on monotone-homogeneous dendrites.

A larger class of mappings than that of monotone ones is the class of confluent mappings. For dendrites monotone homogeneity and confluent homogeneity are equivalent, [30, Theorem 9, p. 363]. It is not known if this equivalence is valid for wider classes of continua, e.g. for (smooth) dendroids, [30, Questions 11, 27 and 28, p. 364 and 367].

A surjective mapping  $f : X \rightarrow Y$  between topological spaces is called a *local homeomorphism* provided that for each point  $x \in X$  there is an open neighborhood  $U$  of  $x$  such that  $f(U)$  is an open subset of  $Y$  and the partial mapping  $f|U : U \rightarrow f(U)$

is a homeomorphism. Let us come back to the Mazurkiewicz result, recalled here as Theorem 1.3. This result can be extended as follows (see [21, Theorem 15, p. 772] and [22, Proposition 44, p. 501]).

**Theorem 4.15.** *If a plane continuum  $X$  is locally connected, then the following conditions are equivalent:*

- (4.15.1)  $X$  is homogeneous;
- (4.15.2)  $X$  is homogeneous with respect to the class of local homeomorphisms;
- (4.15.3)  $X$  is homogeneous with respect to the class of light open mappings;
- (4.15.4)  $X$  is homogeneous with respect to the class of light confluent mappings;
- (4.15.5)  $X$  is a simple closed curve.

**Theorem 4.16.** *If a continuum  $X$  contains a point of order 2, then the following conditions are equivalent:*

- (4.16.1)  $X$  is homogeneous;
- (4.16.2)  $X$  is homogeneous with respect to any class of mappings that do not decrease order of points;
- (4.16.3)  $X$  is homogeneous with respect to any class of mappings that do not increase order of points;
- (4.16.4)  $X$  is a simple closed curve.

Let us recall the following old result due to P. S. Urysohn, see [106, Chapter VI, Section 2, p. 105] and [30, Proposition 4, p. 493].

- If all points of a continuum are of the same order  $\mathfrak{n}$ , then this order can take only four values, namely  $\mathfrak{n} \in \{2, \omega, \aleph_0, \mathfrak{c}\}$ .

If  $\mathfrak{n} = 2$ , then the continuum under consideration is a simple closed curve, see [74, §51, V, Theorem 6, p. 294], and it is homogeneous. With regard to  $\mathfrak{n} = \omega$  and  $\mathfrak{n} = \aleph_0$ , Urysohn constructed in [106, Chapter VI, Sections 6-8 and 9-10, p. 109-115] (for order  $\omega$  see also [81, Chapter VIII, Section 5, p. 279])

two locally connected plane curves  $X(\omega)$  and  $X(\aleph_0)$ . Constructions of these curves are recalled in [30, Examples 5 and 23, p. 494 and 496], respectively. According to Theorem 1.3, they are not homogeneous. Their  $\mathfrak{M}$ -homogeneity for other classes  $\mathfrak{M}$  of mappings (such as open, monotone, etc.) is not known, see [30, Question 39, p. 499].

Let us mention that for *rational* continua (i.e. for continua all points of which are of order at most  $\aleph_0$ ) the class of m.o.-homogeneous continua coincides with the class of homogeneous ones, [29, Corollary, p. 316], and that in [92] a complete classification is given of continua in 2-manifolds containing an arc, which are homogeneous with respect to light open mappings. It is proved that they are exactly those continua, which are homogeneous (with respect to homeomorphisms), i.e., simple closed curves and 2-manifolds without boundary.

Concerning  $\mathfrak{n} = \mathfrak{c}$ , the Sierpiński universal plane curve  $M_1^2$  is a (locally connected, plane) continuum composed exclusively of points of order  $\mathfrak{c}$ , see e.g. [74, §51, I, Theorem 5, p. 275]. Again by Theorem 1.3 it is not homogeneous. It has however very strong homogeneity properties. A study of these properties was started by J. Krasinkiewicz in [61]. Later, in [18] (see also [17]) the author has shown that  $M_1^2$  is homogeneous with respect to the class of *simple* mappings, i.e., mappings whose point inverses are either singletons or two-point sets, and it is monotone-homogeneous, while not homogeneous with respect to the class of local homeomorphisms, see [18, Theorems 1, 4 and 5, p. 128, 130 and 131, respectively]. These results have been augmented and extended by C. R. Seaquist, who has shown that no compact planar manifold with boundary is monotone-homogeneous, [100, Theorem 9, p. 51], and that  $M_1^2$  is homogeneous with respect to the class of mappings which are monotone and open simultaneously (shortly m.o.-homogeneous), [101]. A more general result was obtained by J. R. Prajs. Namely each curve that is locally homeomorphic to the Sierpiński universal plane curve  $M_1^2$  is m.o.-homogeneous, [93, Theorem 23, p. 38]. Both Prajs' and Seaquist's proofs



of these results utilize continuous decompositions of some locally connected plane continua into pseudo-arcs, see [93, Main Theorem 16, p. 34] and [99]; compare also [98]. The existence of such decompositions leads to another important result: the 2-cell (i.e. the plane disk) is open-homogeneous (see [93, Example 19, p. 37]).

Concerning homogeneity of the universal Menger compacta  $M_n^m$  it was mentioned previously that they are homogeneous for  $m \geq 2n + 1$ , [3], and they are not homogeneous for each  $n \in \mathbb{N}$  and  $m \in \{1, 2, \dots, 2n\}$ , [75]. In connection with the latter result recall the following problem (see [24, Problem 3, p. 11]).

**Problem 4.17.** For what classes  $\mathfrak{M}$  of mappings the intermediate Menger compacta  $M_n^m$  for each  $n \in \mathbb{N}$  and  $m \in \{1, 2, \dots, 2n\}$  are  $\mathfrak{M}$ -homogeneous?

One of Effros' theorems, [42, Theorem 2.1, p. 39] was used by G. S. Ungar [104] and C. L. Hagopian [49] and [50] to study homogeneous continua. Namely it has been shown (see [104, (1), p. 397] and [50, Lemma 4, p. 37]) that these continua have a property, called later the  $\varepsilon$ -push property or the *Effros' property* (where  $\rho$  denotes a metric on a homogeneous continuum  $X$ ): for each  $\varepsilon > 0$  and for each point  $x \in X$  there is  $\delta > 0$  such that for every two points  $y, z$  of a  $\delta$ -neighborhood of  $x$  there exists a homeomorphism  $h$  of  $X$  onto  $X$  satisfying  $h(y) = z$  and  $\rho(v, h(v)) < \varepsilon$  for all points  $v \in X$ . In other words, if  $G$  is the group of all homeomorphisms of  $X$  onto itself, then for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that for any two points  $x, y \in X$  with  $\rho(x, y) < \delta$  there is a homeomorphism  $g \in G$  which is  $\varepsilon$ -close to the identity, and such that  $g(x) = y$ . Because of many important applications of this powerful result, it is very attractive to have an analog of the Effros theorem for other classes  $\mathfrak{M}$  of mappings. Some success has been achieved for open mappings. It is shown in [34, Proposition 5.10, p. 590] that if a continuum  $X$  is open-homogeneous, then for each subcontinuum  $K$  of  $X$  with the nonempty interior, and for each

open subset  $V$  containing  $K$ , there exists a subcontinuum  $L$  of  $X$  such that  $K \subset \text{int} L \subset V$  and  $L \subset \text{cl} V$ . To get this result a version of the Effros property is proved for the class of open mappings, [34, Theorem 5.9, p. 509].

**Theorem 4.18.** *Let  $X$  be a compact metric space and let  $\mathfrak{D} \subset X^X$  denote the class of all open mappings of  $X$  onto itself, where the space  $X^X$  of all surjective mappings is equipped with the compact-open topology. If  $X$  is  $\mathfrak{D}$ -homogeneous, then for each pair of points  $x, y \in X$  there exists  $f \in \mathfrak{D}$  such that  $y = f(x) \in (T_x(\mathcal{H}))^*$  for each  $\mathcal{H}$  open in  $\mathfrak{M}$  with  $f \in \mathcal{H}$  (here  $T_x(\mathcal{H}) = \{z \in X : z = h(x) \text{ for some } h \in \mathcal{H}\}$ , and  $(T_x(\mathcal{H}))^*$  denotes its quasi-interior, see [42, p. 39] or [34, (3.1), p. 582]).*

There is considerably more material on other classes of mappings in [34], but the possible extensions of the Effros property to these classes are rather subjects of a further study.

## 5. GENERALIZED $n$ -HOMOGENEITY

Given a natural number  $n$  and a class  $\mathfrak{M}$  of mappings, a space  $X$  is said to be  $\mathfrak{M}$ - $n$ -homogeneous provided that for every pair  $A, B$  of  $n$ -element subsets of  $X$  there is a mapping  $f \in \mathfrak{M}$  of  $X$  onto itself which maps  $A$  onto  $B$ . If this condition holds for  $n$ -element sequences in place of sets, then  $X$  is said to be *strongly  $\mathfrak{M}$ - $n$ -homogeneous*.

As it was mentioned in Theorem 3.2 above, each 2-homogeneous metric continuum is locally connected, [104, Theorem 3.12, p. 397]. A simpler proof of even more general result has been presented in [109]. On the basis of that paper the following results were recently obtained in [32].

**Theorem 5.1.** *Let a metric, complete, separable, m.o.-homogeneous space  $X$  satisfy the condition*

- (5.1.1) *for every two points  $x, y \in X$  there is a point  $z \in X$  such that for each  $\varepsilon > 0$  there is a monotone and open mapping  $f : X \rightarrow X$  and a continuum  $D$  contained in an open  $\varepsilon$ -neighborhood of  $z$  such that  $f(x), f(y) \in D$ .*

*Then  $X$  is locally connected.*

**Theorem 5.2.** *Let a metric, compact m.o.-homogeneous space  $X$  be such that each pair of points of  $X$  can be mapped under monotone and open mappings from  $X$  to  $X$  into connected sets of arbitrarily small diameters. Then  $X$  is locally connected.*

**Corollary 5.3.** *Each m.o.-2-homogeneous metric continuum is locally connected.*

**Question 5.4.** Is every a) monotone-2-homogeneous, b) open-2-homogeneous metric continuum locally connected?

The concept of generalized  $n$ -homogeneity has been defined very recently, and therefore there are a few results only related to this notion. Similarly as for generalized homogeneity, an extensive research program would be to verify which results concerning  $n$ -homogeneity or  $1/n$ -homogeneity of certain spaces can be modified so that the homeomorphisms are replaced by other mappings.

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