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ON s-IMAGES OF METRIC SPACES

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ABSTRACT. In this paper we study the relations among closed k-networks, countably compact k-networks and compact k-networks and construct an example which gives a negative answer to Question 37 of B of the Problem Section in [5].

1. INTRODUCTION

We assume that all spaces are regular T_1 and all maps are continuous and onto.

After A. V. Arkhangelskiǐ's paper [1], maps as an important object is studied. G. Gruenhage, E. Michael and Y. Tanaka in [3] discussed quotient maps and also showed relations between k-networks and quotient s-images of metric spaces. In this paper, we use maps as a tool to study the relations among closed k-networks, countably compact k-networks and compact k-networks, also to study the following question which was raised by S. Lin [6] and was arranged as Question 37 of B of the Problem Section in [5]:

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Question 37. Suppose a space X has a point-countable closed k-network. Is X a space with a point-countable compact k-network if every first countable closed subspace of X is locally compact?

Recall that a collection $\mathcal{C}$ of subsets of a space X is a *k-network* for X if, whenever $C \subset U$ with C compact and U open in X, then $C \subset \cup\{F : F \in \mathcal{F}\} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{C}$. If $\mathcal{C}$ is a k-network for Y, then $\mathcal{C}$ is a closed (countably compact, compact) k-network if each $C \in \mathcal{C}$ is closed (countably compact, compact) in Y.

2. CLOSED k-NETWORKS AND COUNTABLY COMPACT k-NETWORKS

Lemma 2.1. *Let M be a metric space and $f : M \to Y$ be a continuous map. If each metric closed subset of Y is locally compact, then for each point-countable base $\mathcal{B}$ of M, there is a base $\mathcal{B}' \subset \mathcal{B}$ such that $\overline{f(B)}$ is countably compact for each $B \in \mathcal{B}'$.*

Proof: Let $\mathcal{B}$ be a point-countable base of M and $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\} = \{B_n : n \in \omega\} \subset \mathcal{B}$. Then:

1. $\cap_{n\in\omega} f(B_n) = \{f(x)\}$.
2. There is a $B_n \in \mathcal{B}(x)$ such that $\overline{f(B_n)}$ is countably compact.

Indeed, let $y = f(x)$. Suppose $\cap_{i\leq n}\overline{f(B_i)}$ is not countably compact for each $n \in \omega$.

If $\overline{f(B_1)}$ is not countably compact, then there is a discrete closed subset $D_1 = \{y_{1m} : m \in \omega\} \subset \overline{f(B_1)}$ such that $|D_1| = \aleph_0$ and y is not in D_1. Take an open neighborhood O_1 of y in Y such that $\overline{O_1} \cap D_1 = \emptyset$ since Y is a regular T_1 space. Then $f^{-1}(O_1)$ is open in M with $x \in f^{-1}(O_1)$. So there is a $B_{n_1} \in \mathcal{B}(x)$ with $x \in B_{n_1} \subset f^{-1}(O_1)$ and $1 < n_1$. Then $\cap_{i\leq n_1}\overline{f(B_i)}$ is not countably compact by supposition. So there is a discrete closed subset $D_2 = \{y_{2m} : m \in \omega\} \subset \cap_{i\leq n_1}\overline{f(B_i)}$

by $|\cap_{i\leq n_1}\overline{f(B_i)}| \geq \aleph_0$ such that $|D_2| = \aleph_0$, $D_2 \cap D_1 = \emptyset$ and y is not in D_2.

Then, by induction, there is an $n_i \in \omega$ with $n_i < n_{i+1}$ and a discrete closed subset $D_i = \{y_{im} : m \in \omega\} \subset \cap_{j\leq n_i}\overline{f(B_j)}$ for each $i < \omega$ such that $|D_i| = \aleph_0$, $D_i \cap D_j = \emptyset (i \neq j)$ and y is not in D_i. Let $Y_1 = (\cup_{n\in\omega}D_n) \cup \{y\}$. Then Y_1 is a closed metric subspace of Y and is not locally compact, a contradiction. Hence $\cap_{i\leq n}\overline{f(B_i)}$ is countably compact for some $n \in \omega$.

Since $\mathcal{B}(x)$ is a neighborhood base of x in M, then there is a $B_m \in \mathcal{B}(x)$ with $B_m \subset \cap_{i\leq n}B_i$. Since $f(\cap_{i\leq n}B_i) \subset \cap_{i\leq n}f(B_i)$, then $\overline{f(B_m)} \subset \overline{f(\cap_{i\leq n}B_i)}$ is countably compact. Let $\mathcal{B}^*(x) = \{B \in \mathcal{B}(x) : B \subset B_m\}$ and $\mathcal{B}' = \cup\{\mathcal{B}^*(x) : x \in X\}$. Then $\mathcal{B}'$ is a base of M such that $\overline{f(B)}$ is countably compact for each $B \in \mathcal{B}'$.

Recall that a cover $\mathcal{F} = \{B_n : n \in F\}$ of Y is called *irreducible*, if $\cup\{B_n : n \in F_0\} \neq Y$ for each proper subset F_0 of F.

Theorem 2.2. *Let Y have a point-countable closed k-network $\mathcal{B}$ which is closed under finite intersections. Then there is a metric space M and a continuous onto map $f : M \to Y$ such that f is a compact-covering s-map.*

Proof: Let $\mathcal{B}$ be a point-countable closed k-network of Y such that $\mathcal{B}$ is closed under finite intersections. Giving $\mathcal{B}$ the discrete topology, the countable product $\Pi_{n>0}\mathcal{B}$ is a metric space. Pick $x' = (B_i')$ and $x'' = (B_i'')$ from $\Pi_{n>0}\mathcal{B}$. Let $d^*(x', x'') = 1/n$ if n is the first index with $B_i' \neq B_i''$. Then d^* is a metric of $\Pi_{n>0}\mathcal{B}$. Let $M \subset \Pi_{n>0}\mathcal{B}$ be all (B_n) such that there is a $y \in Y$ with $\cap_{n>0}B_n = \{y\}$ and every neighborhood of y contains some B_n. Let $f : M \to Y$ such that, for each $(B_n) \in M$, $f((B_n)) = y$ if $\cap_{n>0}B_n = \{y\}$. We may show that f is an onto continuous s-map just as the proof of Theorem 6.1 of [3]. Let $\mathcal{C}_n = \{C \subset M : C = (\{B_1\} \times \{B_2\} \times ... \times \{B_n\} \times \Pi_{j>n}\mathcal{B}) \cap M$ and $B_i \in \mathcal{B}$ for each $i \leq n\}$. Let $\mathcal{C} = \cup_{n>0}\mathcal{C}_n$. Then $\mathcal{C}$ is a σ-discrete base of M. In the following proof, we show that $f : M \to Y$ is a compact-covering map.

Let K be a compact subset of Y. Then K is a metric subset of Y by Theorem 3.3 in [3]. If K is a finite subset of Y, then there is a finite subset C of M with $f(C) = K$. So we assume that K is infinite in the following proof. Let $\mathcal{F}$ be a finite subcollection of $\mathcal{B}$ which is an irreducible cover of K, and let $\mathcal{F}(y) = \{F \in \mathcal{F} : y \in F\}$.

Claim 1. *If $y \in K$ and O is an open neighborhood of y in Y, then there is a finite subcollection $\mathcal{F}$ of $\mathcal{B}$ which is an irreducible cover of K such that $\cup\mathcal{F}(y) \subset O$.*

Proof: Let $y \in W \subset \overline{W} \subset O \cap K$, where W is open in K. There is a finite irreducible cover $\mathcal{F}_1 \subset \mathcal{B}$ of $\overline{W}$ with $\cup\mathcal{F}_1 \subset O$. For each $F \in \mathcal{F}_1$, choose a point $x_F \in F \cap W \setminus \cup\{F' : F' \in \mathcal{F}_1, F' \neq F\}$. Note that such x_F can be chosen in W, since members of $\mathcal{F}_1$ are closed. Now there is a finite irreducible cover $\mathcal{F}_2 \subset \mathcal{B}$ of $K \setminus W$ with $\cup\mathcal{F}_2 \subset X \setminus (\{x_F : F \in \mathcal{F}_1\} \cup \{y\})$. Then $\mathcal{F}_1 \cup \mathcal{F}_2$ contains an irreducible cover $\mathcal{F}$ of K. It is easy to check that $\mathcal{F}$ has the desired properties.

Claim 2. $|\{\mathcal{F} \subset \mathcal{B} :$ *$\mathcal{F}$ is an irreducible finite cover of K* $\}| = \aleph_0$.

Proof: Mišcenko [4] proved that if $\mathcal{B}$ is a point-countable cover of K, then there are *at most* countably many finite subcollections of $\mathcal{B}$ which are irreducible covers of K.

Because K is infinite, there is a cluster point $y \in K$. Let d be a metric of K. Let $U_n = \{y' \in K : d(y, y') < 1/n\}$ for each $n > 0$. Let O_n be an open set of Y with $K \cap O_n = U_n$ for each $n > 0$. Then there is a finite subcollection $\mathcal{F}_n$ of $\mathcal{B}$ which is an irreducible cover of K such that $\cup\mathcal{F}_n(y) \subset O_n$ by Claim 1 for each $n > 0$. So there must be *at least* countably infinitely many finite subcollections of $\mathcal{B}$ which are irreducible covers of K.

Let $(\mathcal{F}_n)$ enumerate all finite subcollections of $\mathcal{B}$ which are irreducible covers of K. Then $\Pi_{n>0}\mathcal{F}_n$ is a compact subset of $\Pi_{n>0}\mathcal{B}$. Let $D = (\Pi_{n>0}\mathcal{F}_n) \cap M$.

Claim 3. $f(D) = K$.

Proof: Pick an $x = (B_n) \in D$. Suppose $f(x)$ is not in K. Then $Y - K$ is an open neighborhood of $f(x)$. So there is a B_n in (B_n) with $f(x) \in B_n \subset Y - K$ by the definition of the subspace M. Assume $B_n \in \mathcal{F}_n$. Then $\mathcal{F}_n - \{B_n\}$ is still a cover of K. It is a contradiction.

Pick a $y \in K$. Then for each $n > 0$, there is a $B'_n \in \mathcal{F}_n$ with $y \in B'_n$. Let $x = (B'_n)$. Then $x \in \Pi_{n>0}\mathcal{F}_n$ and $y \in \cap_n B'_n$. Pick an open set $O \subset Y$ with $y \in O$. Then, by Claim 1, there is an irreducible finite cover $\mathcal{F}$ of K such that $\cup\mathcal{F}(y) \subset O$. Since $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then there is an $\mathcal{F}_n = \mathcal{F}$. So B'_n is in $\mathcal{F}$. Then $B'_n \in \mathcal{F}(y)$ by $y \in B'_n$. Then $B'_n \subset \cup\mathcal{F}(y) \subset O$. This implies $x \in M$ and $f(x) = y \in f(D)$.

Claim 4. *D is a compact subset of* $\Pi_{n>0}\mathcal{F}_n$.

Proof: One may assume $x = (B_n) \in \Pi_{n>0}\mathcal{F}_n$ since $\Pi_{n>0}\mathcal{F}_n$ is closed. Then $\cap_{i\leq n} B_i \cap K \neq \emptyset$ for every n, since $x \in \overline{D}$. So, as K is compact, there is $y \in \cap_{n>0} B_n$. Let $O \subset Y$ be an open set with $y \in O$. Then, by Claim 1, there is an irreducible finite cover $\mathcal{F}$ of K such that $\cup\mathcal{F}(y) \subset O$. Since $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then there is an $\mathcal{F}_n = \mathcal{F}$. So $B_n \in \mathcal{F}_n = \mathcal{F}$. Then $B_n \in \mathcal{F}(y)$ since $y \in B_n$. Then $y \in B_n \subset \cup\mathcal{F}(y) \subset O$. This implies $x = (B_n) \in D$. So D is a closed subset of compact metric set $\Pi_{n>0}\mathcal{F}_n$.

Proof of Theorem 2.2. (*continued*) If K is an infinite compact subset of Y, then there must be countably infinitely many finite subcollections of $\mathcal{B}$ which are irreducible covers of K by Claim 2. If $(\mathcal{F}_n)$ enumerates the all finite subcollections of $\mathcal{B}$ which are irreducible covers of K, then $D = (\Pi_{n>0}\mathcal{F}_n) \cap M$ is a compact subset of M by Claim 4. Then $f(D) = K$ by Claim 3. So $f : M \to Y$ is a compact-covering map.

Theorem 2.3. *Let Y have a point-countable closed k-network. If each metric closed subset of Y is locally compact, then Y has a point-countable countably compact k-network.*

Proof: Let $\mathcal{B}$ be a point-countable closed k-network. Let $\mathcal{B}_1$ be the collection of all finite intersections of $\mathcal{B}$. Then $\mathcal{B}_1$ is a point-countable closed k-network which is closed under finite intersections. So we may assume that Y has a point-countable closed k-network $\mathcal{B}$ which is closed under finite intersections.

Let $M \subset \Pi_{n>0}\mathcal{B}$ be the metric space, $\mathcal{C} = \cup_{n>0}\mathcal{C}_n$ be the σ-discrete base of M and $f : M \to Y$ be the onto continuous compact-covering s-map as in Theorem 2.2 above. Then there is a subcollection $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is a base of M and $\overline{f(C)}$ is countably compact for each $C \in \mathcal{C}'$ by Lemma 2.1. If $C = (\{B_1\} \times \{B_2\} \times ... \times \{B_n\} \times \Pi_{j>n}\mathcal{B}) \cap M \in \mathcal{C}'$, then $f(C) = \cap_{i\leq n} B_i$ is closed. So $f(C) = \overline{f(C)}$ is countably compact. Then $\{f(C) : C \in \mathcal{C}'\}$ is a point-countable collection of countably compact subsets of Y. We have proved that $f : M \to Y$ is a compact-covering map by Theorem 2.2. Then $\{f(C) : C \in \mathcal{C}'\}$ is a point-countable countably compact k-network since $\mathcal{C}'$ is a base of M.

3. COUNTABLY COMPACT k-NETWORK AND COMPACT k-NETWORKS

We would like to give a proposition about Question 37.

Proposition 3.1. *Let Y be a regular T_1 space. Then the following are equivalent.*

1. *Every metric closed subspace of Y is locally countably compact.*
2. *Every first countable closed subspace of Y is locally countably compact.*
3. *Every metric closed subspace of Y is locally compact.*

Proof: $1 \Rightarrow 2$.

Suppose that there is a first countable closed subset B of Y which is not locally countably compact. Then B contains a closed subset homeomorphic to T by Lemma 3 of [5]. Where $T = (\cup_{n\in\omega} T_n) \cup \{\infty\}$ was introduced by [5] before Lemma 3 such that the points of countably infinite set T_n are isolated for each $n \in \omega$, and a basic open set containing ∞ has the form $\{\infty\} \cup_{n\geq k} T_n$, where $n \in \omega$. Then T is a closed metric subspace of Y and is not locally countably compact, a contradiction. Hence each first countable closed subset B of Y is locally countably compact.

$2 \Rightarrow 1$.

Every metric space is a first countable space. If every first countable closed subspace of Y is locally countably compact, then every metric closed subspace of Y is locally countably compact.

$3 \Leftrightarrow 1$.

Each compact subset is countably compact and each countably compact metric subset is compact.

Proposition 3.2. *Let Y have a point-countable closed k-network. If each first countable closed subset of Y is locally countably compact and each countably compact subset is compact, then Y has a point-countable compact k-network.*

Proof: If every first countable closed subspace of Y is locally countably compact, then every metric closed subspace of Y is locally compact by Proposition 3.1. So Y has a point-countable countably compact k-network by Theorem 2.3. Then Y has a point-countable compact k-network since each countably compact subset is compact.

In the following, we give an example which shows "countably compact "in Theorem 2.3 can not be strengthened to "compact "and the condition "each first countable closed subset of Y is locally countably compact and each countably compact subset is compact "in Proposition 3.2 can not be omitted as "each first countable closed subset of Y is locally compact ". It also gives a negative answer to Question 37 of B of the Problem Section in [6].

Example 3.3. *There is a regular T_1 countably compact space Y such that Y has a point-countable closed k-network and every first countable closed subspace of Y is compact, but Y has no point-countable compact k-network.*

Construction.

Recall Example 9.1 in [3]. There is an infinite, completely regular countably compact space X, all of whose compact subsets are finite. Therefore $\{\{x\} : x \in X\}$ is a point-countable closed k-network for X. Let $Y = \{\infty\} \cup \bigcup_{n\in\omega} X_n$, where $\{X_n : n \in \omega\}$ is a collection of disjoint clopen copies of X, and $\{U_n = \{\infty\} \cup (\bigcup_{m>n} X_m) : n \in \omega\}$ is a neighborhood base of point ∞ in Y. Then Y is countably compact since every infinite subset of Y either meets some X_n in an infinite set, or else ∞ is a limit point. Also, it is easy to see that $\mathcal{P} = \{\{x\} : x \in \cup_{n\in\omega} X_n\} \cup \{U_n : n \in \omega\}$ is a point-contable closed k-network consisting of countably compact sets.

Claim 5. *Each first countable closed subspace of Y is locally compact.*

Proof: Let B be a first countable closed subspace of Y. Suppose that $B \cap X_n = B \cap X$ is not discrete. Then $B \cap X$ must contain a convergence sequence S which converges to x, since $B \cap X$ is first countable. Then $S \cup \{x\}$ is an infinite compact subset of X, a contradiction. So $B \cap X$ is a discrete closed subspace of X. Then the discrete countably compact closed subspace $B \cap X_n$ is compact in X_n. So B is compact in Y.

Claim 6. *Y has no point-countable compact k-network.*

Proof: Suppose that $\mathcal{C}$ is a point-countable compact k-network. Let $\mathcal{C}(\infty) = \{C \in \mathcal{C} : \infty \in C\} = \{C_n : n \in \omega\}$ by $\mathcal{C}$ point-countable. Then $C_n \cap X_m$ is finite for $m, n \in \omega$. So $\cup\mathcal{C}(\infty)$ is a countable subset of Y. Pick an $x_n \in X_n - \cup\mathcal{C}(\infty)$ for each $n \in \omega$ since X_n is uncountable. Let $S = \{x_n : n \in \omega\}$. Then $S \cup \{\infty\}$ is compact. So there is a finite subcollection $\mathcal{F} \subset \mathcal{C}(\infty)$ and an $n' \in \omega$ with $\{x_n : n \geq n'\} \subset \cup\mathcal{F} \subset \cup\mathcal{C}(\infty)$.

It is a contradiction to $S \cap (\cup \mathcal{C}(\infty)) = \emptyset$. So Y has no point-countable compact k-network.

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