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SOME EXAMPLES OF MI-SPACES AND OF SI-SPACES

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ABSTRACT. An MI-space is a crowded space in which every dense subset is open, and an SI-space is a crowded space in which every nonempty subset is irresolvable. E. Hewitt [He] showed that every MI-space is SI. Here, we present a Tychonoff SI-space with dispersion character $\alpha \geq \omega$ which is not MI, for every infinite cardinal α. We also show that if a Tychonoff space contains a non-closed discrete subset, then there is a maximal Tychonoff extension of the topology which is not MI. This provides a general method to construct maximal Tychonoff spaces which are not MI.

0. INTRODUCTION

The topological spaces considered in this paper will be Tychonoff without isolated points (crowded). For a set X, the family of all Tychonoff topologies on X is denoted by $\mathcal{TY}(X)$. The *dispersion character* of a nonempty space X is the cardinal

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number defined by $\Delta(X) = \min\{|U| : \emptyset \neq U \subseteq X$ and U is open$\}$, and the *nowhere density number* of a space X is $nwd(X) = \min\{|A| : A$ is not nowhere dense in $X\}$. The dispersion character was introduced in [He] and the nowhere density number in [CG2]. For every space X, we have that $nwd(X) \leq \Delta(X) \leq |X|$. If $\mathcal{P}$ is a topological property, then a space (X, τ) is said to be $\mathcal{P}$-maximal if for every topology τ' on X finer than τ we have that either (X, τ') has an isolated point or (X, τ') does not have property $\mathcal{P}$. Using Zorn's Lemma, one may show that every (crowded) topology extends to a maximal (crowded) topology, and that every Tychonoff topology extends to a maximal Tychonoff topology (for details and related results see [He], [vD] and [CG1]).

A space is *irresolvable* if it does not contain two disjoint dense subsets, and a space that is not irresolvable is called *resolvable*. E. Hewitt [He] introduced and studied these two classes of spaces. Many of the familiar spaces fail to be irresolvable (for instance, k-spaces [Py] and countably compact spaces [CG1]), and several known examples of irresolvable spaces are constructed via maximal topologies (see [He]). In [He], Hewitt called a space X an *MI-space* (respectively, *SI-space*) if X is crowded and every dense subset is open (respectively, every nonempty subspace is irresolvable). An MI-space is simply called MI and an SI-space is simply called SI. The MI-spaces, not necessary being crowded, were also introduced by Bourbaki [Bo] with the name *submaximal spaces*.

The following relationships were established by Hewitt [He].

1. [He, Th. 23] Every MI-space is an SI-space.
2. [He, Th. 26] Every maximal Tychonoff space is an SI-space.

In this paper, we give the following examples:

1. A Tychonoff SI-space of dispersion character α which is not MI, for every cardinal number $\alpha \geq \omega$.
2. A maximal Tychonoff space which is not MI.

3. A Tychonoff MI-space which is not maximal Tychonoff.

The first listed example is new. Although it is not explicitly stated in his paper, one may check that the van Douwen's example [vD, Ex. 1.9] serves as the second example. Here, we construct such an example in a general way. The third example is almost trivial and it is known to some topologists, but the authors did not find it in the literature. For the sake of completeness, we believe that this paper is the right place to write this example. We shall prove general results (Theorems 1.5, 1.8 and 1.9) and the three examples will follow directly from them. One may see that the second example is also an example of a maximal Tychonoff space which is not maximal, as is van Douwen's example mentioned above.

1. The Examples

The following two lemmas are taken from [Fe].

Lemma 1.1. *For every cardinal $\alpha \geq \omega$, there exists an SI-space X such that $nwd(X) = \Delta(X) = |X| = \alpha$.*

Lemma 1.2. *Let $\{\tau_i : i \in I\}$ be a chain of topologies on a set X (ordered by the usual set-theoretical inclusion), and let τ be the topology on X generated by the chain, that is, τ has $\bigcup_{i \in I} \tau_i$ as a base. If N is nowhere dense in (X, τ_i) for every $i \in I$, then N is nowhere dense in (X, τ).*

It follows from Lemma 1.2 that if $\{\tau_i : i \in I\}$ is a chain of topologies on X and τ is the topology generated by this chain, then

$$nwd(X, \tau) = \min\{nwd(X, \tau_i) : i \in I\}.$$

The method of extending a Tychonoff topology to another Tychonoff topology by using a function (not necessarily continuous) $f : X \to \mathbf{R}$ was considered in [CG1]. The next lemma is the main tool in this technique.

Notation. For a space $X = (X, \tau)$ and a function $f : X \to \mathbf{R}$, the symbol τ_f denotes the topology on X induced by τ and

f, that is, the topology in which $\tau \cup \{f^{-1}(V) : V \text{ is open in } \mathbf{R}\}$ is a subbase.

Lemma 1.3. [CG1] *Let X be a set and let $f : X \to \mathbf{R}$ be a function.*

1. *If $\tau \in \mathcal{TY}(X)$, then $\tau_f \in \mathcal{TY}(X)$; and*
2. *if every $W \in \tau$ and every open $V \subseteq \mathbf{R}$ satisfy either $W \cap f^{-1}(V) = \emptyset$ or $|W \cap f^{-1}(V)| \geq \omega$, then (X, τ_f) is crowded.*

Lemma 1.4. *If X is a topological space such that $X = A \cup B$, where A and $\overline{B}$ are both SI subspaces, then X is also an SI-space.*

Proof: Suppose that R is a nonempty resolvable subset of X. Then, $R \cap (X - \overline{B})$ is an open resolvable subset of R. Since $R \cap (X - \overline{B}) \subseteq A$ and A is SI, $R \cap (X - \overline{B}) = \emptyset$ and then $R \subseteq \overline{B}$, which contradicts the assumption that $\overline{B}$ is an SI subspace. $\square$

The first example follows directly from the next theorem.

Theorem 1.5. *For every infinite cardinal α, there is an SI-space X which is not an MI-space, and $nwd(X) = \Delta(X) = |X| = \alpha$.*

Proof: Fix $\alpha \geq \omega$. Our first claim is the following.

Claim 1. There is a space Y with $nwd(Y) = \Delta(Y) = |Y| = \alpha$ that contains a nowhere dense, closed and crowded subset N which is an SI subspace of Y.

Proof of Claim 1. By Lemma 1.1 there is an SI-space Z such that $nwd(Z) = \Delta(Z) = |Z| = \alpha$. If $Y = \mathbf{Q} \times Z$, where $\mathbf{Q}$ is the space of all rational numbers, then $nwd(Y) = \Delta(Y) = |Y| = \alpha$, and $N = \{0\} \times Z$ is a nowhere dense, closed and crowded subset of Y which is SI. This shows Claim 1.

Fix the space (Y, σ) and an SI subspace N of Y as in Claim 1. Now, we consider the family $\mathcal{C}$ of all $\tau \in \mathcal{TY}(Y)$ such that

1. $\sigma \subseteq \tau$;

2. $nwd(Y, \tau) = \Delta(Y, \tau) = |Y| = \alpha$;
3. N is nowhere dense in (Y, τ); and
4. $(N, \tau|_N)$ is crowded.

Claim 2. $\mathcal{C}$ has a maximal element in the usual set-theoretical inclusion order.

Proof of Claim 2. We have that $\sigma \in \mathcal{C}$. Let $\mathcal{A}$ be a chain in $\mathcal{C}$ and let τ be the topology on Y generated by $\bigcup \mathcal{A}$. It is evident that $\sigma \subseteq \tau$. By Lemma 1.2, N is nowhere dense in (Y, τ) and $nwd(Y, \tau) = \min\{nwd(Y, \tau_i) : i \in I\} = \alpha$. Since $nwd(Y, \tau) \leq \Delta(Y, \tau) \leq |Y| = \alpha$, we must have $nwd(Y, \tau) = \Delta(Y, \tau) = |Y| = \alpha$. To prove that $(N, \tau|_N)$ is crowded, we let $V \in \tau$ such that $\emptyset \neq V \cap N$. Then, there is $\tau' \in \mathcal{A}$ and $W \in \tau'$ such that $\emptyset \neq W \cap N \subseteq V \cap N$. Since $(N, \tau'|_N)$ is crowded, $\omega \leq |W \cap N| \leq |V \cap N|$. So, $\tau \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element.

In what follows, we fix a maximal element τ of $\mathcal{C}$.

Claim 3. Every nonempty open subset of $(Y - N, \tau|_{Y-N})$ is irresolvable.

Proof of Claim 3. Suppose the contrary. Then, there is a nonempty open resolvable subset E of $(Y - N, \tau|_{Y-N})$. Since N is closed in (Y, τ) (in fact, it is σ-closed), by the regularity of Y, we may find $U \in \tau$ such that $\emptyset \neq U \subseteq \overline{U}^{\tau} \subseteq E \subseteq Y - N$. Put $F = \overline{U}^{\tau}$ and equip F with the subspace topology from (Y, τ). Then, F is resolvable and hence $F = D_0 \cup D_1$, where D_0 and D_1 are disjoint and dense in F. Notice that $nwd(F) = \Delta(F) = |F| = \alpha$, hence $|D_0| = |D_1| = \alpha$. Let us consider the topology τ_f on Y induced by the function $f : Y \to \mathbf{R}$ defined by $f(x) = 0$ for any $x \in D_0$; $f(x) = 1$ for any $x \in D_1$ and $f(x) = 2$ for any $x \in Y - F$. We have that the family $\tau \cup \{W \cap D_i : i = 0, 1, W \in \tau\}$ is a base for the topology τ_f. By Lemma 1.3, $\tau_f \in \mathcal{TY}(Y)$. To get a contradiction, it suffices to show that $\tau_f \in \mathcal{C}$ since $\tau_f \neq \tau$. Indeed, it is evident that $\sigma \subseteq \tau_f$. Assume that $A \subseteq Y$ has size $< \alpha$ and there is $V \in \tau_f$ such that $\emptyset \neq V \subseteq \overline{A}^{\tau_f}$. Since A is nowhere dense in (Y, τ) and $\overline{A}^{\tau_f} \subseteq \overline{A}^{\tau}$, $V = W \cap D_i$ for some $W \in \tau$ and some

$i \in \{0,1\}$. Since D_i is dense in $(F, \tau|_F)$ and $F = \overline{U}^{\tau}$, $\emptyset \neq U \cap W \subseteq \overline{U \cap W \cap D_i}^{\tau} \subseteq \overline{W \cap D_i}^{\tau} \subseteq \overline{A}^{\tau}$, which means that A is not nowhere dense in (Y, τ) and $|A| < \alpha$, a contradiction. This shows that $nwd(Y, \tau_f) = \Delta(Y, \tau_f) = |Y| = \alpha$. If N is not nowhere dense in (Y, τ_f), then there is $V \in \tau_f$ such that $\emptyset \neq V \subseteq \overline{N}^{\tau_f} \subseteq \overline{N}^{\tau} = N$. On the other hand, since $N \subseteq Y - F$ and $\tau_f|_{Y-F} = \tau|_{Y-F}$, we must have that $V \in \tau$, which contradicts the hypothesis N is nowhere dense in (Y, τ). Thus, N is nowhere dense in (Y, τ_f). Now, we shall verify that $(N, \tau_f|_N)$ is crowded. Given $y \in N$ and a neighborhood $V_y \in \tau_f$ of y, there is $W \in \tau$ such that $y \in W \cap (Y - F) \subseteq V_y$. Since $W \cap (Y - F) \in \tau$, $\omega \leq |W \cap (Y - F) \cap N| \leq |V_y \cap N|$. So, $(N, \tau_f|_N)$ does not have any isolated points. Thus, we have proved that $\tau_f \in \mathcal{C}$. But this is impossible since D_0, $D_1 \in \tau_f - \tau$. Therefore, every nonempty open subset of $(Y - N, \tau|_{Y-N})$ is irresolvable.

By Theorem 3.7(b) of [FM], the space $(Y - N, \tau|_{Y-N})$ contains an open dense SI subset S. Our space will be the subspace $X = S \cup N$ of (Y, τ). Since N is closed and nowhere dense in (Y, τ), $S \in \tau$ and S is dense in $(X, \tau|_X)$. To show that $nwd(X) = \Delta(X) = |X| = \alpha$, we fix $A \subseteq X$ with $|A| < \alpha$. Then, $A \cap N$ is nowhere dense in $(X, \tau|_X)$. Since $nwd(Y, \tau) = \alpha$ and $S \in \tau$, $A \cap S$ is also nowhere dense in $(X, \tau|_X)$. Hence, A is a nowhere dense subset of $(X, \tau|_X)$. This proves the equality $nwd(X, \tau|_X) = \Delta(X, \tau|_X) = |X| = \alpha$. It remains to show the following.

Claim 4. $(X, \tau|_X)$ is SI-space but not MI.

Proof of Claim 4. It is evident that $(N, \tau|_N)$ is an SI-space, since $(N, \sigma|_N)$ is so and $\sigma \subseteq \tau$. Thus, $(X, \tau|_X)$ is the union of two SI subspaces. Since N is closed, by Lemma 1.4, $(X, \tau|_X)$ is SI. Finally, we shall verify that $(X, \tau|_X)$ cannot be MI. Pick $y \in N$. Since N is nowhere dense in $(X, \tau|_X)$, $(X - N) \cup \{y\}$ is dense in (X, τ). But, $(X - N) \cup \{y\}$ cannot be open since $(N, \tau|_N)$ has no isolated points. Therefore, $(X, \tau|_X)$ is not an MI-space. $\square$

A maximal Tychonoff space which is not maximal was given in [vD, Ex. 1.9]. We shall give a condition on a space (X, τ) which guarantees that τ has a maximal Tychonoff extension that fails to be MI. Thus spaces with such maximal Tychonoff topologies are not maximal. Some basic facts from [vD] will be useful in this task. E. K. van Douwen [vD] called a crowded space X *ultradisconnected* if every nonempty proper subset $A \subset X$ is clopen iff both A and $X - A$ are crowded. He used this concept to characterize the maximal regular spaces. He showed [vD, Th. 1.8] that a space X is maximal regular iff X is regular and ultradisconnected. He also proved that every maximal regular space is extremally disconnected and hence zero-dimensional. Hence we have the following lemma.

Lemma 1.6. *1. Every maximal regular extension of a Hausdorff topology is a maximal Tychonoff topology.*

2. Every maximal Tychonoff topology is maximal regular.

The following characterization of an ultradisconnected MI space can be deduced directly from Fact 1.15, Theorem 2.2 of [vD], and Theorem 24 of [He].

Lemma 1.7. *For an ultradisconnected space X, the following are equivalent.*

1. *X is MI.*
2. *Every discrete subset of X is closed.*

Theorem 1.8. *If (X, σ) is a Tychonoff space that contains a non-closed discrete subset, then there is a maximal Tychonoff extension of σ which is not MI.*

Proof: Suppose that (X, σ) is a Tychonoff space that contains a discrete subset $N \subseteq X$ and a point $x \in X$ such that $x \in \overline{N}^{\sigma} - N$. Let $\mathcal{D}$ be the family of all $\tau \in TY(X)$ such that

1. $\sigma \subseteq \tau$; and
2. $x \in \overline{N}^{\tau} - N$.

First, we observe that if $\sigma \subseteq \tau$ and (X, τ) is crowded, then $int_{(X,\tau)}N = \emptyset$, since N is discrete in (X, σ). It is clear that

$\sigma \in \mathcal{D}$ and if $\mathcal{A}$ is a chain in $\mathcal{D}$, then the topology on X generated by the chain is again an element of $\mathcal{D}$. By Zorn's Lemma, $\mathcal{D}$ has a maximal element. Let us still use τ to denote this maximal element of $\mathcal{D}$. We will show that τ is a maximal regular extension of σ. Suppose the contrary. By van Douwen's characterization, (X, τ) is not ultradisconnected, that is, there is $A \subseteq X$ such that A is not τ-clopen and both A and $X - A$ are τ-crowded. Without loss of the generality, we may assume that $x \in A$. Since $x \in \overline{N}^{\tau} - N$, it is enough to consider the following two cases:

Case 1. $V \cap (A - \{x\}) \cap N \neq \emptyset$ for every neighborhood $V \in \tau$ of x.

Define $f : X \to \mathbf{R}$ by $f(x) = 0$ for any $x \in A$, and $f(x) = 1$ for any $x \in X - A$. By Lemma 1.3, τ_f is a Tychonoff extension of τ which does not have isolated points, since both A and $X - A$ are τ-crowded. A base for τ_f is $\tau \cup \{V \cap A : V \in \tau\} \cup \{V \cap (X - A) : V \in \tau\}$. It follows from our hypothesis that $x \in \overline{N}^{\tau_f}$. That is, $\tau_f \in \mathcal{D}$. But this happens only when $\tau_f = \tau$, which is impossible since $A, X - A \in \tau_f$ and neither $A \in \tau$ nor $X - A \in \tau$.

Case 2. $V \cap (X - A) \cap N \neq \emptyset$ for every neighborhood $V \in \tau$ of x.

Define $B = A - \{x\}$. We shall prove that both B and $X - B$ are τ-crowded. Since (X, τ) is a Hausdorff space, A is τ-crowded implies that $B = A - \{x\}$ is also τ-crowded, and since x is an accumulation point of $X - A$, $X - B = (X - A) \cup \{x\}$ is τ-crowded as well. So, B and $X - B$ are τ-crowded. Define $g : X \to \mathbf{R}$ by $g(x) = 0$ for any $x \in B$, and $f(x) = 1$ for any $x \in X - B$. In virtue of Lemma 1.3, τ_g is a crowded Tychonoff extension of τ, because of B and $X - B$ are both τ-crowded. By the hypothesis, we obtain that $x \in \overline{N}^{\tau_g}$. Thus, $\tau_g \in \mathcal{D}$. Since $\tau \subseteq \tau_g$ and τ is a maximal element of $\mathcal{D}$, $\tau = \tau_g$. But, this is a contradiction, since A is τ-crowded, $X - B \in \tau_g = \tau$ and $A \cap (X - B) = \{x\}$.

Therefore, (X, τ) is maximal regular and hence maximal Tychonoff. Since $x \in \overline{N}^{\tau} - N$ and N is discrete in (X, τ) (in fact, it is discrete in every finer extension of σ), by Lemma 1.7, (X, τ) cannot be an MI-space. □

The above theorem provides us a general method to construct maximal Tychonoff, non-MI spaces. We observe that E. K. van Douwen's example [vD, Ex. 1.9] mentioned above has a countable non-closed, discrete subset and, therefore, is not an MI-space.

In the last example, we require the adjunction spaces (for the definition and basic properties of adjunction spaces, we refer the reader to the books [Du] and [En]). We mainly consider the following special type of adjunction spaces.

Let (X_1, τ_1) and (X_2, τ_2) be two disjoint spaces, $p \in X_1$ and $f : \{p\} \to X_2$ be a function. Then, $X_1 \cup_f X_2$ will denote the adjunction space determined by X_1, X_2 and f. $X_1 \cup_f X_2$ is just the space obtained by identifying p with $f(p)$. The space $X_1 \cup_f X_2$ is homeomorphic to the following one.

Suppose that $X_1 \cap X_2 = \{p\}$. On the set $Z = X_1 \cup X_2$, we define a topology τ on Z as follows: $\tau_0|_{X_0 - \{p\}} \cup \tau_1|_{X_1 - \{p\}} \subseteq \tau$ and the open neighborhoods of p are $\{U \cup V : U \in \tau_0, V \in \tau_1, p \in U \cap V\}$.

It is easy to prove that (Z, τ) is homeomorphic to $X_1 \cup_f X_2$. For simplicity, we work on (Z, τ) rather than $X_1 \cup_f X_2$ and keep the notation $X_1 \cup_f X_2$ for this space. We remark that $X_1 - \{p\}$ and $X_2 - \{p\}$ are open in $X_1 \cup_f X_2$, $(X_1, \tau|_{X_1})$ and $(X_2, \tau|_{X_2})$ are closed in $X_1 \cup_f X_2$, and $(X_1, \tau|_{X_1})$ and $(X_2, \tau|_{X_2})$ are homeomorphic to (X_1, τ_1) and (X_2, τ_2), respectively.

Theorem 1.9. *Let (X_1, τ_1) and (X_2, τ_2) be two disjoint spaces, $p \in X_1$ and $f : \{p\} \to X_2$ a function. Let $X_1 \cup_f X_2$ be the adjunction space determined by X_1, X_2 and f.*

1. *If (X_1, τ_1) and (X_2, τ_2) are Tychonoff, then $X_1 \cup_f X_2$ is also Tychonoff.*
2. *If (X_1, τ_1) and (X_2, τ_2) are MI-spaces, then $X_1 \cup_f X_2$ is also MI.*
3. $\Delta(X_1 \cup_f X_2) = \min\{\Delta(X_1), \Delta(X_2)\}$.

4. *$X_1 \cup_f X_2$ is never extremally disconnected.*
5. *$X_1 \cup_f X_2$ is never maximal Tychonoff.*

Proof: 1. It is easy to verify that $X_1 \cup_f X_2$ is Hausdorff. Let V be an open subset of $X_1 \cup_f X_2$ with $p \in V$. Since $V \cap X_1$ and $V \cap X_2$ are open in X_1 and X_2 respectively, there are two continuous functions $f_1 : X_1 \to [0,1]$ and $f_2 : X_2 \to [0,1]$ such that $f_1(p) = f_2(p) = 0$, $f_1(X_1 - V) = 1$ and $f_2(X_2 - V) = 1$. Define $F : X_1 \cup_f X_2 \to [0,1]$ by $F(z) = f_1(z)$ if $z \in X_1$, and $F(z) = f_2(z)$ if $z \in X_2$. Then, F is continuous, $F(p) = 0$ and $F(X_1 \cup_f X_2 - V) = 1$. Now, let V be an open subset of $X_1 \cup_f X_2$ and let $q \in V \cap (X_1 - \{p\})$. We may find $W \in \tau_1$ such that $q \in W \subseteq V \cap (X_1 - \{p\})$. Since X_1 is completely regular there is a continuous function $g : X_1 \to [0,1]$ such that $g(q) = 0$ and $g(X_1 - W) = 1$. Define $G : X_1 \cup_f X_2 \to [0,1]$ by $G(z) = g(z)$ if $z \in X_1$, and $G(z) = 1$ if $z \in X_2$. Then G is continuous, $G(q) = 0$ and $G(X_1 \cup_f X_2 - V) = 1$. A similar argument shows that there are continuous functions to separate points in $X_2 - \{p\}$ and closed subsets of $X_1 \cup_f X_2$. Therefore, $X_1 \cup_f X_2$ is Tychonoff.

2. Suppose that D is a dense subset of (Z, τ). We will show that D is an open subset of Z. Thus (Z, τ) is an MI-space.

Case 1. $p \notin D$.

Since $X_1 - \{p\}$ and $X_2 - \{p\}$ are open in Z, we have $D \cap X_1 = D \cap (X_1 - \{p\})$ is dense in $X_1 - \{p\}$ and $D \cap X_2 = D \cap (X_2 - \{p\})$ is dense in $X_2 - \{p\}$. Thus, $D \cap X_1$ is dense in X_1 and $D \cap X_2$ is dense in X_2. Since X_1 and X_2 are MI, $D \cap X_1$ and $D \cap X_2$ are open in X_1 and X_2 respectively. So, D is open in Z.

Case 2. $p \in D$.

Since $D \cap X_1$ is dense in X_1 and X_1 is MI, $D \cap X_1$ is open in X_1. Similarly, $D \cap X_2$ is open in X_2. Then, D is open in Z.

3. This is evident.

4. It is clear that $p \in \overline{X_1 - \{p\}} \cap \overline{X_2 - \{p\}}$. Since $X_1 - \{p\}$ and $X_2 - \{p\}$ are disjoint open subsets of $X_1 \cup_f X_2$, we obtain that $X_1 \cup_f X_2$ cannot be extremally disconnected.

5. The discrete union of $X_1 - \{p\}$ and X_2 gives a Tychonoff topology on $X_1 \cup_f X_2$ stronger than the original one. $\square$

Example 1.10. **[Folklore]** *There exists a Tychonoff MI space which is not maximal Tychonoff.*

Proof: Let X be a Tychonoff MI space. Let $X_1 = X \times \{0\}$ and $X_1 = X \times \{1\}$. Fix a point $p \in X$ and define $f : \{(p,0)\} \to X_2$ by $f((p,0)) = (p,1)$. Then by Theorem 1.9, $X_1 \cup_f X_2$ is a Tychonoff MI space which is not maximal Tychonoff. □

In brief, we have seen that the space obtained by identifying two points of a Tychonoff MI space is MI but not maximal Tychonoff. Examples of Tychonoff MI-spaces, in ZFC, are given in [El], [vD] and [LP].

Theorem 1.9 also implies that the union of two maximal Tychonoff subspaces is not necessary maximal Tychonoff. On the other hand, Example 2 shows that, in general, a crowded topology cannot be extended to an MI Tychonoff topology. This suggests the following question:

Question 1.11. *What kind of spaces can be extended to maximal Tychonoff MI spaces?*

We remark that if (X,τ) is MI, then every Tychonoff crowded extension of X is MI. We also notice that, from Lemma 1.6 and [vD, Th. 2.2], a Tychonoff space is Hausdorff maximal iff it is a maximal Tychonoff MI-space.

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