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CONDITIONS WHICH IMPLY METRIZABILITY IN SOME GENERALIZED METRIC SPACES*

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ABSTRACT. In this paper we show that two important generalized metric properties are generalizations of first countability. We give some conditions on these generalized metric properties which imply metrizability. We prove that, a space X is metrizable if and only if X is a strongly–quasi–N–space, quasi–γ–space; a quasi–γ space is metrizable if and only if it is a pseudo wN–space or quasi–Nagata–space with quasi–G_δ^*–diagonal; a space X is a metrizable space if and only if X has a CWBC–map g satisfying the following conditions:

1. g is a pseudo–strongly–quasi–N–map;
2. for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.

1. Introduction

A **COC-map** (= countable open covering map) for a topological space X is a function from $\mathbb{N} \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}, x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$.

Consider the following conditions on g.

(A) If $x \in g(n, x_n)$ for every $n \in \mathbb{N}$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(B) If for each $n \in \mathbb{N}$, $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

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(*C*) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(*D*) If for each $n \in \mathbb{N}$, $\{x, x_n\} \subset g(n, y_n)$ and $y_n \in g(n, x)$, then x is a cluster point of the sequence $\langle x_n \rangle$.

(*E*) If for each $n \in \mathbb{N}$, $x_n \in g(n, y_n)$ and the sequence $\langle y_n \rangle$ converges in X, then the sequence $\langle x_n \rangle$ has a cluster point.

(*F*) If for each $n \in \mathbb{N}$, $y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to x in X, then x is a cluster point of the sequence $\langle x_n \rangle$.

Remark 1.1. *There is no loss of generality in conditions (A), (B), (C), (D), and (F) in assuming that the sequence* $\langle x_n \rangle$ *converges to* x. *(See* [19, Remark 2.1]*).*

Let (S) be any of the conditions (A), (B), (C), (D), (E), or (F), and (S^{-1}) be the statement obtained by formally interchanging all memberships (e.g., (C^{-1}) is the condition: If for each $n \in \mathbb{N}$, $y_n \in g(n, x) \cap g(n, x_n)$, then x is a cluster point of the sequence $\langle x_n \rangle$). If the COC–map g satisfies condition (S)(resp. (S^{-1})) for $S = A$, B, C, D, E, or F, we say that g is an $\boldsymbol{S}$**–map** (resp. $\boldsymbol{S^{-1}}$**–map**). If there is an S–map (resp. S^{-1}–map) for X then we say that (X, τ) is an $\boldsymbol{S}$**–space** (resp. $\boldsymbol{S^{-1}}$**–space**). Corresponding to each of the conditions S above except (E) is the weaker condition, denoted wS, in which 'then x is a cluster point of the sequence $\langle x_n \rangle$' is replaced by 'then the sequence $\langle x_n \rangle$ has a cluster point'. If g satisfies wS, we say that g is an wS–map. If there is an wS–map for X then we say that (X, τ) is a wS–space. wS^{-1}–maps and wS^{-1}–spaces are defined analogously. The following are known, $A =$**semi–stratifiable space**, $B =\boldsymbol{\sigma}$**–space**, $C =$**developable space**, $D =\boldsymbol{\theta}$**–space**, $E =$**quasi–**γ**–space**, $F =$**strongly–quasi Nagata space (= strongly–quasi–N space)**, $A^{-1} =$**first–countable space**, $B^{-1} =\gamma$**–space**, $C^{-1} =$**Nagata space (= N–space)**, $E^{-1} =$**quasi–Nagata space (= quasi–N space)**, $wA =\boldsymbol{\beta}$**–space**, $wB =\boldsymbol{w\sigma}$**–space**, $wD =\boldsymbol{w\theta}$**–space**, $wA^{-1} =\boldsymbol{q}$**–space**, $wB^{-1} =\boldsymbol{w}\gamma$**–space**, $wC^{-1} =\boldsymbol{wN}$**–space**.

A **CWBC–map** (= countable weak base covering map) for a topological space X is a function from $\mathbb{N}\times X$ into $\mathcal{P}(X)$ such that for every $x\in X$ and $n\in\mathbb{N}$ we have $x\in g(n,x), g(n+1,x)\subseteq g(n,x)$ and a subset U of X is open if and only if for every $x\in U$there is an $n\in\mathbb{N}$ such that $g(n,x)$ is contained in U. A space with a CWBC-map is called **weakly first countable**.

H.W. Martin in [34] introduced weakly developable spaces. A space X is called a **weakly developable** space if there is a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of covers of X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ for all n and $\{st(x,\mathcal{G}_n)\}_{n\in\mathbb{N}}$ is a local weak base at x for each $x\in X$; the sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is said to be a weak–development for the space X.

A space X has a **quasi–G_δ^*–diagonal** (resp. **quasi–S_2–diagonal**) (resp. **quasi–α_1–diagonal**) if there exists a countable family $\mathcal{G}=\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of collections of open subsets (resp. of collections of subsets and for each $x\in X, st(x,\mathcal{G}_n)$ is open for all $n\in\mathbb{N}$) (resp. of collections of subsets and for each $x\in X, x\in$ Int $st(x,\mathcal{G}_n)$) such that for any distinct $x,y\in X$, there exists $n\in\mathbb{N}$ such that $x\in\overline{st(x,\mathcal{G}_n)}\subset X-\{y\}$.

A space X is called **c–semi–stratifiable** [35] (**c–stratifiable**) if there a sequence $\langle g(n,x)\rangle$ of open neighborhoods of x such that for each compact set $K\subset X$, if $g(n,K)=\bigcup\{g(n,x):x\in K\}$, then $\bigcap\{g(n,K):n\geq 1\}=K$ ($\bigcap\overline{\{g(n,K):n\geq 1\}}=K$). The COC–map $g:\mathbb{N}\times X\to\tau$ is called a c–semi–stratification (c–stratification) of X.

A space X which has a CWBC-map that satisfies condition (wC^{-1}) is called **pseudo–wN space**.

A space X which has a CWBC-map that satisfies condition (C^{-1}) is called **pseudo–N space**.

A space X which has a CWBC-map that satisfies condition (wB^{-1}) is called **pseudo–quasi–γ space**.

A space X which has a CWBC-map that satisfies condition (B^{-1}) is called **pseudo–γ space**.

From the papers [16], [18], [27] and [32], the relationship between the classes of spaces above can be summarized in the following diagram:

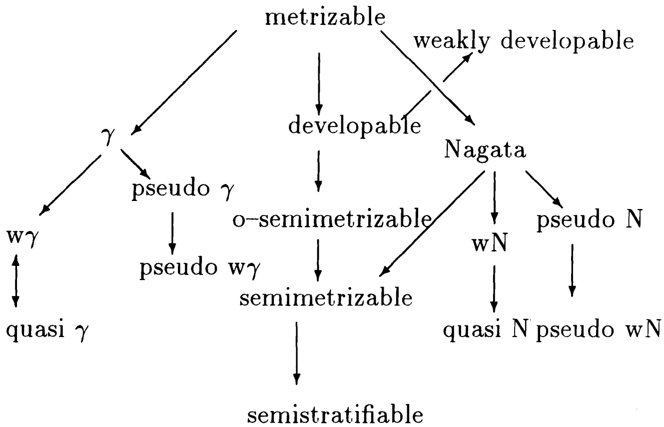


Figure 1: Relationships between some generalized metric spaces.

A space X is called an **ℵ–space** if it has a σ–locally finite K–network, where a collection $\mathcal{B}$ of subsets of X is a **K–network** if for any compact set C and open neighborhood U of C there is a finite subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that $C \subset \mathcal{B}'^* \subset U$, where $\mathcal{B}'^* = \bigcup\{B : B \in \mathcal{B}'\}$. The following implications are well–known.

Frechet $\aleph \Rightarrow$ Lasnev $\Rightarrow$ stratifiable $\Rightarrow$ strongly–quasi-$N \Rightarrow$ $\sigma \Rightarrow$ semi–stratifiable.

In this paper all spaces will be Hausdorff, unless we state otherwise.

2. Generalization of first countable spaces

A space X is **sequential** [7] if every sequentially open set is open, where a set U is said to be sequentially open if every sequence converging to a point in U is eventually in U. A space is **Frechet** [7] if every accumulation point of a set is the limit of a sequence in the set. X is called **strongly Frechet** if, whenever $\{F_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with a cluster point x, then there are $x_n \in F_n, n \in \mathbb{N}$ such that $\langle x_n \rangle$ converges to x.

Lemma 2.1. [41] *A space X is first countable if and only if X is Frechet and weakly first countable.*

Example 2.2. [41] *A Frechet space which is not weakly first countable and so not first countable.*

The space of rational numbers with the integers identified to a point and the quotient (or identification) topology. The one–point compactification of an uncountable discrete space. □

Example 2.3. [41] *A q, and weakly first countable space which is not Frechet and so not first countable.*

Let X be obtained from $[0,\infty)$ by identifying $1/n$ and n for all $n \in \mathbb{N}$. We denote by x_n the point $\{1/n, n\}$ in the identification space X. All other points of X are singleton equivalence classes, i.e. real numbers.

This example is also quasi–N space but neither wN nor strongly–quasi–N. □

Note that every Nagata space is first countable; every γ space is first countable and every Frechet, pseudo wN–space is a wN–space.

The proof of the following theorem is straightforward:

Theorem 2.4. *(1) Every quasi–N–space is β.*
(2).Every quasi–γ space is q.

Theorem 2.5. *The following are equivalent for a first countable space X*

1. *X is a quasi–γ–space.*
2. *X is a pseudo γ–space.*
3. *X is a pseudo quasi–γ–space.*

Proof: It is clear that, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We prove that every Frechet, pseudo quasi–γ–space is a quasi–γ–space. Let $g : \mathbb{N}\times X \to \mathcal{P}(X)$ be a pseudo quasi–γ–map. We can use same proof as for Lemma 2.1 to prove that for each $x \in X, g(n,x)$ is a neighborhood of x for each $n \in \mathbb{N}$. Thus x is in the interior of $g(n,x)$. Now, put $h(n,x) = Int\ g(n,x)$ for each $n \in \mathbb{N}$ and $x \in X$, then $h : \mathbb{N} \times X \to \tau$ satisfies the quasi–γ–condition. □

Y. Inui and Y. Kotake [23] proved the following result:

Theorem 2.6. *The following are equivalent for a first countable space X*

1. *X is a wN–space.*
2. *X is a quasi N–space.*
3. *X is a pseudo N–space.*
4. *X is a pseudo wN–space.*

Lemma 2.7. *A q space with quasi–S_2 is first countable.*

Proof: Let f be a q–map and $\langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ a quasi–S_2–sequence of X. Define g by

$$g(n,x) = \begin{cases} st(x, \mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* \ . \end{cases}$$

For each $x \in X$ and $n \in \mathbb{N}$, let $h(n,x) = f(n,x) \cap g(n,x)$. Then h is a first countable map. Let $x_n \in h(n,x)$. Then $\langle x_n \rangle$ has a cluster point, say y (because g is q–map). For all $n \in \mathbb{N}, y$ is a cluster point of $\{x_m : m \geq n\}_{n \in \mathbb{N}}$, so $y \in \overline{h(n,x)}$ as $x_m \in h(n,x)$ for all m. Thus $y \in \bigcap_{n \in \mathbb{N}} \overline{h(n,x)} \subset \bigcap_{n \in \mathbb{N}} \overline{st(x, \mathcal{G}_n)} = \{x\}$, so $y = x$ and x is a cluster point of $\langle x_n \rangle$. □

Theorem 2.8. *(Lutzer [30]) Let X be a regular q space. If every point in X is a G_δ then X is first countable.*

Corollary 2.9. *A regular q space with quasi–α_1–diagonal is first countable.*

3. Stability of Strongly–quasi–N Spaces

Theorem 3.1. *Every subspace of a strongly–quasi–N–space is a strongly–quasi–N–space.*

Proof: Let g be a COC–map on X satisfying the condition for a strongly–quasi–N–space. Let Y be a subspace. Then the restriction h of g on $\mathbb{N} \times Y$, $h(n,x) = g(n.x) \cap Y$ is a COC–map. □

Theorem 3.2. *Every countable product of strongly–quasi–N–spaces is a strongly–quasi–N–space.*

Proof: For each i, let X_i be a strongly–quasi–N space with a COC–map g_i satisfying the strongly–quasi–N condition. Let $X = \prod X_i$ be the product space, and let $\pi_i : X \to X_i$ be the projection. For each i, n and $x \in X$, let $h_i(n, x) = g_i(n, \pi_i(x))$ if $i \leq j$, and X_i if $i > j$. Now let $g(n, x) = \prod_{i=1}^{\infty} h_i(n, x)$ for each $(n, x) \in \mathbb{N} \times X$. That is, $g(n, x) = g_1(n, x_1) \times g_2(n, x_2) \times g_3(n, x_3) \times ...g_n(n, x_n) \times \prod_{j>n} X_j$ for each $n \in \mathbb{N}$, where $x = (x_1, x_2, x_3, ..)$.

Clearly each $g(n, x)$ is open, $x \in g(n, x)$ and $g(n+1, x) \subset g(n, x)$ for each $(n, x) \in \mathbb{N} \times X$.

To verify g is a strongly–quasi–N–map for X, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two sequences in $X = \prod X_i$ such that $y_n \in g(n, x_n)$ and the sequence $\langle y_n \rangle$ converges to x in X, we only need to prove that x is a cluster point of the sequence $\langle x_n \rangle$. Put, $x_n = (x_n)_i$, $y_n = (y_n)_i$ and $x = (x)_i$. For each fixed $i \in \mathbb{N}$, we have $(y_n)_i \in g(n, (x_n)_i)$ when $n \geq i$ and and the sequence $\langle (y_n)_i \rangle$ converges to x_i in X_i. Since each X_i is a strongly–quasi–N space, $\langle (x_n)_i \rangle$ converges to x_i. Thus, x is a cluster point of $\langle (x_n) \rangle$ in X. Hence, $X = \prod X_i$ is a strongly–quasi–N–space. □

Theorem 3.3. *Closed images of regular strongly–quasi–N–-spaces are strongly–quasi–N–spaces.*

Proof: Let $f : X \to Y$ be closed surjective map such that X is a strongly–quasi–N–space. We want to show that Y is also a strongly–quasi–N–space. Since X is a strongly–quasi–N–space, there is a COC–map g satisfying the strongly–quasi–N–condition. In other words, if $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\langle x_n \rangle$ converges to x, then $\langle y_n \rangle$ converges to x. Define $h(n, y) = Y - f(X - (\bigcup\{g(n, x) : x \in f^{-1}(y)\}))$. It is clear that h is a COC–map. Let $x_n' \in h(n, y_n')$. Suppose $\langle x_n' \rangle$ converges to x'. We want to prove that $\langle y_n' \rangle$ converges to x'.

Let $x_n \in f^{-1}(x_n')$ for each $n \in \mathbb{N}$, so every subsequence of $\langle x_n \rangle$ has at least a cluster point in $f^{-1}(x')$ since f is closed.

Note that, since X is a strongly–quasi–N–space, it is perfect, so, $f^{-1}\{x'\}$ is a G_δ–set for each $x' \in Y$ and since f is closed, $\{x'\}$ is a G_δ–set for every $x' \in Y$. Let $x \in f^{-1}(x')$ be a cluster point of $\langle x_n \rangle$. Note that, $\{x\} = \bigcap_{n=1}^{\infty} G_n$, where G_n is a closed neighborhood of x (X is regular).

Choose $x_{n_m} \in \{x_n\} \cap G_m$ (because x is a cluster point of $\langle x_n \rangle$ and G_n is a neighborhood of x), where we may assume $n_1 < n_2 < \ldots$, then x is a unique cluster point of $\langle x_{n_m} \rangle$, (if $z \neq x$, then there is a G_{m_0} such that $z \notin G_{m_0}$, hence z is not cluster point of $\langle x_{n_m} \rangle$). But $\langle x_{n_m} \rangle$ has a cluster point, therefore, y is a unique cluster point of $\langle x_{n_m} \rangle$. Since every subsequence of $\langle x_{n_m} \rangle$ has cluster point, we have that the sequence $\langle x_{n_m} \rangle$ converges to x.

Now, we have $x_{n_m} \in g(n_m, y_{n_m}) \subset g(m, y_{n_m})$ and $\langle x_{n_m} \rangle$ converges to x is a cluster point of . Since X is a strongly–quasi–N–space, x is a cluster point of $\langle y_{n_m} \rangle$. Since f is closed, x' is a cluster point of $\langle {y_{n_m}}' \rangle$ and hence x' is a cluster point of $\langle {y_n}' \rangle$. This completes the proof that Y is a strongly–quasi–N–space. □

Theorem 3.4. *Every strongly–quasi–N–space is σ–space.*

Proof: Let g be a COC–map on X satisfying the condition for a strongly–quasi–N–space. Let $x \in g(n, y_n)$ and $y_n \in g(n, x_n)$, for each $n \in \mathbb{N}$. Then $\langle y_n \rangle$ converges to x and since g is strongly–quasi–N–map, x is a cluster point of the sequence $\langle x_n \rangle$. □

Example 3.5. *The converse of Theorem 3.4 is not true. There is a σ–space (and so semi–stratifiable) which is not a strongly–quasi–N–space.*

Proof: Let X (Heath space [17]) be the upper half plane including the real axis $\mathbb{R}$. Let each point of $X - \mathbb{R}$ be open and take as a neighborhood basis of points $x \in \mathbb{R}$ a V–vertex at x, sides of slopes $= 1$ and height $1/n$, which a V–vertex at x is the set $W = \{(\xi, \eta) : \eta = |\xi - x| \text{ and } \eta < \frac{1}{n}\}$.

We define a COC–map by:

$$h(n,x) = \begin{cases} \{x\} & \text{if } x \in X - \mathbb{R}. \\ \text{the } V\text{–vertex at } x \text{ of height } 1/n & \text{if } x \in \mathbb{R}\ . \end{cases}$$

Clearly h is a COC–map and satisfies the condition for a σ–space. Thus X is a σ–space. It is known that X is a Moore space [17], and hence first countable. If X is a strongly–quasi–N–space, it would be stratifiable by 4.1 and hence it would be paracompact. However X is not even normal: consider the two closed sets consisting of the rationals and irrationals in $\mathbb{R}$ respectively. □

In [13], Z. Gao proved the following result:

Theorem 3.6. *Every regular k–semi–stratifiable space is a strongly–quasi–N–space.*

Example 3.7. *There is a strongly–quasi–N–space which is not an N–space (it is not even stratifiable).*

Proof: In [39], O'Meara constructs an example of a non–normal (and hence not stratifiable) $\aleph$–space which is completely regular, and by Lemma 2.4 [30], any $\aleph$–space is k–semi–stratifiable and hence a strongly–quasi–N–space. □

4. Metrizability Results

Theorem 4.1. *A space X is N if and only if it is a first countable strongly–quasi–N–space.*

Proof: It is well–known that every Nagata–space is a paracompact first countable space. Now, let f and g be, respectively, a first countable–map and a strongly–quasi–N–map on X. Let $h(n,x) = f(n,x) \cap g(n,x)$. It is easy to see that h is a first countable and strongly–quasi–N–map. To prove h is a Nagata–map, suppose that $h(n,x_n) \cap h(n,x) \neq \emptyset$. Then there is a sequence $\langle y_n \rangle$ such that $y_n \in h(n,x_n) \cap h(n,x)$. Since h is a first countable–map, $\langle y_n \rangle$ converges to x and since h is strongly–quasi–N–map, $\langle x_n \rangle$ converges to x. Hence X is a Nagata space. □

Corollary 4.2. *A space X is N (and stratifiable) if and only if it is a q strongly–quasi–N–space with quasi–G_δ^*–diagonal.*

Proof: The 'only if' part is obvious. The ' if' follows from Theorem 4.1 and Lemma 2.7 . □

From Theorem 2.8, Theorem 3.6 and Theorem 4.1 we get the following result:

Corollary 4.3. *A space X is N (and stratifiable) if and only if it is a regular q k–semi–stratifiable space.*

Theorem 4.4. *A space X is metrizable if and only if X is a strongly–quasi–N–space and quasi–γ–space.*

Proof: Suppose that X is strongly–quasi–N–space and quasi–γ–space. We shall show that the space X is developable. This will complete the proof since developable spaces are first countable and first countable strongly–quasi–Nagata spaces are Nagata, hence paracompact, and paracompact developable spaces are metrizable [3]. Let $f : \mathbb{N} \times X \to \tau$ and $g : \mathbb{N} \times X \to \tau$ be, respectively, quasi–γ and strongly–quasi–N maps for X. Let $h(n,x) = f(n,x) \cap g(n,x)$ for each $(n,x) \in \mathbb{N} \times X$. Then $h : \mathbb{N} \times X \to \tau$ is both a quasi–γ and a strongly–quasi–N map for X. Suppose $\{p, x_n\} \in h(n, y_n)$ for each $n \in \mathbb{N}$. Since h is a semistratifiable map, p is a cluster point to $\langle y_n \rangle$. Also there is a point q such that q is a cluster point of $\langle x_n \rangle$ (because h is quasi–γ) and since h is a strongly–quasi–N map, q is a cluster point of $\langle y_n \rangle$, so $p = q$. Hence p is a cluster point of $\langle x_n \rangle$. □

From [18, Corollary 4.6 (a space X is a Moore space if and only if it is a regular semi–stratifiable $w\theta$–space)], Corollary 4.3 and Nagata's famous double sequence theorem (every Nagata developable space is metrizable), we have the following:

Corollary 4.5. *A space is metrizable if and only if it is a regular k–semi–stratifiable $w\theta$–space.*

Martin proved the following result:

Theorem 4.6. [32] *Every γ quasi–Nagata–space is metrizable.*

He asked in [33, Question 1]: Is every quasi–N, quasi–γ space with a G^*_δ–diagonal metrizable?

Noting that, every space with a G^*_δ–diagonal has a quasi–S_2–diagonal, we answer this question in the affirmative by the following:

Theorem 4.7. *A quasi–γ space is metrizable if and only if it is a pseudo wN–space or quasi–Nagata–space with quasi–S_2–diagonal.*

Proof: Let X be a quasi-γ, pseudo wN-space or quasi–Nagata–space with quasi–S_2. Since every quasi-γ space is a q-space [23] then by lemma 2.7, X is first countable. From Theorem 2.6 and [18, Proposition 3.2], X is countably paracompact, so by [2], X is regular. Since every wN–space is β and every β space with quasi–S_2–diagonal is a semistratifiable space [37], X is a Nagata-space which is therefore a strongly–quasi–Nagata space. Applying Theorem 4.4 completes the proof. □

The following is a well–known characterization of γ–spaces (see [10, Section 7.18]:

Proposition 4.8. *A space X is γ if and only if X has a COC–map g such that if $\langle x_n\rangle$ and $\langle y_n\rangle$ are sequences in X such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$ and $\langle y_n\rangle$ converges to x in X, then x is a cluster point of the sequence $\langle x_n\rangle$.*

Definition 4.9. *A space X has a* **quasi–$G^*_\delta(2)$–diagonal** *if there exists a sequence $\langle \mathcal{G}_n : n \in \mathbb{N}\rangle$ of open families of X such that for distinct points x, y there exists some $\mathcal{G}_m$ such that $y \notin \overline{st^2(x, \mathcal{G}_m)}$.*

Theorem 4.10. *A space X with a quasi-$G^*_\delta(2)$–diagonal is Nagata if and only if it is a q, quasi–N–space.*

Proof: Suppose that X is a q quasi–N–space with a quasi–$G^*_\delta(2)$ sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$. Since the space X is a q and has a quasi–G^*_δ–diagonal, by Lemma 2.7, X is a first countable space. From Theorem 4.1, we need only to prove that X is a strongly–quasi–N space. Let f be a quasi–N–map. Define $g : \mathbb{N} \times X \to \tau$ as follows:

$$g(n,x) = \begin{cases} st(x,\mathcal{G}_n) & \text{if } x \in \mathcal{G}_n^*. \\ X & \text{if } x \notin \mathcal{G}_n^* . \end{cases}$$

Let $h(n,x) = \bigcap_{i=1}^{n} g(i,x)$. Set $k(n,x) = f(n,x) \cap h(n,x)$. We show that k is a strongly–quasi–N–map for X. Let $y_n \in k(n,x_n)$ and suppose $\langle y_n \rangle$ converges to p. Since f is a quasi–N–map, $\langle x_n \rangle$ has a cluster point, say q. The proof ends if $p = q$. Suppose $p \neq q$. Fix $n \in c_{\mathcal{G}}(q) = \{m \in \mathbb{N} : q \in \mathcal{G}_m^*\}$. Then there are infinitely many integers $m \geq n$ such that $x_m \in k(n,q)$. Let $m \geq n$ with $x_m \in k(n,q)$. Then $x_m \in g(n,q) = st(q,\mathcal{G}_n)$. Thus $\{y_m : m \geq n\} \subseteq st^2(q,\mathcal{G}_n)$ for all $n \in c_{\mathcal{G}}(x)$.. So, $p \in \overline{\{y_m : m \geq n\}} \subseteq \overline{st^2(q,\mathcal{G}_n)}$ for all $n \in c_{\mathcal{G}}(x)$. It follows that $p \in \bigcap_{n \in c_{\mathcal{G}}(x)} \overline{st^2(q,\mathcal{G}_n)} = \{q\}$. Thus $p = q$, as required. □

From Theorem 4.2 and Theorem 4.10 we get the following result:

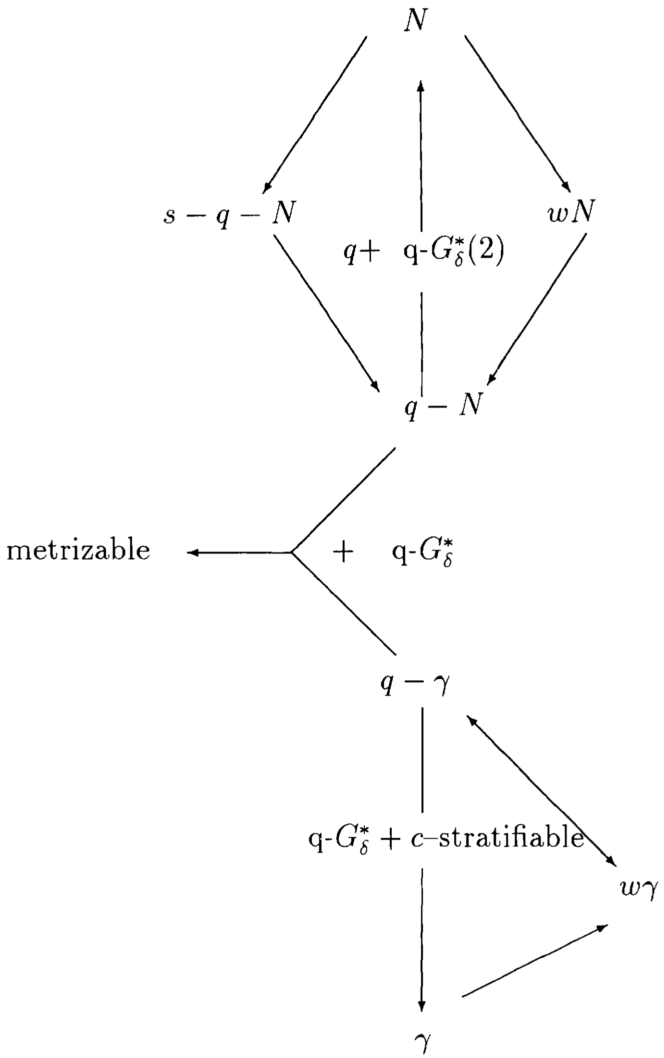
Corollary 4.11. *Let X be a q–space with quasi–G_δ^*–diagonal, then the following are equivalent:*

1. *X is a quasi–N space.*
2. *X is a strongly–quasi–N.*
3. *X is a wN space.*
4. *X is a N space.*

Theorem 4.12. *A space X with a quasi–G_δ^*–diagonal is γ if and only if it is a c–stratifiable space and quasi–γ–space.*

Proof: Suppose that X is a quasi–γ–space with a quasi–G_δ^*–diagonal. From Lemma 2.7, X is a first countable space. Since every first countable quasi–γ–space is $w\gamma$, by Lemma 2.2 [29], X is γ–space. □

The relationships between the classes of spaces considered in this section can be summarized in the following diagram:



Relationships between generalizations of N and γ spaces.

5. Difference Between Metrizability and Strongly–quasi–N and γ Spaces

In this section we discuss and answer the question: What is the difference (in terms of g–maps) between metrizable spaces and various generalized metric spaces, like strongly–quasi–N and γ spaces. First we start with the following result which gives the difference between Lasnev (= the closed continuous image of a metric space) and strongly–quasi–N–spaces.

The proofs of the following theorems can be found in [13] and [38].

Theorem 5.1. *A space X is Lasnev (metrizable) if and only if X is Frechet (strongly Frechet), strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that if the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy either:*

1. *$x_i \in g(n, y_i)$ for all $i \in \mathbb{N}$, and $x_j \in X - g(n, y_i)$ for all $j > i$ or*
2. *$x_i \in X - g(n, y_i)$ for all $i \in \mathbb{N}$, and $x_j \in g(n, y_i)$ for all $j > i$,*

then $\{x_i : i \in \mathbb{N}\}$ is discrete in X.

Theorem 5.2. *A space X is metrizable if and only if X is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.*

Theorem 5.3. *(Nagata) A space X is metrizable if and only if X is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g^2(n,x) : x \in A\}$, where $g^2(n,x) = \bigcup\{g(n,y) : y \in g(n,x)\}$.*

Theorem 5.4. *A space X is an $\aleph$–space if and only if it is strongly–quasi–N and there is a COC–map $g : \mathbb{N} \times X \to \tau$ such that if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$ and for each $x \in X, n \in \mathbb{N}$ $|\{g(n,y) : y \in g(n,x), x \notin g(n,y)\}| < \aleph_0$.*

The following theorem is due (independently) to Hung [21] and Hodel [20].

Theorem 5.5. *A space X is metrizable if and only if X has a COC–map g satisfying the following conditions:*

1. *g is a γ–map;*
2. *for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.*

The proof of our next results relies on a metrisation theorem of H. Martin [34].

Theorem 5.6. *(Martin) A necessary and sufficient condition that a topological space X be metrizable is that X has a weak development $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ such that $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X.*

Definition 5.7. *A space X is called a* **pseudo–strongly–quasi–N–space** *if there is a $CWBC$–map $g : \mathbb{N}\times X \to \mathcal{P}(X)$ such that if for each $n \in \mathbb{N}, y_n \in g(n,x_n)$ and the sequence $\langle y_n\rangle$ converges to p in X, then p is a cluster point of the sequence $\langle x_n\rangle$ (it is equivalent to say $\langle x_n\rangle$ converges to p). The $CWBC$–map g is called a pseudo–strongly–quasi–map.*

Theorem 5.8. *A space X is metrizable if and only if X has a $CWBC$–map g satisfying the following conditions:*

1. *g is a pseudo–strongly–quasi–N–map;*
2. *for any $A \subseteq X, \overline{A} \subseteq \bigcup\{g(n,x) : x \in A\}$.*

Proof: The only if part is obvious. We now prove the if part. Assume that X has a a $CWBC$–map g satisfying the conditions (1) and (2). Let $h(n,x) = \{y \in X : x \in g(n,y)\}$ and $k(n,x) = g(n,x) \cap h(n,x)$ for each $(n,x) \in \mathbb{N}\times X$. Let $\mathcal{G}_n = \{k(n,x) : (n,x) \in \mathbb{N}\times X\}$. Then $st(x,\mathcal{G}_n) = \bigcup\{k(n,y) : x \in k(n,x)\}$ and $st^2(x,\mathcal{G}_n) = \bigcup\{k(n,y) : k(n,y) \cap st(x,\mathcal{G}_n) \neq \emptyset, (n,x) \in \mathbb{N}\times X\}$.

By condition (2), $h(n,x)$ is a neighborhood (not necessarily open) of x and so is $k(n,x)$. Therefore, in virtue of the Martin metrization theorem 5.6, we only need prove that $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}, x \in X\}$ is a weak base of X. If $\{st^2(x,\mathcal{G}_n) : n \in \mathbb{N}\}$ is not a local weak base for some $x \in X$, then there exists an open neighbourhood U of x such that $st^2(x,\mathcal{G}_n) - U \neq \emptyset$ for each $n \in \mathbb{N}$. Take $y_n \in st^2(x,\mathcal{G}_n) - U, n \in \mathbb{N}$. That means we can find $z_n, w_n \in X$ such that $y_n \in k(n,z_n), k(n,z_n) \cap k(n,w_n) \neq \emptyset, x \in k(n,w_n)$. Take $v_n \in k(n,z_n) \cap k(n,w_n)$. By $x \in k(n,w_n) \subseteq g(n,w_n)$ and condition (1), we conclude that $\langle w_n\rangle$ converges to x, and by $v_n \in k(n,w_n) \subseteq h(n,w_n)$

and the definition of h, we get $w_n \in g(n, v_n)$. Using condition (1) again, we have that $\langle v_n \rangle$ converges to x. Similarly, from $v_n \in k(n, z_n) \subseteq g(n, z_n)$, we have that $\langle z_n \rangle$ converges to x, and by $y_n \in k(n, z_n) \subseteq h(n, z_n)$, we get that $\langle y_n \rangle$ converges to x. But $y_n \notin U$ for each $n \in \mathbb{N}$, which is a contradiction. □

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