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ARC-LIKE AND CIRCLE-LIKE ALMOST CONTINUOUS IMAGES OF PEANO CONTINUA

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ABSTRACT. Three arc connectedness properties and a mapping property are considered: (1) Y is almost arcwise connected; (2) Y is almost Peano; (3) Y has a dense arc component; (4) Y is an almost continuous image of any nondegenerate Peano continuum. The following results are proved: If Y is an arc-like continuum, then (1), (2), and (4) are equivalent; if Y is a decomposable, unicoherent, a-triodic continuum, then (1)-(4) are equivalent; if Y is a decomposable circle-like continuum, then (2)-(4) are equivalent; if Y is a circle-like continuum such that each indecomposable subcontinuum of Y is nowhere dense in Y, then (1)-(4) are equivalent.

1. INTRODUCTION

We are concerned with two problems — determining when continua have dense arc components, and finding the best necessary or sufficient condition for special types of continua to be

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almost continuous images of Peano continua. The two problems are related by the following interesting theorem due to Kellum [6]:

Kellum's Theorem. A second countable T_1-space, Y, is an almost continuous image of any nondegenerate Peano continuum if and only if Y is almost Peano.

Our statement of Kellum's theorem and the statement in [6] are formally slightly different but are equivalent (as is seen by applying the Hahn-Mazurkiewicz Theorem [10, p. 126] and Proposition 4 of [11, p. 261]).

After giving definitions, we discuss relationships between our results and the literature.

A function $f : X \to Y$ is *almost continuous* provided that each neighborhood of the graph of f in $X \times Y$ contains the graph of a continuous function $g : X \to Y$ [11, p. 252].

A *continuum* is a nonempty, compact, connected metric space. A *Peano continuum* is a locally connected continuum. A continuum, Y, is *arc-like* (respectively, *circle-like*) provided that for each $\epsilon > 0$, there is an ϵ-map of Y onto [0,1] (respectively, onto the unit circle in R^2). Arc-like and circle-like continua can also be defined in terms of certain types of chains (e.g., [2] and 12.11 of [10]).

A *path* is a continuous image of [0,1]; we remark that the paths and the Peano continua in a Hausdorff space are identical (8.18 of [10]). A space, Y, is *almost Peano* provided that for each finite set of nonempty open sets in Y, there is a path that intersects each of them. A space, Y, is *almost arcwise connected* provided that for each two nonempty open sets in Y, there is an arc that intersects each of them. (The last two definitions are from [6]-[8].)

Finally, an *arc component* of a space is a maximal arcwise connected subset of the space.

It is easy to see that having a dense arc component implies almost Peano, and that almost Peano implies almost arcwise connected. Known examples, whose properties we state below, show that the converse implications are false. Our paper is concerned with when the converse implications are true for arc-like continua and circle-like continua. All our results in sections 2 and 3 contrast, in one way or another, with each of the three examples below; at the end of each example, we refer (by number) to the results in our paper that are in sharpest contrast to the example.

1.1 Example. There is an almost arcwise connected continuum that is not almost Peano, hence does not have a dense arc component (Example 3 of [6, p. 296]). In addition, the continuum is decomposable and circle-like. Compare with our results in 2.1, 2.2, and 3.3.

1.2 Example. There is an almost Peano, indecomposable, arc-like continuum that does not have a dense arc component (Example 2 of [5, p. 260]). In addition, every nondegenerate, proper subcontinuum is decomposable, and the continuum is circle-like (Theorem 7 of [2, p. 657]). Compare with our results in 2.2 and 3.4.

1.3 Example. There is an almost Peano, λ-dendroid (hereditarily decomposable hereditarily unicoherent continuum) that does not have a dense arc component (Example 3 of [5, p. 262]). Compare with our results in 2.2 and 3.4.

We remark that our results in sections 2 and 3 (except for 2.1) provide answers, in special cases, to the three parts of question 4 of [7, p. 259]. For some known results about when continua have dense arc components, see [3], [4], and [5].

In section 4, we apply Kellum's Theorem and the results in sections 2 and 3 to obtain our results about almost continuous images.

We use $\overline{A}$ to denote the closure of A, and we use $\mathrm{int}(A)$ to denote the interior of A.

For a decomposable continuum Y, we write $Y = A \oplus B$ to mean that A and B are proper subcontinua of Y whose union is Y [10, p. 208].

2. Almost Arcwise Connected Arc-like Continua

2.1 Theorem. *Let Y be an arc-like continuum. Then, Y is almost arcwise connected if and only if Y is almost Peano.*

Proof: The "if" half of the theorem follows immediately since any path is arcwise connected (8.18 and 8.23 of [10]).

To prove the other half of the theorem, assume that the arc-like continuum Y is almost arcwise connected. Let $W_1, \ldots, W_n$ be finitely many nonempty open subsets of Y. Then there is an $\epsilon > 0$ with the following property: For any open cover, $\mathcal{U}$, of Y by sets of diameter $< \epsilon$, each W_i contains some member of $\mathcal{U}$. Since Y is arc-like, Y is chainable (12.11 of [10, p. 235]); hence, there is an ϵ-chain, $\mathcal{C} = \{U_1, \ldots, U_m\}$, covering Y — being an ϵ-chain means that $m < \infty$, each U_i is open and has diameter $< \epsilon$, and

$$(*) \quad U_i \cap U_j \neq \varnothing \text{ if and only if } |i - j| \leq 1.$$

Since Y is almost arcwise connected, there is an arc, A, in Y such that $A \cap U_1 \neq \varnothing$ and $A \cap U_m \neq \varnothing$. Thus, since $\mathcal{C}$ is an open cover of Y satisfying (*), it follows easily from the connectedness of A that $A \cap U_i \neq \varnothing$ for each $i = 1, \ldots, m$ (12.12 of [10]). Now, note from our choice of ϵ that each W_i contains a member, $U_{k(i)}$, of $\mathcal{C}$. Thus, since $A \cap U_{k(i)} \neq \varnothing$ for each i, $A \cap W_i \neq \varnothing$ for each i. Therefore, we have shown that Y is almost Peano. ■

Our next theorem is for continua that are more general than arc-like continua.

2.2 Theorem. *Let Y be a decomposable, unicoherent, a-triodic continuum. Then, Y is almost arcwise connected if and only if Y has a dense arc component.*

Proof: The "if" half of the theorem is obvious.

To prove the other half of the theorem, assume that Y is almost arcwise connected.

Since Y is decomposable, $Y = A \oplus B$. Let $C = A \cap B$; since Y is unicoherent, C is a continuum.

Assume that J is an arc in Y such that $J \cap (A - B) \neq \varnothing$ and $J \cap (B - A) \neq \varnothing$. We show that $J \supset C$. Let p and q denote the end points of J. Some subarc of J has one end point in $A - B$ and the other end point in $B - A$; hence, we can assume without loss of generality that $p \in A - B$ and $q \in B - A$. Let K be a subarc of J such that $K \supset J \cap C$ and $p, q \notin K$. Now, suppose that $J \not\supset C$. We obtain a contradiction as follows: First, note that since C separates Y into $A - B$ and $B - A$, $J \cap C \neq \varnothing$; hence, $J \cup C$ is a continuum. Next, note that since $J \not\supset C$, $C - K \neq \varnothing$. It now follows that $J \cup C$ is a triod since $(J \cup C) - K$ is the union of the three nonempty, mutually separated sets $(J - K) \cap (A - B)$, $(J - K) \cap (B - A)$, and $C - K$. This contradicts the assumption in our theorem that Y is a-triodic. Therefore, $J \supset C$.

Now, let $\mathcal{U}$ and $\mathcal{V}$ denote the collections of all nonempty open subsets of $A - B$ and $B - A$, respectively. Since Y is almost arcwise connected, there is, for each $U \in \mathcal{U}$ and $V \in \mathcal{V}$, an arc, $J_{U,V}$, in Y such that $J_{U,V} \cap U \neq \varnothing$ and $J_{U,V} \cap V \neq \varnothing$. Let

$$G = \cup\{J_{U,V} : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}.$$

By what we showed in the preceding paragraph, each $J_{U,V}$ contains C. Therefore, it follows easily that G is arcwise connected and dense in Y. ■

2.3 Corollary. *Let Y be a decomposable arc-like continuum. Then, Y is almost arcwise connected if and only if Y has a dense arc component.*

Proof: The corollary follows from 2.2 since arc-like continua are unicoherent and a-triodic (12.2 and 12.4 of [10]). ■

3. Circle-like Continua With Dense Arc Components

We prove two theorems about when decomposable circle-like continua have dense arc components. Recall the example in 1.1; it shows that our result about decomposable arc-like continua in 2.3 does not extend to decomposable circle-like continua without imposing additional conditions. Our theorems are in 3.3 and 3.4.

It is convenient to state the following general, easy-to-prove lemma; we use the lemma in the proofs of both theorems in this section.

3.1 Lemma. *An almost arcwise connected space that contains an arcwise connected subset with nonempty interior must have a dense arc component.*

The next lemma is probably well known (in the hereditarily decomposable case, the lemma follows quickly from Theorem 2.6 of Miller [9, p. 187]; use Theorems 4, 5, and 7 of [2] to see that Miller's theorem can be applied).

3.2 Lemma. *Let Y be a circle-like continuum such that each indecomposable subcontinuum of Y is nowhere dense in Y. Then $Y = Y_1 \oplus Y_2$ with $Y_1 \cap Y_2 = K \cup L$, where K and L are disjoint continua such that int$(K) \neq \varnothing$ and int$(L) \neq \varnothing$.*

Proof: The continuum Y is decomposable, say $Y = A \oplus B$. Since Y is also not the union of two indecomposable continua, Y is not arc-like (Theorem 7 of [2, p. 657]). Hence, $A \cap B$ has exactly two components, E and F (Theorems 4 and 5 of [2]).

Consider the quotient space $Q = A/_{\{E,F\}}$ obtained by shrinking E and F to different points. Since A is arc-like (12.51 of [10]), Q is arc-like (12.13 of [10]); furthermore, since $A - (E \cup F)$ is open in Y, each indecomposable subcontinuum of Q is nowhere dense in Q (since Y has this property by assumption in our lemma). Hence, there is a decomposition, G, of Q as in Theorem 8 of [1] (see comment at the bottom of p. 658 of [1]); let $\pi : Q \to [0,1]$ denote the quotient map associated with G. Then, by an easy argument using that π is monotone, we see that there is a subcontinuum, M, of Q such that $\{E\} \in M$, $\{F\} \notin M$, and the interior of M in Q is nonempty. Let $K = \cup M$. Then, K is a subcontinuum of A, $K \supset E$, $K \cap F = \varnothing$, and K has nonempty interior in Y.

Similarly, there is a subcontinuum, L, of B such that $L \supset F$, $L \cap E = \varnothing$, and L has nonempty interior in Y.

Now, let $Y_1 = A \cup L$ and $Y_2 = B \cup K$. Then it follows easily that Y_1, Y_2, K, and L satisfy the conclusion to our lemma. ■

3.3 Theorem. *Let Y be a circle-like continuum such that each indecomposble subcontinuum of Y is nowhere dense in Y. Then, Y is almost arcwise connected if and only if Y has a dense arc component.*

Proof: The "if" half of the theorem is clear.

To prove the other half of the theorem, assume that Y is almost arcwise connected. Let Y_1, Y_2, K, and L be as in 3.2. Then, since Y is almost arcwise connected, there is an arc, A, in Y such that $A \cap \operatorname{int}(K) \neq \varnothing$ and $A \cap \operatorname{int}(L) \neq \varnothing$. Hence, there is a subarc, B, of A with end points p and q such that $B \cap K = \{p\}$ and $B \cap L = \{q\}$. Since $K \cup L$ separates Y into $Y_1 - Y_2$ and $Y_2 - Y_1$, we see that $B \subset Y_1$ or $B \subset Y_2$, say $B \subset Y_1$. Then, $B \cap Y_2 = \{p, q\}$; hence, $B \cup Y_2$ is a nonunicoherent continuum. Thus, $B \cup Y_2 = Y$ (12.2 and 12.51 of [10]). Therefore,

$$\operatorname{int}(B) = B - \{p, q\} \neq \varnothing.$$

Hence, Y has a dense arc component by 3.1. ■

3.4 Theorem. *Let Y be a decomposable circle-like continuum. Then, Y is almost Peano if and only if Y has a dense arc component.*

Proof: The "if" half of the theorem is clear.

To prove the other half, assume that Y is almost Peano.

Since Y is decomposable, $Y = A \oplus B$. If Y is arc-like, then we are done by 2.3. Thus, we assume for the proof that Y is not arc-like. Hence, $A \cap B$ has exactly two components, C_1 and C_2 (Theorems 4 and 5 of [2]).

Now, assume that there is an arc, D, in Y such that $D \cap C_1 \neq \varnothing$ and $D \cap C_2 \neq \varnothing$. Then there is a subarc, E, of D with end points p and q such that $E \cap C_1 = \{p\}$ and $E \cap C_2 = \{q\}$. Since $C_1 \cup C_2$ separates Y into $A - B$ and $B - A$, we see that $E \subset A$ or $E \subset B$, say $E \subset A$. Then, $E \cup B$ is a nonunicoherent continuum. Hence, $E \cup B = Y$ (12.2 and 12.51 of [10]). Thus,

$$\operatorname{int}(E) = E - \{p, q\} \neq \varnothing.$$

Therefore, by 3.1, Y has a dense arc component. Thus, we assume for the rest of the proof that an arc such as D does not exist:

(1) No arc in Y intersects both C_1 and C_2.

Next, suppose that there is an arc, F, in Y such that F intersects $A - B$, $B - A$, and C_1, but $F \not\supset C_1$. We obtain a contradiction as follows: By (1), there is an open subset, U, of Y such that $U \supset C_2$ and $(C_1 \cup F) \cap U = \varnothing$. Hence, $C_1 \cup F \neq Y$. Thus, $C_1 \cup F$ is arc-like (12.51 of [10]). Therefore, $C_1 \cap F$ is a continuum (12.2 of [10]). Hence, $C_1 \cup F$ is a triod since $(C_1 \cup F) - (C_1 \cap F)$ is the union of the three nonempty, mutually separated sets $C_1 - F$, $F \cap (A - B)$, and $F \cap (B - A)$. However, since $C_1 \cup F$ is arc-like, $C_1 \cup F$ is not a triod (12.4 of [10]). In view of the contradiction just obtained (and since a contradiction occurs when C_1 is replaced by C_2), we have proved the following fact:

(2) Any arc in Y that intersects $A - B$, $B - A$, and C_i for some i must contain C_i.

(Note: Regarding the use of (1) to prove (2), see 3.5.)

Now, to complete the proof that Y has a dense arc component, we define G_1 and G_2 as follows: For each $i = 1$ and 2,

$$G_i = \cup\{J : J \text{ is an arc in } Y \text{ and } J \text{ intersects } A - B, B - A, \text{ and } C_i\}.$$

It follows immediately from (2) that each G_i is arcwise connected (possibly empty for some i). Hence, we will be done once we show that G_i is dense in Y for some i.

Suppose that $\overline{G}_1 \neq Y$ and $\overline{G}_2 \neq Y$. Then, since Y is almost Peano, there is a path, P, in Y such that P intersects $Y - \overline{G}_1$, $Y - \overline{G}_2$, $A - B$, and $B - A$. Note that P is arcwise connected (8.17 and 8.23 of [10]); hence, by (1), $P \neq Y$; therefore, P is an arc (12.6 and 12.51 of [10]). Since P intersects $A - B$ and $B - A$, P must also intersect $A \cap B = C_1 \cup C_2$; thus, since P is an arc, $P \subset G_1$ or $P \subset G_2$. However, this contradicts the fact that P intersects $Y - \overline{G}_1$ and $Y - \overline{G}_2$. Therefore, $\overline{G}_1 = Y$ or $\overline{G}_2 = Y$. ■

3.5 Remark. In essence, we took two cases in the proof of 3.4: when (1) is false and when (1) is true. We then used (1) to prove (2). Indeed, (2) can be false without (1): In polar coordinates, let $Y = \{(1, \theta) : 0 \leq \theta \leq 2\pi\}$, $A = \{(1, \theta) : 0 \leq \theta \leq \pi\}$, and $B = \{(1, \theta) : \frac{\pi}{2} \leq \theta \leq 2\pi\}$; then the arc given by $\{(1, \theta) : \frac{3\pi}{4} \leq \theta \leq \frac{9\pi}{4}\}$ does not satisfy (2).

We also used (1) near the end of the proof of 3.4 to show that the path P is an arc; this was merely a convenience: Any nondegenerate path in a circle-like continuum is an arc or a simple closed curve (by, e.g., 2.13, 2.21, and 8.40(b) of [10]), and the last part of the proof of 3.4 can be easily carried out for the case when P is a simple closed curve.

4. Almost Continuous Images

We apply previous results to obtain four theorems about almost continuous images of Peano continua. The first two theorems concern arc-like images; the last two theorems concern circle-like images.

4.1 Theorem. *Let Y be an arc-like continuum. Then, Y is an almost continuous image of any nondegenerate Peano continuum if and only if Y is almost arcwise connected.*

Proof: Combine 2.1 with Kellum's Theorem in section 1. ■

4.2 Theorem. *Let Y be a decomposable arc-like continuum or, more generally, a decomposable, unicoherent, a-triodic continuum. Then, Y is an almost continuous image of any nondegenerate Peano continuum if and only if Y has a dense arc component.*

Proof: Combine 2.2 with Kellum's Theorem in section 1. ■

4.3 Theorem. *Let Y be a decomposable circle-like continuum. Then, Y is an almost continuous image of any nondegenerate Peano continuum if and only if Y has a dense arc component.*

Proof: Combine 3.4 with Kellum's Theorem in section 1. ■

4.4 Theorem. *Let Y be a circle-like continuum such that each indecomposable subcontinuum of Y is nowhere dense in Y. Then the following are equivalent:*

(1) Y is an almost continuous image of any nondegenerate Peano continuum;
(2) Y is almost arcwise connected;
(3) Y has a dense arc component.

Proof: Combine 3.3 and 4.3. ■

The examples in section 1 together with Kellum's Theorem show why the assumptions at the beginnings of 4.1-4.4 are necessary.

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