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HEREDITARILY NORMAL, LOCALLY COMPACT DOWKER SPACES

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ABSTRACT. Using the set-theoretic axiom $\diamondsuit$, a method of constructing hereditarily normal, locally compact Dowker spaces (a space X is *Dowker* if it is normal but not countably paracompact) is given. The spaces produced are, in addition, hereditarily separable, locally countable, of cardinality $\aleph_1$, and have Fréchet-Urysohn one-point compactifications. A special case of the method results in a one-point compactification that is α_1, and this cannot be done by assuming CH alone. It is also shown how MA(ω_1) implies the existence of a locally compact, hereditarily countably paracompact anti-Dowker space.

On the last day of the 1999 Spring Topology Conference in Salt Lake City, I offered cash prizes, in the Paul Erdős tradition, for the first person to give certain kinds of solutions to the following problem.

Problem 1. In a compact Hausdorff space, which of the following implies the other: (a) hereditary normality and (b) hereditary countable paracompactness?

One could put "locally" in front of "compact" in this problem and have an equivalent one: on the one hand, every compact Hausdorff space is locally compact; on the other, both hereditary normality and hereditary countable paracompactness are preserved in the taking of one-point compactifications, so that a locally compact counterexample to either (a) implying (b) or the converse also gives a compact one.

Problem 1 was posed by Boris Shapirovskiĭ shortly before his death in 1991. The cash awards announced in Salt Lake City included:

(1) for a consistent "Yes" to (a) $\rightarrow$ (b): \$200.
(2) for a ZFC counterexample to (a) $\rightarrow$ (b): \$500.
(3) for a ZFC counterexample to (b) $\rightarrow$ (a): \$50.
(4) for a consistent "Yes" to (b) $\rightarrow$ (a): \$400.

Originally there was also a \$50 award for a consistent "No" answer to (a) $\rightarrow$ (b), and one of \$500 for a ZFC answer of "Yes" to the same implication, but that was withdrawn a week later in the wake of the main example in this paper: a hereditarily normal (and Hausdorff, hence T_5), locally compact space that is not countably paracompact, using the set-theoretic axiom $\diamondsuit$. Since the one point compactification of a T_5 locally compact space is likewise T_5, this provides a consistent "No" answer to (a) $\rightarrow$ (b). There have long been known consistent "No" answers to (b) $\rightarrow$ (a), which will be discussed in Section 3.

Both implications are actually part of the theme of "Dowker and anti-Dowker spaces," a theme long popular with set-theoretic topologists. A *Dowker space* is a T_4 space space whose product with the closed unit interval is not normal. Dowker spaces can also be characterized as those T_4 spaces which are not countably paracompact. The term "anti-Dowker" is based on this latter characterization: an *anti-Dowker space* is a countably paracompact T_3 space that is not T_4. [Throughout this article, T_i-spaces with $i \geq 2$ are taken to be Hausdorff, and we use the convention that a T_5 space is one that is hereditarily T_4.] Recently there has been a great advance, thanks mostly to Zoltan Balogh, in our techniques of constructing Dowker spaces using only the usual (ZFC) axioms of set theory, but there are still many unsolved problems in this area and even in the area of consistency results. For example, the following problem is still open:

Problem 2. Is ZFC, or even ZFC+CH, enough to imply the existence of a locally compact Dowker space?

In contrast, examples have long been known of locally compact anti-Dowker spaces obtained using just ZFC; for example, the 1976 example of Eric van Douwen [1] is even locally countable and of cardinality ω_1. Problem 1, however, asks for a hereditarily countably paracompact and (locally) compact example. Such examples do exist under the assumption of MA(ω_1)—see Section 3—but we lack a ZFC example.

Our $\diamondsuit$ example is also of cardinality $\aleph_1$ and is hereditarily separable, hence (hereditarily) ω_1-compact. [A space is ω_1-*compact* if every closed discrete subspace is countable.] Normality and ω_1-compactness imply collectionwise normality, and so this example is hereditarily strongly collectionwise Hausdorff (abbreviated "hereditarily scwH"). A space is said to be *scwH* if every closed discrete subspace D can be expanded to a discrete collection of open sets, each of which meets D in a single point. The following result of [2] is in sharp contrast to our $\diamondsuit$ construction:

Theorem [PFA] *Every locally compact, T_4, hereditarily scwH, ω_1-compact space is countably paracompact.*

In other words, there are no locally compact, ω_1-compact, hereditarily scwH Dowker spaces if the Proper Forcing Axiom (PFA) is assumed. Of course, this is still a far cry from a solution to Problem 2, but this is among the very few consistency results that say such-and-such a kind of Dowker space does not exist. All earlier known ones are, if anything, even more specialized. Also, there will be other theorems in [5] of which this theorem is a corollary, that begin to fill the gap between it and Problem 2; one will be given in Section 2.

Our $\diamondsuit$ example is a special case of a construction done by Juhász in [3]. This Juhász construction, in turn, is generalized by Example 3.1 (ii) in [7], which actually gives a Tychonoff space which is locally compact and hereditarily separable, but not countably metacompact; extra conditions along the lines of Juhász's example are needed to make it normal and thus Dowker. There is a statement accompanying Example 3.1 (ii) in [7] which could be construed to imply that this example is hereditarily normal, but that requires yet more extra conditions along the lines of our main example in Section 1.

In Section 2, we will construct a yet more special case of our construction, which has the extra property of having an α_1-space for its one-point compactification. This extra property requires more than just CH: see Section 2, Theorem A.

1. The basic $\diamondsuit$ construction

We will follow Juhász's construction in [3] closely in the following three paragraphs, only making two notational changes: we use Λ in place of L_1 to designate the set of countable limit ordinals, and $R(\alpha, k)$ where [3] uses $C(\alpha, k)$.

1.1. Basic properties. Our space X uses $\omega_1 \times \omega$ as the underlying set. It is locally countable and so, being locally compact, it is first countable. Each 'horizontal line' $\omega_1 \times \{n\} = Y_n$ is a sub-Ostaszewski space in its relative topology, meaning that every relatively open subset is either countable or has countable complement in Y_n. It follows easily that X is hereditarily separable and, since it is hereditarily normal, it is hereditarily collectionwise normal.

Each subspace of the form $\alpha \times n$ $(\alpha < \omega_1, n \in \omega)$ is open, and from this it follows that $\omega_1 \times n$ and $\alpha \times \omega$ are open, while each upper right rectangle $R(\alpha, k) = (\omega_1 \setminus \alpha) \times (\omega \setminus k)$ is closed. Each 'vertical line' $\{\alpha\} \times \omega$ is closed discrete and each set of the form $[\alpha, \alpha + \omega) \times \{k\}$ is discrete and has $R(\alpha + \omega, k)$ as its derived set. From these facts it follows that our space is not countably metacompact. Recall that a space X is countably metacompact if, and only if, each ascending sequence of open sets U_n whose union is X can be followed up by closed subsets; that is, there exists a sequence $\langle F_n : n \in \omega \rangle$ of closed sets whose union is X and which satisfy $F_n \subset U_n$. In our example, if we let $U_n = \omega_1 \times n$ then this proves impossible. Indeed, if F_n is any closed subset of $X = \omega_1 \times \omega$ that is a subset of U_n, then F_n is actually countable, because the alternative is that $Y_k \setminus F_n$ is countable for some $k < n$, but then some interval $[\alpha, \alpha+\omega) \times \{k\}$ would be in $F_n \cap Y_k$, and then all of $R(\alpha+\omega, k)$ would be a subset of F_n, a contradiction.

The facts in the preceding paragraph also help make the proof of hereditary normality easier. For example, if A_0 and A_1 are uncountable subsets of X, then each meets some horizontal line in an uncountable set, and if the lowest such line for A_i is no higher than the lowest for A_{1-i}, then A_i has all but at most countably many of the points of A_{1-i} in its closure. So if F_0 and F_1 are disjoint closed sets in any subspace of X, one must be countable. The case where both are countable is easily handled: both are subsets of some open subspace $X_\alpha = \alpha \times \omega$ of X and this is metrizable, being second countable and regular. Normality of X can thus be achieved by insuring that each countable closed set K and each uncountable closed F disjoint from K can be put into disjoint open sets; for hereditary normality, it is enough to verify this for open subspaces of X since if every open subspace of a topological space is normal, then every subspace is normal, cf. [4]. To achieve normality, we use CH to list all countably infinite subsets of $X = \omega_1 \times \omega$ as $\langle A_\lambda : \lambda \in \Lambda \rangle$ in such a way that A_λ is always a subset of $\alpha \times \omega$ for some $\alpha < \lambda$. Whenever A_λ is closed in $X_\lambda = \lambda \times \omega$, we use the fact that X_λ is metrizable and countable to define a relatively clopen subset Z_λ of X_λ such that $A_\lambda \subset Z_\lambda \subset (\alpha \times \omega)$; note that $\alpha \times \omega$ is open in X_λ and that $Ind(X_\lambda) = 0$. If A_λ is not closed, let $Z_\lambda = \emptyset$. Assuming we can handle the later stages of the inductive construction of X to keep all Z_λ clopen in X itself, normality follows thus: if K is closed and countably infinite and F is closed and disjoint from K, we have $K = A_\lambda$ for some λ, and Z_λ is a countable clopen set containing K; using normality of Z_λ, let U and V be disjoint open subsets of Z_λ

containing K and $F \cap Z_\lambda$ respectively; then U and $V \cup Z_\lambda^c$ are disjoint open subsets of X containing K and F respectively.

To achieve hereditary normality, we define additional open sets. One kind of open subspace that needs special attention is the 'State of Utah' formed by removing some $R(\alpha, k)$ from X. In the resulting space, the sets $\alpha \times (\omega \setminus k)$ and $(\omega_1 \setminus \alpha) \times k$ are disjoint relatively closed sets. To take care of them, we will define open sets $G(\alpha, k)$ such that $\alpha \times (\omega \setminus k) \subset G(\alpha, k) \subset X_\alpha$, and such that $G(\alpha, k) \cap (\alpha \times k)$ is relatively closed in $(\alpha + \omega) \times k$, and we make sure that $G(\alpha, k) \cap (\alpha \times k)$ remains relatively closed in $\omega_1 \times k$. Also, at stage λ, if A_λ is not closed in X but is a subset of $\lambda \times k$ for some $k \in \omega$ and is relatively closed in the subspace $\lambda \times k$ of X_λ, we let $W(\lambda, k)$ be an open subset of $\alpha \times k$ that is relatively closed in $\lambda \times k$ for all integers k for which this applies, otherwise we let $W(\lambda, k) = \emptyset$. We insure that later stages of the induction leave $W(\lambda, k)$ relatively closed in $\omega_1 \times k$ even though it may acquire new limit points outside this subspace. The proof of hereditary normality, assuming this can be done, is similar to that for normality and will be given at the end of the inductive construction.

1.2. The construction of X. Begin by fixing a ladder system $\mathcal{L} = \{L_\lambda : \lambda \in \Lambda\}$ witnessing $\clubsuit$. That is, each ladder L_λ is a set of ordinals of order type ω whose supremum is λ, and for each uncountable $S \subset \omega_1$ there exists λ such that $L_\lambda \subset S$. Let $X_\omega = \omega \times \omega$ be given the discrete topology. Our induction hypothesis at λ is that X_ρ has been defined for all limit $\rho < \lambda$ with topology τ_ρ and underlying set $\rho \times \omega$, in such a way that

(1) τ_ρ is locally compact and Hausdorff.
(2) X_σ is an open subspace of X_ρ whenever $\sigma < \rho$; in other words, $\tau_\sigma = \{U \in \tau_\rho : U \subset \sigma \times \omega\}$.
(3) If $\langle \alpha, n \rangle \in X$ and $\alpha = \gamma + m$ where $\gamma \in \Lambda$ and $m \in \omega$, then $\{V_k(\alpha, n) : k \in \omega\}$ is a descending neighborhood base at $\langle \alpha, n \rangle$ consisting of compact open sets, such that $V_k(\alpha, n) \subset (\gamma \times (n+1)) \cup \{\langle \alpha, n \rangle\}$.

(4) For all limit $\sigma < \rho$, the sets of the form Z_σ, $G(\sigma, k)$ and $W(\sigma, k)$ have been defined to satisfy the foregoing description; in particular, Z_σ is clopen in X_ρ, while $W(\sigma, k) \cap (\rho \times k)$ and $G(\sigma, k) \cap (\rho \times k)$ are relatively clopen in $\rho \times k$.

In (4), if λ is of the form $\gamma + \omega$ for some limit γ, the induction hypothesis includes the assumption that Z_γ, etc. have not yet been defined, nor has Z_δ, etc. for any $\delta \geq \lambda$. In (3), "descending" does not mean "strictly descending," so we let $V_k(m, n) = \{\langle m, n\rangle\}$ for all $\langle m, n\rangle$ in X_ω, and the induction hypothesis is clearly satisfied at $\lambda = \omega$. Suppose it is true for all $\rho \in \Lambda$ such that $\rho < \lambda$.

Case 1: $\lambda \in \Lambda'$ where Λ' denotes the derived set of Λ; that is, Λ' is the set of all countable ordinals that are limits of limit ordinals. In this case, we simply let τ_λ be the topology on $\lambda \times \omega$ whose base is the union of all the earlier τ_ρ, and the induction hypothesis is clearly satisfied; in particular, (1) and (2) are satisfied because they were satisfied at all earlier stages of the induction.

Case 2: λ is of the form $\gamma + \omega$ where $\gamma \in \Lambda$. This case encompasses all the remaining possibilities. We define Z_γ and $W(\gamma, k)$ as explained earlier, letting $\alpha = sup(\pi_1^{\rightarrow} A_\gamma)$. [As usual, π_1 is the canonical projection from $\omega_1 \times \omega$ to ω_1.] There is no problem with doing this since X_γ is countable and metrizable, and $\alpha \times \omega$ is an open subset of X_γ while A_γ is closed whenever $Z_\gamma \neq \emptyset$; similarly for $W(\gamma, k)$. Of course, at most one of $\{Z_\gamma,\ W(\gamma, k)\}$ is nonempty. We hold off defining the open sets $G(\gamma, k)$ until after the neighborhoods of the points $\langle \gamma + m, n\rangle$ have been defined. The definition begins by partitioning the ladder L_γ into infinitely many infinite subsets. Let $\{\xi(p, q, n) : \langle p, q, n\rangle \in \omega^3\}$ be a one-to-one listing of L_γ. The local base at $\langle \gamma + p, q\rangle$ will be defined using the sets

$$D(p, q, i) = \{\langle \xi(p, q, n), i\rangle : n \in \omega\}$$

for which $i \leq q$. Since all the sets Z_σ, $G(\sigma, j)$, and $W(\sigma, k)$ thus far defined are subsets of $\alpha \times \omega$ with $\alpha < \gamma$, and since

$\xi(p,q,n) < \alpha$ for at most finitely many triples of integers, it follows that at most finitely many points of each $D(p,q,i)$ are in any of these Z_σ, etc. Let $\{B_m : m \in \omega\}$ list all sets of the form $Z_\sigma, W(\sigma,k)$, and $G(\sigma,j)$ thus far defined. [The first two kinds have been defined for all $\sigma \leq \gamma$; the third, for all $\sigma < \gamma$.] Given $x = \langle \xi(p,q,n), i\rangle$ in $L_\gamma \times \omega$, let $N(x)$ be a neighborhood of x that misses all B_m such that $m \leq p+q+n$ and such that x is not in the closure of B_m. [By induction hypothesis (3), the only way x could be in the closure of B_m without being in B_m itself is for B_m to be either of the form $G(\sigma,j)$ with $j \leq i$ or of the form $W(\sigma,j)$ with $j \leq i$.] In this way, there are at most finitely many $x \in L_\gamma \times \omega$ outside the closure of B_m for which $N(x)$ meets B_m.

Now $L_\gamma \times \omega$ is a closed discrete subspace of X_γ : this is clear from what induction hypothesis (3) says about the sets $V_k(\alpha,m)$ and from the fact that L_γ is of order type ω with supremum γ. Since X_γ is metrizable and countable, thère is a discrete-in-X_γ family of open sets $\{U(x) : x \in L_\gamma \times \omega\}$ such that $U(x) \cap (L_\gamma \times \omega) = \{x\}$. For each $x \in L_\gamma \times \omega$, we now let $K(x)$ be a basic compact open neighborhood $V_k(x) \subset N(x) \cap U(x)$. Let $V_0(\gamma+p,q)$ be the union of $\{\langle \gamma+p,q\rangle\}$ with

$$\bigcup\{K(x) : x = \langle \xi(p,q,n), i\rangle \text{ for some } n \in \omega,\ i \leq q\}$$

and define $V_k(\gamma+p,q)$ in the same way except with the requirement that $n \geq k$. In this way, $V_k(\gamma+p,q)$ is the one-point compactification of the locally compact subspace

$$\bigcup\{K(x) : x = \langle \xi(p,q,n), i\rangle \text{ for some } n \geq k,\ i \leq q\}$$

of X_γ. For each natural number k, let $\mathcal{V}(\lambda,k) = \{V_k(\gamma+p,q) : \langle p,q\rangle \in \omega^2\}$ and let $\mathcal{V}_\lambda$ be the union of all the $\mathcal{V}(\lambda,k)$ $(k \in \omega)$. The topology τ_λ on X_λ is the one whose base is $\tau_\gamma \cup \mathcal{V}_\lambda$. In X_λ, the family $\mathcal{V}(\lambda,0)$ is easily seen to be a discrete family of clopen sets. So we simply let $G(\gamma,k)$ be the complement in $X_\lambda \setminus R(\gamma,k)$ of $\bigcup\{V_k(\gamma+p,q) : q < k\}$. It is easy to see that the induction hypotheses (1) through (3) continue to be satisfied for $\rho = \lambda$. As for (4): if $Z_\sigma = B_m$ then only finitely many $\xi(p,q,n)$ are less than σ, and there are only finitely many points $x = \langle \xi(p,q,n), i\rangle$ with $\gamma(p,q,n) \geq \sigma$ such that $N(x)$ meets Z_σ, because they all have to satisfy $p+q+n < m$; hence Z_σ remains clopen in X_λ, and the rest of (4) is similarly verified.

The final topology τ on $\omega_1 \times \omega$ is the one whose base is the union of the τ_ρ, $\rho < \omega_1$. The resulting space X is locally countable and first countable by (2) and (3). It is locally compact and Hausdorff because (1) is satisfied at each stage. Its one-point compactification $X+1$ is Fréchet-Urysohn; that is, if a point p is in the closure of $A \subset X+1$, then there is a sequence from A converging to p. This is obvious for the points of X by first countability. If p is the extra point of $X+1$, then either $c\ell_X(A)$ is countable, in which case it is noncompact, or $c\ell_X(A)$ contains points from infinitely many $Y_n = \omega_1 \times \{n\}$–see 1.3. below. In either case, $c\ell_X(A)$ contains an infinite discrete subspace that is closed in X. Now an infinite subset of X is closed in X and discrete if, and only if, some (hence every) one-to-one sequence from it converges to the extra point of $X+1$: this phenomenon holds for all one-point compactifications of locally compact Hausdorff spaces.

1.3. Proofs of key properties. The facts in the beginning of the third paragraph of 1.1 follow easily from (2) and (3) of 1.2, except for the fact that each set of the form $[\alpha, \alpha+\omega) \times \{k\}$ has $R(\alpha+\omega, k)$ as its derived set. This is a consequence of the following crucial property, which is also the key to showing that each horizontal line Y_k is a sub-Ostaszewski space:

$$(\star) \quad \langle \xi, m\rangle \text{ is in the closure of } L_\gamma \times \{k\} \text{ whenever } \xi \geq \gamma \text{ and } m \geq k.$$

Since all but finitely many points of $L_\gamma \times \{k\}$ are also in $[\alpha, \alpha+\omega) \times \{k\}$ whenever $\gamma = \alpha+\omega$, it is immediate from $(\star)$ that $[\alpha, \alpha+\omega) \times \{k\}$ has all of $R(\alpha+\omega)$ in its closure. On the other hand, every point of $X \setminus R(\alpha+\omega, k)$ has a neighborhood that contains at most finitely many points of $[\alpha, \alpha+\omega) \times \{k\}$:

this is immediate from (3) of 1.2. The sub-Ostaszewski property of each Y_k is an easy consequence of $(\star)$ and the fact that the ladder system $\langle L_\lambda : \lambda \in \Lambda\rangle$ witnesses $\clubsuit$: if Y is an uncountable subset of Y_k, then the projection of Y into ω_1 contains some L_γ; in other words, $L_\gamma \times \{k\}$ is a subset of Y, and hence $(\omega_1 \setminus \gamma) \times \{k\}$ is a subset of the closure of Y; and thus every uncountable closed-in-Y_k subset of Y_k is co-countable.

Proof of $(\star)$ All it takes is a simple transfinite induction. The construction in 1.2. insures that all of $[\gamma, \gamma+\omega) \times (\omega \setminus k)$ is in the closure of $L_\gamma \times \{k\}$. Given any limit ordinal $\lambda > \gamma$, if $[\gamma, \lambda) \times (\omega \setminus k)$ is in the closure of $L_\gamma \times \{k\}$, then all but finitely many points of $L_\lambda \times \{k\}$ are in the closure, and therefore all of $[\lambda, \lambda+\omega) \times (\omega \setminus k)$ is in the closure of $L_\gamma \times \{k\}$. If λ is a limit of limit ordinals and $[\gamma, \delta) \times (\omega \setminus k)$ is in the closure of $L_\gamma \times \{k\}$ for all $\delta < \lambda$, then so is the union of these subspaces, namely $[\gamma, \lambda) \times (\omega \setminus k)$. Hence this also holds when $\lambda = \omega_1$, giving us all of $R(\gamma, k)$. $\square$

The proof of hereditary normality, as explained earlier, consists of showing all open sets to be normal. Countable open sets are no problem, of course, while the co-countable case follows easily from normality of X [already shown in 1.1] and the fact that every countable closed set is a subset of some Z_σ: if U is a co-countable open subset of X, its complement is a subset of some clopen Z_σ, and any two disjoint closed subsets of U can only have their closures in X meeting in the metrizable Z_σ, where their traces can easiy be put into disjoint open sets. The remaining open sets fall into two cases.

Case 1. U meets some horizontal line Y_k in a countable set. Then if U meets Y_j in an uncountable set, it follows that $j < k$: if $i < j$ then any open set containing $U \cap Y_j$ must meet Y_i in an unbounded set. Thus there exists $m \in \omega$ such that U meets the first m horizontal lines in a co-countable set and the rest in a countable set. Let

$$\alpha = sup\{\xi : \langle \xi, n\rangle \in U \text{ for some } n \geq m\}$$

and use $G(\alpha, m)$ to break U into two relatively clopen sets. One of them is $G(\alpha, m)$ itself, which is countable, and so the traces on it of any pair of relatively closed disjoint subsets of U can easily be put into disjoint open sets there. The other, $U \setminus G(\alpha, m)$, falls into our remaining case.

Case 2. U is a co-countable subset of $Y_0 \cup Y_1 \cup \cdots \cup Y_k$ for some $k < \omega$. This is handled similarly to the co-countable open subsets of X itself, using the sets $W(\sigma, k)$ in the same way the sets Z_σ were used in that earlier case.

2. A REFINEMENT

There are various properties of compact spaces related to the Fréchet-Urysohn property. One of the most interesting ones is the property of being α_1.

2.1. Definition. A point p of a space Y is an *α_1-point* if, whenever $\{\sigma_n : n \in \omega\}$ is a countable family of sequences in Y converging to p, then there is a sequence σ converging to p such that $ran(\sigma_n) \subseteq^* ran(\sigma)$ for all n. A space is an *α_1-space* if every point is an α_1-point.

As usual, $A \subseteq^* B$ means that $A \setminus B$ is finite. A simple diagonal construction shows that every first countable space is an α_1-space. In this section, we will add some details to the basic $\diamondsuit$ construction in order to make the one-point compactification $X + 1$ of X an α_1-space. Our argument generalizes without change to the more general construction in [3], since we only make use of the sets Z_σ and not the sets $G(\alpha, k)$ nor $W(\lambda, k)$. The following theorem from [2] tells us that we cannot weaken $\diamondsuit$ to CH if we want to construct such a space, even if we do not ask for hereditary normality or even normality.

Theorem A. *The CH is compatible with the statement that every locally compact T_2, Fréchet-Urysohn α_1-space is either first countable or contains the one-point compactification of an uncountable discrete space.*

In [2] we will see much more general theorems which show that such a version of our space cannot be obtained using CH alone, for example:

Theorem. *Modulo large cardinal axioms, CH is compatible with the statement that every locally compact Hausdorff space satisfying wD must satisfy either (i) or (ii) below, if the extra point in the one-point compactification $X+1$ of X is an α_1-point:*

(i) *X is countably metacompact and is the countable union of closed countably compact subspaces; or*
(ii) *X has an uncountable closed discrete subspace.*

If one wishes to do without large cardinals, the extra hypothesis that X can be covered by $\aleph_1$ relatively compact open sets does the trick. The $\diamondsuit$ construction we will now describe satisfies the hypotheses of this theorem along with this extra one, but does not satisfy either conclusion. The additional details in the construction of this X consist of defining, after the construction of each individual X_γ, a clopen set $H_\gamma \subset X_\gamma$ such that:

(1) $Z_\gamma \subset H_\gamma$;
(2) for each $\rho < \gamma$, there is a compact clopen subset $C(\rho,\gamma)$ of X_γ such that $H_\rho \setminus C(\rho,\gamma) \subset H_\gamma$;
(3) $\{x \in L_\gamma \times \{i\} : K(x) \cap H_\gamma \neq \emptyset\}$ is finite for all $i \in \omega$; and
(4) if $\rho < \gamma$ then H_ρ is a clopen subset of X_γ.

Any version of our space X that possesses such a sequence $\{H_\alpha : \alpha \in \Lambda\}$ has a one-point compactification $X+1$ which is an α_1-space. To see this, it is clearly enough to take care of the extra point ∞ of $X+1$. Let σ_n be a sequence converging to ∞ for each n. We may assume $ran(\sigma_n) \subset X$ for all n. Then since $ran(\sigma)$ is closed in X, there exists α_n such that $ran(\sigma_n) \subset Z_{\alpha_n} \subset H_{\alpha_n}$. Letting $\alpha = sup_n \alpha_n$, we have $ran(\sigma_n) \subseteq^* H_\alpha$ for all n, because every compact subset of $ran(\sigma_n)$ is finite. Since H_α is clopen in X, the relative topology on $H_\alpha \cup \{\infty\}$ is the same as the one-point compactification topology. In the latter topology, ∞ is a point of first countability: let $\{V_n(n \in \omega)\}$ be a cover of of H_α by relatively open subsets of H_α with compact closures. The complements G_n of the sets $\overline{V_0 \cup \cdots \cup V_n}$ form a local base at ∞ in $H_\alpha \cup \{\infty\}$ by the usual argument for the elementary fact that every G_δ in a compact Hausdorff space is a point of first countability. Hence the desired $\sigma \to \infty$ exists.

2.2. Construction of the H_α. We begin by letting $H_\omega = \emptyset$. If $\gamma = \beta + \omega$ and H_β has been defined, let $H_\gamma = H_\beta \cup Z_\gamma$. Clearly (1) through (4) hold if they held for β in place of γ. In defining the topology on X_γ, we require that $N(x)$ miss not just the sets Z_σ $(\sigma \leq \gamma)$ specified [in the preceding section, with the notation shift $\lambda \to \gamma$, $\gamma \to \beta$] but also the corresponding larger set H_σ when $\sigma < \gamma$. We do this by altering the definition of $\{B_n : n \in \omega\}$ so that H_σ is substituted for Z_σ whenever $\sigma < \gamma$. Since we did not put any restrictions on how small $N(x)$ can be, this is still subsumed in the general construction. In this way, each H_σ remains clopen in X_γ, assuming it was clopen in X_β. Also, if if λ is a limit of limit ordinals, the topology on X_λ is defined just as before, and if $\sigma < \lambda$, then H_σ remains clopen in X_λ if it was clopen in each earlier X_ρ.

If γ is a limit of limit ordinals, we hold off defining H_γ until τ_λ has been defined for $\lambda = \gamma + \omega$. Then we let $\{\xi_k : k \in \omega\}$ list L_γ in ascending order. For each $\alpha < \gamma$ and $i \in \omega$, define

$$M(\alpha, i) = \{k \in \omega : K(\langle \xi_k, i\rangle) \cap H_\alpha \neq \emptyset\}.$$

Since $H_\alpha \subset X_\alpha$ and H_α is clopen in X_λ, it follows that at most finitely many $\langle \xi_k, i\rangle$ are in the closure of H_α for any given i. It therefore follows from the remark at the end of the first paragraph of 1.2, Case 2, that $M(\alpha, i)$ is finite. Define $f_\alpha : \omega \to \omega$ by $f_\alpha(i) = max(M(\alpha, i))$, and now let $f : \omega \to \omega$ be eventually above all f_α, $\alpha < \gamma$; that is, whenever $\alpha < \gamma$, then $f_\alpha(i) < f(i)$ for all but finitely many i. For each $i \in \omega$ let $H(\gamma, i) = \bigcup\{K(\langle \xi_k, i\rangle) : k < f(i)\}$ and let

$$H_\gamma = (X_\gamma \setminus \bigcup \mathcal{V}(\gamma, 0)) \cup \bigcup\{H(\gamma, i) : i \in \omega\}.$$

Now, $X_\gamma \setminus (\bigcup \mathcal{V}(\gamma, 0))$ is a clopen subset of X_λ, and $\{H(\gamma, i) : i \in \omega\}$ is a discrete collection of compact open sets in X_λ; hence H_γ is clopen not just in X_γ but also in X_λ. It is routine to verify that (1) through (4) of this section hold; in particular, (2) holds because $f_\sigma(i) < f(i)$ for all but finitely many i. $\square$

In [5], we will relax the conditions on the H_α, as well as the topological conditions on X, and still obtain the conclusion that such a space cannot be obtained using CH alone.

3. A LOCALLY COMPACT ANTI-DOWKER SPACE THAT IS HEREDITARILY COUNTABLY PARACOMPACT

The remainder of this paper is devoted to a simple example of a locally compact non-normal space, constructed using MA(ω_1), which is hereditarily countably paracompact. The construction works whenever there is a Q-set—an uncountable subspace Q of $\mathbb{R}$ such that every subset of Q is a G_δ in the relative topology of Q. A seminal result in set-theoretic topology is that MA(ω_1) implies every subset of $\mathbb{R}$ of cardinality $\aleph_1$ is a Q-set [6].

3.1 Example. In [8, pp57–58] it is shown that for any uncountable set of branches of the full binary tree of height ω, there is a 2-coloring of each branch which is not uniformizable. That is, each point of B_α is colored one of two ways, and it is impossible to color the points of the tree in such a way that the coloring agrees with the coloring of each branch B_α on all but finitely many elements of B_α.

Now let X be the topological space whose underlying set is the tree together with a pair of points $p(\alpha, i)$ ($i \in \{0, 1\}$) associated with each B_α. Points of X are isolated while a neighborhood of $p(\alpha, i)$ is any set containing $p(\alpha, i)$ along with all but finitely many points of B_α that were colored with the color that goes with i. The resulting space is not normal, because a pair of disjoint open sets containing the points $p(\alpha, 0)$ on the one hand and the points $p(\alpha, 1)$ on the other hand would give a 2-coloring of the tree that would uniformize the 2-colorings of the branches.

On the other hand, if the branches corresponding to a Q-set in the Cantor set are 2-colored, the resulting space is hereditarily countably paracompact. This follows routinely from the well-known fact [6] that the quotient space obtained by identifying the points in each pair is hereditarily (indeed, perfectly) normal, and from the fact that countable paracompactness of any subspace of X only requires being able to expand any countable partition of the subspace of nonisolated points to a locally finite collection of open sets.

The idea of the proof is this: given a partition, we may as well assume that each member P_n contains either only points $p(\alpha, 0)$ or only points $p(\alpha, 1)$. Then the partition members of the first kind can be expanded to a disjoint collection of open sets whose images are clopen in the quotient space—do the expansion in the quotient space using its normality as in [6]. If V_n is a member of the expansion then the only points in $\overline{V_n} \setminus V_n$ are the "twins" of the points in V_n itself; hence this is a discrete expansion. Repeat the argument for the partition members of the second kind. The result is an expansion of all the P_n such that every point has a neighborhood meeting at most two members of the expansion.

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