Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY
PROCEEDINGS
Volume 24, Spring 1999

THE LINDELÖF PROPERTY OF COUNTABLE PRODUCTS OF LINDELÖF DC–LIKE SPACES

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ABSTRACT. If X_n is Lindelöf regular **DC**–like space for $n \in N$, then $\prod_{n \in N} X_n$ is Lindelöf space.

INTRODUCTION

Let **C** be the class of all compact spaces, **DC** be the class of all spaces which have a discrete cover by compact sets. The topological game $G(\mathbf{C}, X)$ was introduced and studied by R. Telgársky [1]. The games are played by two persons called Players I and Players II. Player I and Player II choose closed subsets of II's previous play (or of X, if $n = 0$): Player I's choice must be in the class **C** and II's choice must be disjoint from I's. We say that Player I wins if the intersection of II's choices is empty. Recall from [1] that a space X is said to be **C**–like if Player I has a winning strategy in $G(\mathbf{C}, X)$. From reference [2] we know that the finite products of **C**–like spaces is Lindelöf space. In this paper, the authors prove that the countable products of Lindelöf **C**–like spaces is also Lindelöf space.

Paracompactness, Lindelöf and metacompact property of countable products have been studied by several authors. E. Michael [4] proved that if X is separable metric space, then

$X^\omega \times Y$ is Lindelöf for every regular hereditarily Lindelöf space Y. Z. Frolik [5] and K. Alster [6] proved that it has the same result if X is a regular Čech–complete Lindelöf space or X is a regular $\mathbf{C}$–scattered Lindelöf space. K. Alster [7] also proved that if Y is a perfect paracompact Hausdorff space and X_n is scattered paracompact Hausdorff space for each $n \in \omega$, then $Y \times \prod_{n\in\omega} X_n$ is paracompact. H. Tanaka [9] proved that if Y is a perfect subparacompact space and X_n is regular subparacompact $\mathbf{DC}$–like space for $n \in \omega$, then $Y \times \prod_{n\in\omega} X_n$ is subparacompact, and he also proved that $\prod_{n\in\omega} X_n$ is metacompact, if X_n is a regular metacopmpact $\mathbf{DC}$–like space for each $n \in \omega$. By now, we don't know the result about Lindelöf $\mathbf{DC}$–like spaces. In this paper, the authors prove that the countable products of regular Lindelöf $\mathbf{DC}$–like spaces is Lindelöf.

1. The Main Result

A function S from 2^X into $2^X \cap \mathbf{C}$ is said to be a stationary strategy for Player I in $G(\mathbf{C}, X)$ if $S(F) \subset F$ for each $F \in 2^X$. We say that the S is winning if he wins every play $\langle S(X), F_0, S(F_0), F_1, S(F_1), \cdots \rangle$. That is, a function S from 2^X into $2^X \cap \mathbf{C}$ is a stationary winning strategy if and only if it satisfies

(i) $S(F) \subset F$ for each $F \in 2^X$,
(ii) if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed subsets of X such that $S(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n\in\omega} F_n = \emptyset$.

X is a $\mathbf{C}$–like space if and only if Player I has a stationary winning strategy in $G(\mathbf{C}, X)$ (cf.[2]).

Theorem 1. *If X_n is a $\mathbf{C}$–like space for each $n \in N$, then $\prod_{n\in N} X_n$ is a Lindelöf space.*

Proof: Let $Y = \bigoplus_{n\in N} X_n$, then Y is **C**–like space (cf. [1]). So we only prove that $\prod_{i\in N} Y_i$ is Lindelöf space, where $Y_i = Y$ for $i \in N$. Let $X = \prod_{i\in N} Y_i$. Let $\mathfrak{B}'$ is base of X, $\mathcal{U}$ is any open cover of X. $\mathcal{U}_{\mathcal{F}} = \{\cup\mathcal{U}' : \mathcal{U}' \subset \mathcal{U}, |\mathcal{U}'| < \omega\}$. $\mathfrak{B} = \{B : B \in \mathfrak{B}', B \subset O$ for some $O \in \mathcal{U}_{\mathcal{F}}\}$. For any $B \in \mathfrak{B}'$, denote $n(B) = \inf\{i : i \in N$, and for any $j \geq i$, $p_j(B) = Y_j\}$. $E = \prod_{i\in N} E_i \subset X$, where $E_i \subset X_i$ is closed compact set of X_i, $i \in N$. Then there is $O \in \mathcal{U}_{\mathcal{F}}$, such that $E \subset O$. From reference [11], we know that there is $B \in \mathfrak{B}'$ such that $E \subset B \subset O$, so $B \in \mathfrak{B}$. Denote $n(E) = \inf\{n(B) : E \subset B, B \in \mathfrak{B}\}$. Then there is $B \in \mathfrak{B}$, such that $n(E) = n(B)$, $B \subset O$ for some $O \in \mathcal{U}_{\mathcal{F}}$. Let S be the stationary winning strategy of I in $G(\mathbf{C}, Y)$.

Denote $\mathcal{F}_1 = \{X\}$, $E_F = \prod_{i\in N} S(Y_i)$ for $F \in \mathcal{F}_1$, $\widetilde{E}_1 = \{E_F : F \in \mathcal{F}_1\}$. For E_F, there exists $B_F \in \mathfrak{B}$, such that $E_F \subset B_F$ and $n(E_F) = n(B_F)$, so there is $O_F \in \mathcal{U}_{\mathcal{F}}$, satisfying $B_F \subset O_F$. Let $\mathcal{U}_1 = \{O_F : F \in \mathcal{F}_1\}$, $\mathfrak{B}_1 = \{B_F : F \in \mathcal{F}_1\}$. Thus $X \setminus O_F \subset X \setminus B_F$. Let $\Delta(F, E_F, B_F) = \mathcal{P}\{1, 2, \cdots, n(B_F) - 1\}$ for $F \in \mathcal{F}_1$. For $A \in \Delta(F, E_F, B_F)$, denote $F_A = \prod_{i\in N} F_i'$, where $F_i' = p_i(F) \setminus p_i(B_F) = Y_i \setminus p_i(B_F)$ if $i \in A$, $F_i' = p_i(F) = Y_i$ if $i \notin A$. Thus F_A is closed set of X. Let $\mathcal{F}_2 = \{F_A : A \in \Delta(F, E_F, B_F), F \in \mathcal{F}_1\}$, $|\mathcal{F}_2| < \omega$. $\widetilde{E}_2 = \{E_F : E_F = \prod_{i\in N} S(p_i(F)), F \in \mathcal{F}_2\}$. For any E_F, there is $B_F \in \mathfrak{B}$, such that $E_F \subset B_F$ and $n(B_F) = n(E_F)$, so there is $O_F \in \mathcal{U}_{\mathcal{F}}$, such that $B_F \subset O_F$. Denote $\mathcal{U}_2 = \{O_F : F \in \mathcal{F}_2\}$, $|\mathcal{U}_2| < \omega$. $\mathfrak{B}_2 = \{B_F : F \in \mathcal{F}_2\}$, $|\mathfrak{B}_2| < \omega$.

For $n \in N$, we assume for every $j \leq n$, $j \geq 2$, there are finite closed family $\mathcal{F}_j$, compact family $\widetilde{E}_j$, and finite open finite $\mathcal{U}_j \subset \mathcal{U}_{\mathcal{F}}$, $\mathfrak{B}_j \subset \mathfrak{B}'$. Satisfying: For any $F \in \mathcal{F}_{j-1}$, $F = \prod_{i\in N} F_i$, where F_i is closed subset of Y_i for $i \in N$. There is $E_F \in \widetilde{E}_{j-1}$, $E_F = \prod_{i\in N} S(p_i(F))$ is compact set of X. There is $B_F \in \mathfrak{B}_{j-1} \subset \mathfrak{B}$, such that $E_F \subset B_F$ and $n(B_F) = n(E_F)$. $B_F \subset O_F$ for some $O_F \in \mathcal{U}_{j-1} \subset \mathcal{U}_F$. $\Delta(F, E_F, B_F) = \mathcal{P}\{1, 2, \cdots, n(B_F) - 1\}$ for $F \in \mathcal{F}_{j-1}$. $\mathcal{F}_j = \{F_A : A \in \Delta(F, E_F, B_F),\ F \in \mathcal{F}_{j-1}\}$, where $F_A = \prod_{i \in N} F_i'$, $F_i' = p_i(F) \setminus p_i(B_F)$, if $i \in A$, otherwise $F_i' = p_i(F)$. $p_i(F_A)$ is closed set of Y_i. For any $F \in \mathcal{F}_{j-1}$, $O_F \supset B_F \supset E_F$, $F \setminus O_F \subset F \setminus B_F = \cup\{F_A : A \in \Delta(F, E_F, B_F)\}$. For $i \in A$, $p_i(F_A) \cap p_i(E_F) = \emptyset$. $\widetilde{E}_j = \{E_F : E_F = \prod_{i \in N} S(p_i(F)),\ F \in \mathcal{F}_j\}$ compact family of X. For $F \in \mathcal{F}_j$, there is $B_F \in \mathfrak{B}$, such that $E_F \subset B_F$ and $n(B_F) = n(E_F)$, and exists $O_F \in \mathcal{U}_{\mathcal{F}}$, $B_F \subset O_F$, $\mathfrak{B}_j = \{B_F : F \in \mathcal{F}_j\}$, $\mathcal{U}_j = \{O_F : F \in \mathcal{F}_j\}$.

For every $F \in \mathcal{F}_n$, let $\Delta(F, E_F, B_F) = \mathcal{P}\{1, 2, \cdots, n(B_F) - 1\}$, where $E_F \in \widetilde{E}_n$, $B_F \in \mathfrak{B}_n$, $E_F \subset B_F \subset O_F$, $O_F \in \mathcal{U}_n$. For any $A \in \Delta(F, E_F, B_F)$, let $F_A = \prod_{i \in N} F_i'$, where $F_i' = p_i(F) \setminus p_i(B_F)$, if $i \in A$, otherwise, $F_i' = p_i(F)$. So $p_i(F_A)$ is closed set of Y_i for $i \in N$. $F \setminus O_F \subset F \setminus B_F = \cup\{F_A : A \in \Delta(F, E_F, B_F)$ for $F \in \mathcal{F}_n$. Let $\mathcal{F}_{n+1} = \{F_A : A \in \Delta(F, E_F, B_F),\ F \in \mathcal{F}_n\}$. Then $|\mathcal{F}_{n+1}| < \omega$. Let $\widetilde{E}_{n+1} = \{E_F : E_F = \prod_{i \in N} S(p_i(F)), F \in \mathcal{F}_{n+1}\}$, $\widetilde{E}_{n+1}$ is a closed compact family of X. For any $F \in \mathcal{F}_{n+1}$, there is $B_F \in \mathfrak{B}$, such that $E_F \subset B_F$ and $n(E_F) = n(B_F)$. So $B_F \subset O_F$ for some $O_F \in \mathcal{U}_{\mathcal{F}}$. Let $\mathcal{U}_{n+1} = \{O_F : F \in \mathcal{F}_{n+1}\}$, $\mathfrak{B}_{n+1} = \{B_F : F \in \mathcal{F}_{n+1}\}$. They are all finite families.

By induction, we have closed family $\mathcal{F}_i$, compact family $\widetilde{E}_i$, and open family $\mathcal{U}_i \subset \mathcal{U}_{\mathcal{F}}$, $\mathfrak{B}_i \subset \mathfrak{B}'$. For any $F \in \mathcal{F}_n$, $E_F \subset F$, $E_F \in \widetilde{E}_n$. $B_F \in \mathfrak{B}_n$, $O_F \in \mathcal{U}_n$, such that $E_F \subset B_F$ and $n(E_F) = n(B_F)$, $B_F \subset O_F$, $F \setminus O_F \subset F \setminus B_F = \cup\{F_A : A \in \Delta(F, E_F, B_F)\}$, where $\Delta(F, E_F, B_F) = \mathcal{P}\{1, 2, \cdots, n(B_F) - 1\}$. Let $\mathcal{U}' = \cup\{\mathcal{U}_n : n \in N\}$. We want to prove $\mathcal{U}'$ is countable open cover of X. Now, we prove a claim which will be used following.

Claim. For any $j \in N$, $F_j \in \mathcal{F}_j$, if $F_{j+1} = F_{A_j}$ for some $A_j \in \Delta(F, E_F, B_F)$, where $F = F_j$, then $\bigcap_{j \in N} F_j$ is empty.

Proof of the claim: For $j \in N$, $F = F_j$, $F_{j+1} = F_{A_j} = \prod_{i \in N} F'_i$, $F'_i = p_i(F) \setminus p_i(B_F) = p_i(F_j) \setminus p_i(B_{F_j})$ if $i \in A_j$, otherwise $F'_i = p_i(F) = p_i(F_j)$. So $F_{j+1} \subset F_j$. Suppose for each $i \in N$, there is natural number $t_i > i$, such that $p_i(F_t) = p_i(F_{t_i})$ for any $t \geq t_i$. Let $F = \prod_{i \in N} p_i(F_{t_i})$. $E = \prod_{i \in N} S(p_i(F_{t_i}))$. Then E is a compact set of X. So there exists $B \in \mathfrak{B}$ such that $E \subset B$ and $n(B) = n(E)$. We choose a natural number t, $t > \{t_i : i < n(B)\}$. Thus $p_i(F_t) = p_i(F_{t_i}) = p_i(F)$ for $i < n(B)$. Then $S(p_i(F_t)) = S(p_i(F))$ for $i < n(B)$. So we have $E_{F_t} = \prod_{i \in N} S(p_i(F_t) \subset B$, $p_i(E_{F_t}) = S(p_i(F_t)) = S(p_i(F)) = p_i(E)$ for $i < n(B)$, and $E \subset B$, $n(E) = n(B)$. So $E_{F_t} \subset B$ and $n(E_{F_t}) = n(B)$. So $n(B) = n(B_{F_t})$. $F_{t+1} = F_{A_t}$ for some $A_t \in \Delta(F, E_F, B_F)$, $F = F_t$. So there is $i \in A_t$, such that $p_i(F_{t+1}) = p_i(F_t) \setminus p_i(B_{F_t})$. Thus $P_i(F_{t+1}) \neq p_i(F_t)$ for some $i < n(B_{F_t}) = n(B)$, and $t > t_i$. Contradicts with $p_i(F_{t+1}) = p_i(F_{t_i})$. So there is some $i \in N$, an increasing natural number sequence $\{m_n\}_{n \in N}$, satisfying $p_i(F_{m_n}) \neq p_i(F_{m_p})$ if $p \neq n$. So $p_i(F_{m_{n+1}}) \subsetneq p_i(F_{m_n})$. From the assumption, we know that $S(p_i(F_{m_n})) \cap p_i(F_{m_{n+1}}) = \emptyset$. So $\cap\{p_i(F_{m_n}) : n \in N\} = \emptyset$. Thus $\bigcap_{n \in N} F_n \subset \bigcap_{n \in N} F_{m_n} = \emptyset$.

Now we prove that $\mathcal{U}' = \cup\{\mathcal{U}_i : i \in N\}$ is a countable cover of X. For $|\mathcal{U}_i| < \omega$, $i \in N$, so $\mathcal{U}'$ is countable. For any $x \in X$, and any $i \in N$. $\mathcal{F}^i_x = \{F : x \in F, F \in \mathcal{F}_i\}$ is finite family. Suppose $\mathcal{F}^i_x$ is not empty for any $i \in N$. For any $i \in N$, and any $F_j \in \mathcal{F}^j_x$, where $2 \leq j \leq i$. There is a $F_{j-1} \in \mathcal{F}^{j-1}_x$, $B_{F_{j-1}} \in \mathfrak{B}_{j-1}$, $O_{F_{j-1}} \in \mathcal{U}_{j-1}$, such that $E_{F_{j-1}} \subset B_{F_{j-1}} \subset O_{F_{j-1}}$, where $E_{F_{j-1}} \in \widetilde{E}_{j-1}$, $n(E_{F_{j-1}}) = n(B_{F_{j-1}})$, and some $A_{j-1} \in \Delta(F_{j-1}, E_{F_{j-1}}, B_{F_{j-1}})$, satisfying $p_i(F_j) = p_i(F_{j-1}) \setminus p_i(B_{F_{j-1}})$, if $i \in A_{j-1}$, otherwise $p_i(F_j) = p_i(F_{j-1})$. By König's Lemma (cf. [10]), we have a closed sets sequence $\{F_j\}_{j \in N}$, $F_j \in \mathcal{F}^j_x$, and $E_{F_j} \in \widetilde{E}_j$, $B_{F_j} \in \mathfrak{B}_j$, $O_{F_j} \in \mathcal{U}_j$, $E_{F_j} \subset B_{F_j} \subset O_{F_j}$ for $j \in N$, such that for any $j \in N$, there is some $A_j \in \Delta(F_j, E_{F_j}, B_{F_j})$, satisfying $p_i(F_{j+1}) = p_i(F_j) \setminus p_i(B_{F_j})$, if $i \in A_j$, otherwise $p_i(F_{j+1}) = p_i(F_j)$. So $F_{j+1} = F_{A_j}$, where $F = F_j$, $A_j \in \Delta(F_j, E_{F_j}, B_{F_j})$. But $x \in \bigcap_{j \in N} F_j$, this contradicts the claim. Thus there exists $i \in N$, such that $\mathcal{F}_i^x = \emptyset$. Let n is the smallest number i, such that F_x^n is empty. Then $\mathcal{F}_x^{n-1} \neq \emptyset$. For any $F \in \mathcal{F}_x^{n-1}$, $x \in F$, $F \setminus O_F \subset F \setminus B_F$, where $O_F \in \mathcal{U}_{n-1}$, $B_F \in \mathfrak{B}_{n-1}$. $F \setminus B_F = \cup\{F_A : A \in \Delta(F, E_F, B_F)\}$. For any $A \in \Delta(F, E_F, B_F)$, $x \notin F_A$, so $x \notin F \setminus B_F$. Thus $x \in O_F$ for some $F \in \mathcal{F}_x^{n-1}$. Then $x \in \cup \mathcal{U}_{n-1}$. Thus $\mathcal{U}' = \cup\{\mathcal{U}_i : i \in N\}$ is countable open cover of X. So X is Lindelöf space.

Theorem 2. [3] *X is a regular space, if X is a Lindelöf* **DC**–*like space, then X is a* **C**–*like space.*

Theorem 3. *If X_n is a regular Lindelöf* **DC**–*like space for each $n \in N$, then $\prod_{n \in N} X_n$ is a Lindelöf space.*

Proof: By Theorem 2, we know that X_n is a **C**–like space for each $n \in N$. So $\prod_{n \in N} X_n$ is a Lindelöf space from Theorem 1.

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