Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

TOPOLOGY PROCEEDINGS
Volume 24, Spring 1999

A SUBSPACE OF THE UPPER STONE-CECH COMPACTIFICATION

VRUNDA PRABHU

ABSTRACT. We construct a space ρX which contains an unique copy of every strict H-closed extension of $e[X]$ (and no others). ρX is compact, homeomorphic to the set of all open filters on X with the Alexandroff topology, and an atomic complete upper semi-lattice.

1. Background and Introduction

We explain some of the terms required for this sequel in this section. A detailed treatment can be found in [6].

A Hausdorff space X is **H-closed** if it is closed in every Hausdorff space containing X as a subspace.

$\mathcal{H}(X) = \{Y \in \mathcal{E}(X) : Y \text{ is H-closed}\}$ is a set of H-closed extensions of X such that no two are equivalent and each H-closed extension of X is equivalent to some $Y \in \mathcal{H}(X)$.

$\kappa X = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$ and $\{U : U$ is open in $X\} \cup \{U \cup \{\mathcal{U}\} : U \in \mathcal{U}, \mathcal{U} \in \kappa X \setminus X\}$ is an open base for the open sets in κX, called the **Katetov extension** of X. A characterization of κX is the following:

1991 *Mathematics Subject Classification.* 54, 04.

Key words and phrases. embeddings, H-closed, upper Stone-Cêch compactification, Banaschewski-Fomin-Sanin extension, Katetov extension, strict extension, minimal Hausdorff extension.

Theorem 1.1. *Let X be Hausdorff. Then*

(a) *κX is an H-closed extension of X and X is open in κX*
(b) *If $Y \in \mathcal{H}(X)$, there is a unique continuous function $f : \kappa X \to Y$ such that $f|_X = id_X$, i.e., $\kappa X \geq Y$ and*
(c) *If $Z \in \mathcal{H}(X)$ and $Z \geq Y$ for all $Y \in \mathcal{H}(X)$, then $\kappa X \equiv_X Z$, in particular $\kappa X = \vee\mathcal{H}(X)$.*

Let Y be a Hausdorff extension of X. For $U \in \tau(X)$, $oU = \cup\{W : W \in \tau(Y) \text{ and } W \cap X \subseteq U\}$. $\{oU : U \in \tau(Y)\}$ is an open base for a Hausdorff topology $\tau^{\#}$ on Y that is contained in the original topology of Y, called the **strict extension topology** on Y. The Hausdorff extension Y with the strict extension topology $\tau^{\#}$ is denoted by $Y^{\#}$. The strict extension $(\kappa X)^{\#}$ of X is denoted as σX and called the Fomin extension of X. The **Fomin extension** σX of X has the strict topology, and as a set $\sigma X = \kappa X$.

A Hausdorff space is **minimal Hausdorff** if X has no strictly coarser Hausdorff topology.

Proposition 1.2. *Let X be Hausdorff. T.F.A.E.:*

(a) *X is minimal Hausdorff*
(b) *X is semiregular and H-closed*
(c) *Every open filter with an unique adherent point converges.*

The semiregularization $\kappa X(s)$ of the Katetov extension κX is the set κX with the topology generated by $\{oU : U \in \mathcal{RO}\ (X)\}$ ($\mathcal{RO}\ (X)$ is the set of regular open sets of X). The semiregular H-closed extension $\kappa X(s)$ of X is denoted as μX and called the **Banaschewski-Fomin-Sanin minimal Hausdorff extension** for the semiregular Hausdorff space X.

In this article, we will work with the unit interval with a special topology. Let I^+ be the unit interval with $\tau(I^+) = \{\emptyset, I^+\} \cup \{[0, a) : 0 < a < 1\}$. The class of all T_0 spaces can be embedded in a product of copies of I^+. For a T_0 space X, the T_0 compactification $\beta_{I^+}X$ of X is denoted as β^+X and called the upper Stone-Cêch compactification of X [3]. I^+ is called the generating space for the class of T_0 spaces. Let $C^+(X) =$

$C(X, I^+)$ and $\prod_{C^+(X)} I^+$ denote the product of $C^+(X)$ copies of I^+.

There is no T_1 space that works as a generating space for the class of Hausdorff or H-closed spaces [2]. But, there are T_0 spaces, namely I^+ and $\mathbf{S}$ ($\mathbf{S} = \{0,1\}$ with $\tau(\mathbf{S}) = \{\emptyset, \{0\}, \mathbf{S}\}$) that are generating spaces for all H-closed spaces. The usual embedding function embeds a Tychonoff space X in $\prod_{C^*(X)} I$ in such a way that its closure is the Stone-Cêch compactification βX of X. A natural question is whether there is a parallel analogue of embedding a Hausdorff space X in $\prod_{C^+(X)} I^+$. In 1976, Porter [4] asked if it possible to construct in terms of $\prod_{C^+(X)} I^+$, the Fomin H-closed extension σX for a Hausdorff space X or the Banaschewski-Fomin-Sanin minimal Hausdorff extension μX for a semiregular space X. In 1993, we [7] showed that it is possible to embed σX in $\prod_{C^+} I^+$ in such a way that $\sigma X \subseteq \beta^+ X$. We showed that σX, μX, and in fact a very large class of extensions of X are embedded in $\beta^+ X$.

2. $\beta^+ X$

The results in this section are proved in [8], and are stated here to demonstrate the need for finding the space described in section 3.

Definition 2.1. *Let X be a Hausdorff space and $f \in C^+(X)$. For $y \in \sigma X \setminus X$ (recall that y is a free open ultrafilter on X), let $\tilde{f}(y)$ be the unique point in I to which y converges in the usual topology on I, and for $y \in X$ let $\tilde{f}(y) = f(y)$.*

Proposition 2.2. [8] *Let X be a Hausdorff space and $f \in C^+(X)$. Then $\tilde{f} \in C^+(\sigma X)$. In particular, X is C^+-embedded in σX.*

Notation 2.3. For $A \subseteq X$, the reverse characteristic function, $\chi_A : X \to \{0,1\}$ is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \in \mathrm{A} \\ 1 & \text{if } x \notin \mathrm{A} \end{cases}$$

Theorem 2.4. [8][9] *For a Hausdorff space X, the function $\tilde{e} : \sigma X \hookrightarrow \prod_{C^+(X)} I^+$ defined by $\tilde{e}(y)(f) = \tilde{f}(y)$ is an embedding.*

Corollary 2.5. [1]A Hausdorff space X is H-closed iff $e[X]$ is a maximal Hausdorff subspace of $\beta^+ X$.

The following notation is helpful in showing that μx can be embedded in $\beta^+ X$ when X is a Hausdorff space.

Notation 2.6. Let Y be an extension of X and $f : X \to I^+$ be a function. For each $0 < r < 1$, let $U_r = Y \setminus cl_Y f^{\leftarrow}[[r, 1]]$. Then U_r is open in Y. Define $\chi_{f,r} : Y \to I^+$ by

$$\chi_{f,r}(y) = \begin{cases} r & \text{if } y \in U_r \\ 1 & \text{if } y \in Y \setminus U_r \end{cases}$$

This is a modification of the characteristic functions of U_r in X.

Proposition 2.7. [8]*Let Y be an extension of X, $f \in C^+(X)$, then, $\hat{f} : Y^{\#} \to I^+$, $\hat{f} = \wedge\{\chi_{f,r} : 0 < r < 1\}$ is continuous. (Note that in this conclusion the domain of $\hat{f}$ is changed to $Y^{\#}$ and is not Y.)*

Theorem 2.8. [8]*Let Y be a strict T_0 extension of X. The function $\hat{e}_Y : Y \to \prod_{C^+(X)} I^+$ defined by $\hat{e}_Y(y)(f) = \hat{f}(y)$ is an embedding and $e[X] \subseteq \hat{e}_Y[Y] \subseteq \beta^+ X$.*

As a semiregular extension is a strict extension (7.1(e)(4) of [6]), the next result is a consequence of 2.8.

Proposition 2.9. [8]*Let H be an H-closed extension of X. Then H_0 is semiregular iff $H_0 = H(s)$.*

Theorem 2.10. [13]*Every minimal Hausdorff extension of a semiregular space can be embedded in $\prod_{C^+(X)} I^+$. In particular, μX can be embedded in $\prod_{C^+(X)} I^+$.*

Another application is that Tikoo, in 1984, extended the definition of the Banaschewski-Fomin-Sanin minimal Hausdorff extension μX for semiregular Hausdorff spaces X to arbitrary Hausdorff spaces. The extended definition of the extension μX is a strict extension. Thus, μX can be embedded in $\beta^+ X$. This answers a 1976 question posed by Porter [4].

Thus, $\beta^+ X$ contains all the strict H-closed extensions of X and may in fact contain other non-strict H-closed extensions. As noted in [7], many copies of the same strict H-closed extension of X are contained in $\beta^+ X$.

In the next section we construct an extension ρX of X which contains all the strict H-closed extensions of $e[X]$ (and no others) corresponding to the open filters on X, inside of $\prod_{C^+(X)} I^+$, and in fact exactly one copy of each strict H-closed extension. The reverse characteristic functions of open sets on X still play an important role as the basic open sets of the new structure ρX formed inside of $cle[X]$ can be expressed in terms of these functions. We show that ρX is homeomorphic to the set of all open filters on X with the Alexandroff topology.

3. Construction of ρX

We describe here some facts that are useful for later results.

Remarks 3.1. Let X be a space, $f \in C^+(X)$ and $\mathcal{F}$ an open filterbase on X.

(a) Then $f(\mathcal{F})$ always converges to 1. The filterbase $f(\mathcal{F})$ may converge to other points but 1 is always in $c(f(\mathcal{F}))$ (= set of convergent points of $f(\mathcal{F})$ in I^+).

(b) If $f(\mathcal{F})$ converges to $b \in I^+$ and d is such that $b \leq d \leq 1$, then $f(\mathcal{F})$ converges to d.

(c) There is some $a \in I^+$, such that $c(f(\mathcal{F})) = [a, 1]$. This can be easily justified using the above two facts. Let $a = \inf(c(f(\mathcal{F})))$. Then $(a, 1] \subseteq c(f(\mathcal{F}))$. It suffices to show that $f(\mathcal{F})$ converges to a. We are done if $a = 1$, so assume that $a < 1$. An open neighborhood of a is of the form $[0, b)$, where $a < b$. Let $a < d < b$, $f(\mathcal{F})$ converges to d, thus there is a $F \in \mathcal{F}$ such that $f[F] \subseteq [0, b)$. But, $[0, b)$

is an arbitrary neighborhood of a; so, $f(\mathcal{F})$ converges to a.

(d) If $\alpha \in \prod_{C^+(X)} I^+$, then $cl\{\alpha\}$ is denoted as $cl\alpha$. Thus,

$$\begin{aligned} cl\alpha &= cl\textstyle\prod_{C^+(X)}\{\alpha(f)\} = \prod_{C^+(X)} cl\{\alpha(f)\} \\ &= \textstyle\prod_{C^+(X)}[\alpha(f), 1]. \end{aligned}$$

For α, $\beta \in \prod_{C^+(X)} I^+$, define $\alpha \leq \beta$ iff $\alpha(f) \leq \beta(f)$ for every $f \in C^+(X)$. The binary relation $\leq$ is a partial order on $\prod_{C^+(X)} I^+$ with $\vec{\mathbf{0}}$ as its smallest element and $\vec{\mathbf{1}}$ as its largest element. For α, $\beta \in \prod_{C^+(X)} I^+$, it follows that $\alpha \vee \beta$ exists and is defined by $(\alpha \vee \beta)(f) = \alpha(f) \vee \beta(f)$. The product $\prod_{C^+(X)} I^+$ with this partial order is a complete lattice. Note that $\alpha \leq \beta$ iff $\beta \in cl\alpha$. This is a partial order which can be defined on a T_0 space but which becomes trivial for T_1 spaces.

Let X be a T_0 space and e be the map that embeds X in $\prod_{C^+(X)} I^+$. It follows that the partially ordered set $(\beta^+X, \leq)$ is a complete upper semilattice. The maximum element of β^+X is denoted as $\vec{\mathbf{1}}$, i.e., $\vec{\mathbf{1}}(f) = 1$ for all $f \in C^+(X)$. For $\alpha \in \prod_{C^+(X)} I^+$, define

$$\mathcal{G}(\alpha) = \{\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[X] : f \in C^+(X), n \in \mathbf{N}\}.$$

Proposition 3.2. *Let $\alpha \in \prod_{C^+(X)} I^+$. Then $\alpha \in \beta^+X$ iff $\mathcal{G}(\alpha)$ is an open filter subbase on $e[X]$.*

Proof: Let F be a finite subset of $C^+(X)$, $\{n_f : f \in F\} \subseteq \mathbf{N}$, and $T = \bigcap\{\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n_f)] : f \in F\}$. Now, T is a basic open set of $\alpha \in \prod_{C^+(X)} I^+$. Thus, $\alpha \in \beta^+X$ iff for each T, $T \cap e[X] \neq \emptyset$. This shows that $\alpha \in \beta^+X$ iff $\mathcal{G}(\alpha)$ is an open filter subbase on $e[X]$. □

For $\alpha \in \beta^+X$, $\mathcal{G}(\alpha)$ need not be a base as is shown in the following example.

Example 3.3. Let $X = \mathbf{R}$ and $\chi_{(-1,2)}$ and $\chi_{(-2,1)}$ be the reverse characteristic functions of the sets $(-1, 2)$ and $(-2, 1)$.

Define $\alpha \in \prod_{C^+(X)} I^+$ as follows:

$$\alpha(f) = \begin{cases} 0 & \text{if } f = \chi_{(-1,2)} \text{ or } \chi_{(-2,1)} \\ 1 & \text{otherwise} \end{cases}$$

Now, $e(0) \in e[\mathbf{R}]$ and $e(0) \leq \alpha$, so $\alpha \in cle(0) \subseteq cle[\mathbf{R}]$. If $f = \chi_{(-1,2)}$ (resp. $\chi_{(-2,1)}$), then $\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[\mathbf{R}] = e[(-1,2)]$ (resp. $\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[\mathbf{R}] = e[(-2,1)]$) for all $n \in \mathbf{N}$. If $f \neq \chi_{(-1,2)}$ or $\chi_{(-2,1)}$, then $\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[\mathbf{R}] = e[\mathbf{R}]$ for all $n \in \mathbf{N}$. So, $\mathcal{G}(\alpha) = \{e[(-1,2)], e[(-2,1)], e[\mathbf{R}]\}$ is an open filter subbase but is not an open filterbase. □

Let $f \in C^+(X)$, $\alpha \in \beta^+X$, and $\langle \mathcal{G}(\alpha) \rangle$ be the open filter generated by $\mathcal{G}(\alpha)$ on $e[X]$. By 3.1(a), $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ converges to 1. Moreover, since $\pi_f[e[X] \cap \pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)]] \subseteq [0, \alpha(f) + 1/n)$ for all $n \in \mathbf{N}$ and $e[X] \cap \pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \in \mathcal{G}(\alpha)$, then $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ converges to $\alpha(f)$.

We now introduce two notational definitions.

Definition 3.4. *For $f \in C^+(X)$ and $\alpha \in \beta^+X$, let $\alpha^\star(\mathbf{f}) = \inf(c(\pi_f[\langle \mathcal{G}(\alpha) \rangle]))$, i.e., the infimum of the set of convergent points of the filterbase $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ on I^+, and let $\rho\mathbf{X} = \{\alpha^\star : \alpha \in \beta^+X\}$.*

Proposition 3.5. *Let $\alpha \in \beta^+X$. Then $\alpha^\star \in \beta^+X$, $\alpha^\star \leq \alpha$ and for $f \in C^+(X)$, $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ converges to $\alpha^\star(f)$.*

Proof: Let F be a finite set. Suppose $\alpha^\star \in \bigcap\{\pi_{f_i}^{\leftarrow}[[0, b_i)] : i \in F\}$. Then $\alpha^\star(f_i) < b_i$ and $\pi_{f_i}^{\leftarrow}[[0, b_i)] \cap e[X] \in \langle \mathcal{G}(\alpha) \rangle$. As $\langle \mathcal{G}(\alpha) \rangle$ has the finite intersection property, $e[X] \cap \bigcap\{\pi_{f_i}^{\leftarrow}[[0, b_i)] : i \in F\} \neq \emptyset$. Thus, $\alpha^\star \in \beta^+X$. By the note preceding Definition 3.4, for $f \in C^+(X)$ and $\alpha \in \beta^+X$, $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ converges to $\alpha(f)$. Also, $\alpha^\star(\mathbf{f}) = \inf(c(\pi_f[\langle \mathcal{G}(\alpha) \rangle]))$. Hence, $\alpha^\star(f) \leq \alpha(f)$ and so, $\alpha^\star \leq \alpha$. □

The partial order on ρX is the one induced by the natural partial order on $\prod_{C^+(X)} I^+$. The maximum element $\vec{\mathbf{1}} \in \beta^+X$ and $\mathcal{G}(\vec{\mathbf{1}}) = \{e[X]\}$ is an open filter on $e[X]$. Also, $\pi_f[e[X]] = f[X] \subseteq [0,1]$, $c(\pi_f\langle \mathcal{G}(\vec{\mathbf{1}}) \rangle) = [sup f[X], 1]$ and $\inf(c(\pi_f\langle \mathcal{G}(\vec{\mathbf{1}}) \rangle)) =$

$supf[X]$. Thus, $\vec{\mathbf{1}}^\star(f) = supf[X]$. When $f = \chi_X$ (the reverse characteristic function of X), $\vec{\mathbf{1}}^\star(f) = 0$. This means $\vec{\mathbf{1}}^\star \neq \vec{\mathbf{1}}$ and $\vec{\mathbf{1}} \notin \rho X$. By 3.1(c), $\pi_f[\langle\mathcal{G}(\alpha)\rangle]$ converges to $\alpha(f)$.

Remark 3.6. (a) First, we show that $e[X] \subseteq \rho X$, i.e., for $x \in X$, $e(x)^\star = e(x)$. By Proposition 3.5, we have that $e(x)^\star \le e(x)$. To show that $e(x) \le e(x)^\star$, let $f \in C^+(X)$ and $n \in \mathbf{N}$, and note that each element of $\mathcal{G}(e(x))$, e.g., $\pi_f^{\leftarrow}[[0, f(x) + 1/n)] \cap e[X]$, always contains $e(x)$. If $g \in C^+(X)$ and $\pi_g[\langle\mathcal{G}(e(x))\rangle]$ converges to $a \in I^+$, then for $\varepsilon > 0$, there are $U_1, \cdots, U_n \in \mathcal{G}(e(x))$ such that $\pi_g[U_1 \cap \cdots \cap U_n] \subseteq [0, a + \varepsilon)$. As, $e(x) \in U_1 \cap \cdots \cap U_n$, $\pi_g(e(x)) < a + \varepsilon$, i.e., $e(x)(g) < a + \varepsilon$ for all $\varepsilon > 0$. Hence, $e(x)(g) \le a$. Since $\pi_g[\langle\mathcal{G}(e(x))\rangle]$ converges to $e(x)^\star(g)$ by Proposition 3.5, it follows that $e(x)(g) \le e(x)^\star(g)$. This shows that $e(x) \le e(x)^\star$.

(b) By (a), we have that $e[X] \subseteq \rho X \subseteq \beta^+ X$ and $cl\rho X = \beta^+ X$.

(c) Note that the minimal element $\vec{\mathbf{0}}$ in $\prod_{C^+(X)} I^+$ need not be in $cl\rho X$. To see this let U, V be nonempty disjoint open sets in X. Then $\pi_{\chi_U}^{\leftarrow}[[0, 1/2)] \cap \pi_{\chi_V}^{\leftarrow}[[0, 1/2)] \cap e[X] = e[U] \cap e[V] = e[\emptyset] = \emptyset$. But, $\vec{\mathbf{0}} \in \pi_{\chi_U}^{\leftarrow}[[0, 1/2)] \cap \pi_{\chi_V}^{\leftarrow}[[0, 1/2)]$; hence $\vec{\mathbf{0}} \notin \beta^+ X$. If $\alpha \in \beta^+ X$, then $\alpha^\star \le \alpha$. We have that $\alpha \in cl\alpha^\star \subseteq cl\rho X$. Hence $cl\rho X = \beta^+ X$.

The next four propositions describe the structure of ρX and establish some of its properties. Earlier, when trying to embed σX in $\prod_{C^+(X)} I^+$, in order to define a function from $\sigma X \to \prod_{C^+(X)} I^+$ we used the fact that I with the usual topology is Hausdorff. Here, in a T_0 setting we are able to determine a unique point in I that corresponds to an open filter on X and so can extend functions and have a correspondence between points of ρX and the set of open filters on X.

Proposition 3.7. *Let $\alpha \in \beta^+ X$. Then $\mathcal{G}(\alpha^\star) = \langle\mathcal{G}(\alpha)\rangle$; in particular $(\alpha^\star)^\star = \alpha^\star$.*

Proof: As $\alpha^\star \leq \alpha$, it follows that for all $f \in C^+(X)$, and $n \in \mathbf{N}$, $\pi_f^{\leftarrow}[[0, \alpha^\star(f) + 1/n)] \cap e[X] \subseteq \pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[X]$. Thus, $\langle \mathcal{G}(\alpha) \rangle \subseteq \langle \mathcal{G}(\alpha^\star) \rangle$. Also, for $f \in C^+(X)$, as $\pi_f[\langle \mathcal{G}(\alpha) \rangle]$ converges to $\alpha^\star(f)$, for each $n \in \mathbf{N}$, $\pi_f^{\leftarrow}[[0, \alpha^\star(f) + 1/n)] \cap e[X] \in \langle \mathcal{G}(\alpha) \rangle$. Thus, $\mathcal{G}(\alpha^\star) \subseteq \langle \mathcal{G}(\alpha) \rangle$ and, hence, $\langle \mathcal{G}(\alpha^\star) \rangle \subseteq \langle \mathcal{G}(\alpha) \rangle$. So, we have established that $\langle \mathcal{G}(\alpha^\star) \rangle = \langle \mathcal{G}(\alpha) \rangle$. Next, we show that $\mathcal{G}(\alpha^\star) = \langle \mathcal{G}(\alpha^\star) \rangle$. The first step is to show that $\mathcal{G}(\alpha^\star)$ is closed under finite intersections. Suppose f, $g \in C^+(X)$ and n, $m \in \mathbf{N}$. There is some $U \in \tau(X)$ such that $e[U] = \pi_f^{\leftarrow}[[0, \alpha^\star(f) + 1/n)] \cap \pi_g^{\leftarrow}[[0, \alpha^\star(g) + 1/m)] \cap e[X]$. Now, the reverse characteristic function of U, $\chi_U \in C^+(X)$. As $e[U] \in \langle \mathcal{G}(\alpha) \rangle$ and $\pi_{\chi_U}[e[U]] = \{0\}$, it follows that $\pi_{\chi_U}[\langle \mathcal{G}(\alpha) \rangle]$ converges to 0. So, $\alpha^\star(\chi_U) = 0$ and $e[U] \in \mathcal{G}(\alpha^\star)$. The final step in showing that $\mathcal{G}(\alpha^\star)$ is an open filter, is to show that if W is an open set in $e[X]$ and $W \supseteq V$ for some $V \in \mathcal{G}(\alpha^\star)$, then $W \in \mathcal{G}(\alpha^\star)$. As W is open in $e[X]$, there is an open set R in X such that $e[R] = W$. As $\pi_{\chi_R}[V] \subseteq \pi_{\chi_R}[e[R]] \subseteq \{0\}$, it follows that $\pi_{\chi_R}[\langle \mathcal{G}(\alpha) \rangle]$ converges to 0. So, $\alpha^\star(\chi_R) = 0$ and $W = e[R] = \pi_{\chi_R}^{\leftarrow}[[0, \alpha^\star(\chi_R) + 1/2)] \cap e[X] \in \mathcal{G}(\alpha^\star)$. Finally, to show that $(\alpha^\star)^\star = \alpha^\star$. For $f \in C^+(X)$, $(\alpha^\star)^\star(f) = \inf(c(\pi_f(\langle \mathcal{G}(\alpha^\star) \rangle))) = \inf(c(\pi_f(\langle \mathcal{G}(\alpha) \rangle))) = \alpha^\star(f)$. So, $(\alpha^\star)^\star = \alpha^\star$ $\square$

The next result is a corollary to the proof of Proposition 3.7 but is stated separately as it is used frequently in the sequel. This result characterizes those open filters on $e[X]$ that converge to points in ρX.

Corollary 3.8. *For an open set U in X and $\alpha \in \beta^+X$, $e[U] \in \mathcal{G}(\alpha^\star)$ iff $\pi_{\chi_U}(\alpha^\star) = 0$, i.e., $\mathcal{G}(\alpha^\star) = \{e[U] : U \in \tau(X), \alpha^\star(\chi_U) = 0\}$.*

Proposition 3.9. *Let $\alpha, \gamma \in \beta^+X$. Then $\langle \mathcal{G}(\gamma) \rangle = \langle \mathcal{G}(\alpha) \rangle$ iff $\alpha^\star = \gamma^\star \leq \gamma$.*

Proof: Suppose $\alpha^\star = \gamma^\star$. Then by Proposition 3.7, $\langle \mathcal{G}(\gamma) \rangle = \mathcal{G}(\gamma^\star) = \mathcal{G}(\alpha^\star) = \langle \mathcal{G}(\alpha) \rangle$. Conversely, suppose $\alpha, \gamma \in \beta^+X$ and $\langle \mathcal{G}(\gamma) \rangle = \langle \mathcal{G}(\alpha) \rangle$. For $f \in C^+(X)$, $\gamma^\star(f) = \inf(c(\pi_f(\langle \mathcal{G}(\gamma) \rangle))) =$

$\inf(c(\pi_f(\langle \mathcal{G}(\alpha)\rangle))) = \alpha^\star(f)$. So, $\alpha^\star = \gamma^\star$. As, $\gamma^\star \le \gamma$ by Proposition 3.5, the conclusion follows. $\square$

Proposition 3.10. *Let $\alpha \in \beta^+X$, $\beta \in \prod_{C^+(X)} I^+$ and $\alpha \le \beta$. Then $\beta \in \beta^+X$, $\alpha^\star \le \beta^\star$ and $\langle \mathcal{G}(\beta)\rangle \subseteq \langle \mathcal{G}(\alpha)\rangle$.*

Proof: $\beta \in cl\{\alpha\} \subseteq cl(\beta^+X) = \beta^+X$. Next, let $f \in C^+(X)$ and $n \in \mathbf{N}$. $[0, \alpha(f) + 1/n) \subseteq [0, \beta(f) + 1/n)$, therefore, $\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[X] \subseteq \pi_f^{\leftarrow}[[0, \beta(f) + 1/n)] \cap e[X]$. Thus, $\pi_f^{\leftarrow}[[0, \beta(f) + 1/n)] \cap e[X] \in \langle \mathcal{G}(\alpha)\rangle$ i.e.,$\langle \mathcal{G}(\beta)\rangle \subseteq \langle \mathcal{G}(\alpha)\rangle$. So, $c(\pi_f(\langle \mathcal{G}(\beta)\rangle)) \subseteq c(\pi_f(\langle \mathcal{G}(\alpha)\rangle))$ and $\alpha^\star(f) = \inf c(\pi_f(\langle \mathcal{G}(\alpha)\rangle)) \le \inf c(\pi_f(\langle \mathcal{G}(\beta)\rangle)) = \beta^\star(f)$. Hence, $\alpha^\star \le \beta^\star$. $\square$

Another way of viewing ρX is that it arises from a partition of β^+X. For $\alpha \in \beta^+X$, let $P_\alpha = \{\gamma \in \beta^+X : \gamma^\star = \alpha^\star\}$. Then $\{P_\alpha : \alpha \in \beta^+X\}$ is a partition of β^+X. By Proposition 3.5, for $\alpha \in \beta^+X$, $\alpha^\star$ is the smallest element of P_α. For $\alpha \in \beta^+X$, $\vee P_\alpha$ exists in β^+X since β^+X is a complete upper semilattice. One question is whether $\vee P_\alpha \in P_\alpha$. This is answered in the negative by the next results.

Proposition 3.11. *Let $\alpha \in \rho X$ and $f \in C^+(X)$. Define $\beta(g) = \alpha(g)$ for $g \ne f$ and $\beta(f) = 1$. Then $\alpha^\star = \beta^\star$.*

Proof: Since $\alpha \le \beta$, by Proposition 3.10, $\beta \in \beta^+X$, $\alpha = \alpha^\star \le \beta^\star$, and $\langle \mathcal{G}(\beta)\rangle \subseteq \mathcal{G}(\alpha)$. By Proposition 3.9, it suffices to show that $\mathcal{G}(\alpha) \subseteq \langle \mathcal{G}(\beta)\rangle$. A typical element of $\mathcal{G}(\alpha)$ is of the form $\pi_g^{\leftarrow}[[0, \alpha(g) + 1/n)] \cap e[X]$ where $g \in C^+(X)$ and $n \in \mathbf{N}$. If $f \ne g$, $\pi_g^{\leftarrow}[[0, \alpha(g) + 1/n)] \cap e[X] \in \mathcal{G}(\beta)$. We need to consider the case when $f = g$. There is an open set U in X such that $e[U] = \pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[X]$. Since $e[X] = \pi_f^{\leftarrow}[[0, \beta(f) + 1/m)] \cap e[X]$ for any $m \in \mathbf{N}$, we can assume that $U \ne X$. If $f \ne \chi_U$, then $\beta(\chi_U) = \alpha(\chi_U)$ and $e[U] = \pi_{\chi_U}^{\leftarrow}[[0, \beta(\chi_U) + \frac{1}{2})] \cap e[X] \in \mathcal{G}(\beta)$. So, suppose that $f = \chi_U$. Since, $X \setminus U \ne \emptyset$, $f \ne \frac{1}{2}\chi_U$. But for $g = \frac{1}{2}\chi_U$, $\pi_g^{\leftarrow}[[0, \beta(g) + \frac{1}{3})] \cap e[X] = \pi_g^{\leftarrow}[[0, \alpha(g) + \frac{1}{3})] \cap e[X] = e[U] \in \mathcal{G}(\beta)$. $\square$

Remark 3.12. By Proposition 3.11, we have that for each $\alpha \in \rho X$, $\vee P_\alpha = \vec{\mathbf{1}}$. This emphasizes that $\beta^+ X$ is too unstructured.

Proposition 3.13. *Let α, $\beta \in \beta^+ X$ such that $\langle \mathcal{G}(\beta) \rangle \subseteq \langle \mathcal{G}(\alpha) \rangle$. Then $\alpha^\star \leq \beta^\star \leq \beta$.*

Proof: By Proposition 3.5, $\beta^\star \leq \beta$; so, it suffices to prove that $\alpha^\star \leq \beta^\star$. Let $f \in C^+(X)$. Then $\beta^\star(f) = \inf c(\pi_f(\langle \mathcal{G}(\beta) \rangle))$. By Proposition 3.5, $\beta^\star(f) \in c(\pi_f(\langle \mathcal{G}(\beta) \rangle))$. Since $\langle \mathcal{G}(\beta) \rangle \subseteq \langle \mathcal{G}(\alpha) \rangle$, it follows that $\beta^\star(f) \in c(\pi_f(\langle \mathcal{G}(\alpha) \rangle))$. As $\alpha^\star(f) = \inf c(\pi_f(\langle \mathcal{G}(\alpha) \rangle))$, $\alpha^\star(f) \leq \beta^\star(f)$. $\square$

Now, $X \approx e[X] \subseteq \rho X \subseteq \beta^+ X$. The extension $\beta^+ X$ of $e[X]$ is closed in the compact space $\prod_{C^+(X)} I^+$ and hence is compact. On the other hand, ρX is not closed in $\prod_{C^+(X)} I^+$; so, it is natural to ask if ρX is also compact. We answer this in the affirmative in the next theorem.

Proposition 3.14. *Let $f \in C^+(X)$ and $\varepsilon > 0$. Then $\pi_f^{\leftarrow}[[0, 1^\star(f) + \varepsilon)] \cap \rho X = \rho X$.*

Proof: Let $\alpha \in \rho X$. Then $\alpha \leq \vec{\mathbf{1}}$; so, $\alpha = \alpha^\star \leq \vec{\mathbf{1}}^\star$ and $\alpha(f) \leq \vec{\mathbf{1}}^\star(f)$. Hence, $\alpha(f) \in [0, \vec{\mathbf{1}}^\star(f) + \varepsilon)$. Thus, $\alpha \in \pi_f^{\leftarrow}[[0, \vec{\mathbf{1}}^\star(f) + \varepsilon)]$. $\square$

Theorem 3.15. *ρX is compact.*

Proof: From the paragraph after Proposition 3.5, $\vec{\mathbf{1}}^\star \in \rho X$. Let $\mathcal{C}$ be an open cover of ρX. There is some $U \in \mathcal{C}$ such that $\vec{\mathbf{1}}^\star \in U$. There is a finite set $F \subseteq C^+(X)$ and $\varepsilon > 0$ such that $\vec{\mathbf{1}}^\star \in \bigcap \{\pi_f^{\leftarrow}[[0, \vec{\mathbf{1}}^\star(f) + \varepsilon)] : f \in F\} \cap \rho X \subseteq U$. By Proposition 3.14, $\pi_f^{\leftarrow}[[0, \vec{\mathbf{1}}^\star(f) + \varepsilon)] \cap \rho X = \rho X$. Thus, $\vec{\mathbf{1}}^\star \in \rho X \subseteq U$. Hence, ρX can be covered by exactly one element. $\square$

Remark 3.16. Actually, we show in the proof of Theorem 3.15, that if $e[X] \subseteq Y \subseteq \rho X$, then $Y \cup \{(\vec{\mathbf{1}})^\star\}$ is compact. An improvement of Proposition 3.10 would be: for α, $\beta \in \beta^+ X$, $\alpha \leq \beta$ iff $\alpha^\star \leq \beta^\star$. This is false. For example, let $\alpha \in \rho X$ such that $\alpha \neq \vec{\mathbf{1}}$. Then there is some $f \in C^+(X)$ such that

$\alpha(f) \neq \vec{\mathbf{1}}(f)$. Define $\beta \in \beta^+X$ by $\beta(g) = \alpha(g)$ for $g \neq f$ and $\beta(f) = 1$. Now, $\alpha^\star = \beta^\star$ and $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$. However, $\beta \not\leq \alpha$ even though $\alpha < \beta$.

Proposition 3.17. *Let $\alpha \in \beta^+X$ such that $\langle\mathcal{G}(\alpha)\rangle$ is an open ultrafilter on $e[X]$. Then $\alpha^\star$ is a minimal element of β^+X.*

Proof: Suppose $\langle\mathcal{G}(\alpha)\rangle$ is an open ultrafilter on $e[X]$. To show that $\alpha^\star$ is a minimal element of β^+X, let $\gamma \in \beta^+X$ and $\gamma \leq \alpha^\star$. By Proposition 3.10, $\mathcal{G}(\alpha^\star) = \langle\mathcal{G}(\alpha)\rangle \subseteq \langle\mathcal{G}(\gamma)\rangle$. As, $\mathcal{G}(\alpha^\star)$ is an open ultrafilter, $\mathcal{G}(\alpha^\star) = \langle\mathcal{G}(\alpha)\rangle = \langle\mathcal{G}(\gamma)\rangle$. By Proposition 3.9 $\alpha^\star = \gamma^\star \leq \gamma$. Thus, $\gamma = \alpha^\star$. □

Proposition 3.18. *Let $\alpha \in \beta^+X$. Then α is a minimal element of β^+X iff $\alpha = \alpha^\star$ and $\mathcal{G}(\alpha)$ is an open ultrafilter on $e[X]$.*

Proof: Suppose α is a minimal element of β^+X. By Proposition 3.5 $\alpha^\star \leq \alpha$ and $\alpha^\star \in \beta^+X$. Hence, $\alpha = \alpha^\star$. Let $\mathcal{H}$ be an open filter such that $\mathcal{H} \supseteq \mathcal{G}(\alpha)$. Define $\beta \in \prod_{C^+(X)} I^+$ as follows

$$\beta(f) = \begin{cases} 0 & \text{if } f = \chi_U \text{ and } U \in \mathcal{H} \\ 1 & \text{otherwise} \end{cases}$$

It follows that $\mathcal{G}(\beta) = \mathcal{H}$. Now, $c(\pi_f\langle\mathcal{G}(\beta)\rangle) \supseteq c(\pi_f\langle\mathcal{G}(\alpha)\rangle)$. So, $\beta^\star(f) = \inf c(\pi_f\langle\mathcal{G}(\beta)\rangle) \leq \inf c(\pi_f(\mathcal{G}(\alpha)) = \alpha^\star(f) \leq \alpha(f)$. Hence, $\beta^\star \leq \alpha$. As, α is a minimal element of β^+X, $\alpha = \beta^\star \leq \beta$. By Proposition 3.10, $\mathcal{H} = \mathcal{G}(\beta) \subseteq \mathcal{G}(\alpha)$. This completes the proof that $\mathcal{G}(\alpha)$ is an open ultrafilter on $e[X]$.

Conversely, suppose that $\alpha = \alpha^\star$ and $\mathcal{G}(\alpha)$ is an open ultrafilter on $e[X]$. By Proposition 3.17, $\alpha = \alpha^\star$ is a minimal element of β^+X. □

Remark 3.19. Let θX denote the set of open ultrafilters on X. If $\alpha \in \beta^+X$ is a minimal element, then by Proposition 3.18, $\mathcal{G}(\alpha)$ is an open ultrafilter on $e[X]$ and $\alpha = \alpha^\star$. In particular, there is an open ultrafilter $\mathcal{U}$ on X (i.e., there is $\mathcal{U} \in \theta X$) such that $e[\mathcal{U}] = \{e[U] : U \in \mathcal{U}\} = \mathcal{G}(\alpha)$. If, also, $e[\mathcal{U}] = \mathcal{G}(\beta)$ for some $\beta \in \beta^+X$, then $\mathcal{G}(\beta^\star) = \mathcal{G}(\beta)$; by Proposition 3.7,

$\alpha^\star = \beta^\star$. In particular, there is a unique element $\alpha_{\mathcal{U}} \in \rho X$ such that $e[\mathcal{U}] = \mathcal{G}(\alpha_{\mathcal{U}})$. Thus, the set of minimal elements of $\beta^+ X = \{\alpha_{\mathcal{U}} : \mathcal{U} \in \theta X\}$.

Proposition 3.20. $\beta^+ X = \bigcup\{\prod_{C^+(X)} cl\alpha_{\mathcal{U}}(f) : \mathcal{U} \in \theta X\}$.

Proof: By Remark 3.1[d],

$$\bigcup\{\textstyle\prod_{C^+(X)} cl\alpha_{\mathcal{U}}(f) : \mathcal{U} \in \theta X\} = \bigcup\{cl\alpha_{\mathcal{U}} : \mathcal{U} \in \theta X\} \subseteq \beta^+ X.$$

Conversely, let $\gamma \in \beta^+ X$. The open filter $\langle \mathcal{G}(\gamma) \rangle$ is contained in $e[\mathcal{U}]$ for some $\mathcal{U} \in \theta X$. By Remark 3.19, $\mathcal{G}(\alpha_{\mathcal{U}}) = e[\mathcal{U}]$, $\alpha_{\mathcal{U}}^\star = \alpha_{\mathcal{U}}$, and $\alpha_{\mathcal{U}}$ is a minimal element of $\beta^+ X$. Now, $\langle \mathcal{G}(\gamma) \rangle \subseteq \mathcal{G}(\alpha_{\mathcal{U}})$. By Proposition 3.13, $\alpha_{\mathcal{U}} \leq \gamma^\star \leq \gamma$. For $f \in C^+(X)$, $\gamma(f) \in [\alpha_{\mathcal{U}}(f), 1] = cl\{\alpha_{\mathcal{U}}(f)\}$. This implies that $\gamma \in \prod_{C^+(X)} cl\alpha_{\mathcal{U}}(f)$. □

Theorem 3.21. *ρX is an atomic, complete upper semilattice.*

Proof: By Propositions 3.17 and 3.18, it is seen that the minimal elements of ρX are $\{\alpha_{\mathcal{U}} : \mathcal{U} \in \theta X\}$. $\vec{\mathbf{1}}^\star$ is the largest element of ρX.

To show that ρX is atomic, let $\alpha \in \rho X$. Now, $\mathcal{G}(\alpha)$ is an open filter on $e[X]$ and there is some $\mathcal{U} \in \theta X$ such that $e[\mathcal{U}] \supseteq \mathcal{G}(\alpha)$. But, $e[\mathcal{U}] = \mathcal{G}(\alpha_{\mathcal{U}})$ for some $\alpha_{\mathcal{U}} \in \rho X$, by Remark 3.19. Hence, $\mathcal{G}(\alpha_{\mathcal{U}}) \supseteq \mathcal{G}(\alpha)$. By Proposition 3.13, $\alpha_{\mathcal{U}} \leq \alpha^\star$. Since, $\alpha_{\mathcal{U}}, \alpha \in \rho X$, $\alpha_{\mathcal{U}} = \alpha_{\mathcal{U}}^\star \leq \alpha^\star = \alpha$.

Thus, it only remains to prove that ρX is closed under arbitrary joins. We denote by $\vee_{cl}$ and $\vee_{\rho}$, the joins in the spaces $\beta^+ X$ and ρX respectively. It follows that $\vee_{\rho}\alpha_i$ $(\vee_{cl}\alpha_i)^\star$. Let $\{\alpha_i : i \in J\} \subseteq \rho X$. Then, for $i \in J$, $(\vee_{cl}\alpha_i) \geq (\vee_{cl}\alpha_i)^\star \geq \alpha_i^\star = \alpha_i$. Let $\gamma \in \rho X$ such that $\gamma \geq \alpha_i$, for every $i \in J$. Then $\gamma \geq \vee_{cl}\alpha_i$, so, $\gamma^\star = \gamma \geq (\vee_{cl}\alpha_i)^\star$ and thus $\vee_{\rho}\alpha_i = (\vee_{cl}\alpha_i)^\star$. Hence, $\vee_{\rho}\alpha_i \in \rho X$. □

Example 3.22. Let $X = \omega$ with the discrete topology and let $e : \omega \to \prod_{C^+(\omega)} I^+$ be the usual embedding function. Let $\alpha \in \beta\omega$ and let $\mathcal{U}$ be the neighborhood trace filter of α on

ω. It is easy to describe $\alpha(f)$ for every $f \in C(\omega)$ in terms of the neighborhood trace filter $\mathcal{U}$. In fact, $f(\mathcal{U})$ converges to an unique point in the usual topology in I, and that point is $\alpha(f)$.

Thus, $\{\alpha(f)\} = c_I(f[\mathcal{U}]) = \inf c_{I^+}(f[\mathcal{U}]) = \alpha_{\mathcal{U}}(f)$. Hence, $\alpha = \alpha_{\mathcal{U}}$. Now $\alpha_{\mathcal{U}}$ is a minimal element in $\beta^+\omega$ by Remark 3.19. Also, as $cl\alpha_{\mathcal{U}} = \prod_{C^+(\omega)}[\alpha_{\mathcal{U}}(f), 1]$, $\beta^+\omega = \bigcup\{cl\alpha : \alpha \in \beta\omega\}$. □

4. Embedding Strict Extensions of X in ρX

In this section we look at the conclusion of Theorem 2.8 with respect to the space ρX. Let Y be an extension of X and $f \in C^+(X)$, $\hat{f} \in C^+(Y)$ be the continuous extension of f defined in 2.7. Consider the continuous function $\hat{e}_Y : Y \to \prod_{C^+(X)} I^+$ (defined in Remark 2.8) such that $e[X] \subseteq \hat{e}_Y[Y] \subseteq \beta^+X$. In this section, we show that $\hat{e}_Y[Y] \subseteq \rho X$. For $y \in Y$, let $O^y = \{U \cap X : y \in U \in \tau(Y)\}$. For $V \in \tau(X)$, let $o_Y V = \{y \in Y : V \in O^y\}$.

Proposition 4.1. *Let Y be an extension of X. For $y \in Y$, $\mathcal{G}(\hat{e}_Y(y)) = \{e[U] : U \in O^y\}$, in particular, $\mathcal{G}(\hat{e}_Y(y))$ is an open filter on $e[X]$.*

Proof: Now, $\mathcal{G}(\hat{e}_Y(y)) = \{\pi_f^{\leftarrow}[[0, \hat{e}_Y(y)(f) + 1/n)] \cap e[X] : f \in C^+(X), n \in \mathbf{N}\}$. First note that for $f \in C^+(X)$ and $n \in \mathbf{N}$,

$$\pi_f^{\leftarrow}[[0, \hat{e}_Y(y)(f) + 1/n)] \cap e[X]$$

$$= \pi_f^{\leftarrow}[[0, \hat{f}(y) + 1/n)] \cap e[X]$$

$$= e[\{x \in X : f(x) < \hat{f}(y) + 1/n\}]$$

$$= e[f^{\leftarrow}[[0, \hat{f}(y) + 1/n)]].$$

As, $f^{\leftarrow}[[0, \hat{f}(y)+1/n)] = X \cap \hat{f}^{\leftarrow}[[0, \hat{f}(y)+1/n)] \in O^y$, we have shown that $\mathcal{G}(\hat{e}_Y(y)) \subseteq \{e[U] : U \in O^y\}$. Conversely, suppose that $U \in O^y$ and let χ_U denote the reverse characteristic function of U in X. It is easy to verify that $\widehat{\chi}_U$ is the reverse characteristic function of $o_Y U$ in Y. As $y \in o_Y U$, $\widehat{\chi}_U(y) = 0$. Also, $\chi_U^{\leftarrow}[[0, \widehat{\chi}_U(z) + 1/n)] = U$ for all $z \in o_Y U$. Now,

$$e[U] = e[\chi_U^{\leftarrow}[[0, \widehat{\chi}_U(y) + 1/n)]]$$

$$= \pi_{\chi_U}^{\leftarrow}[[0, \hat{e}_Y(y)(\chi_U) + 1/n)] \cap e[X].$$

So, $e[U] \in \mathcal{G}(\hat{e}_Y(y))$. Hence, $\{e[U] : U \in O^y\} \in \mathcal{G}(\hat{e}_Y(y))$. This completes the proof of the Proposition. $\square$

For an extension Y of X, $y \in Y$ and $f \in C^+(X)$, $(\hat{e}_Y(y))^\star(f) = \inf c(\pi_f[\langle \mathcal{G}(\hat{e}_Y(y))\rangle]) = \inf c(\pi_f[e(O^y)])$ by Proposition 4.1. But as $\pi_f \circ e = f$, we have that $(\hat{e}_Y(y))^\star(f) = \inf c(f(O^y))$.

Proposition 4.2. *Let Y be an extension of X and $f \in C^+(X)$. Define $F : Y \to I^+$ by $F(y) = \inf(c(f(O^y)))$ for $y \in Y$. Then F is continuous, $F|_X = f$ and $F(y) \in c(f(O^y))$ for $y \in Y$.*

Proof: Let $x \in X$. Since $f(O^x)$ converges to $f(x)$, it follows that $f(x) \in c(f(O^x))$ and $F(x) \leq f(x)$. Let $n \in \mathbf{N}$. Then $f(O^x)$ converges to $F(x) + 1/n$. There is an open set $U \in O^x$ such that $f[U] \subseteq [0, F(x)+1/n)$. As $x \in U$, $f(x) < F(x)+1/n$. Therefore, $f(x) \leq F(x)$. By the above, $f(x) = F(x)$. To show that F is continuous, let $y \in Y$ and $n \in \mathbf{N}$. Since $f(O^y)$ converges to $F(y) + 1/2n$, there is an open set $U \in O^y$ such that $f[U] \subseteq [0, F(y) + 1/2n)$. Suppose $U \in O^z$ for some $z \in Y$. As $f[U] \subseteq [0, F(y) + 1/2n)$, $f(O^z)$ converges to $F(y) + 1/2n$. Hence, $F(z) \leq F(y) + 1/2n$. Thus, $F[o_Y U] \subseteq [0, F(y) + 1/2n] \subseteq [0, F(y) + 1/n)$. This completes the proof that F is continuous.

By the continuity of F, for $y \in Y$, $F(O^y)$ converges to $F(y)$. But $F(O^y) = f(O^y)$. This shows that $F(y) \in c(f(O^y))$. $\square$

Theorem 4.3. *Let Y be an extension of X. Consider the continuous function $\hat{e}_Y$ such that $e[X] \subseteq \hat{e}_Y[Y] \subseteq \beta^+ X$. Then $\hat{e}_Y[Y] \subseteq \rho X$.*

Proof: For $y \in Y$ and $f \in C^+(X)$, we must show that $(\hat{e}_Y(y))^\star(f) \geq (\hat{e}_Y(y)(f))$ by Proposition 3.5. But $\hat{e}_Y(y)(f) = \hat{f}(y)$ by Theorem 2.8.

Fix f and let $F(y) = (\hat{e}_Y(y))^\star(f) = \inf c(f(O^y))$ as in the above Proposition. Since F is continuous and $F\,|_X = f$, by Proposition 4.2, $\hat{f} \leq F$. So, $\hat{f}(y) \leq F(y)$. $\square$

For an extension Y of X and $x \in X$, since $\hat{e}_Y(x) = e(x)$, it follows by the proof of the above theorem that $e(x)^\star = e(x)$. Using Theorem 4.3, we have the following improvement of Theorem 2.8.

Theorem 4.4. *Let Y be an extension of X. Define $\hat{e}_Y : Y \to \prod_{C^+(X)} I^+$ by $\hat{e}_Y(y)(f) = \hat{f}(y)$. If the strict extension $Y^\#$ of X is T_0, then $\hat{e}_Y : Y^\# \to \prod_{C^+(X)} I^+$ is an embedding and $e[X] \subseteq \hat{e}_Y[Y^\#] \subseteq \rho X$.*

5. ρX and the Open Filters on X

In this section we establish the properties of ρX further. We show that ρX is homeomorphic with the set of all open filters on X with the Alexandroff topology.

Proposition 5.1. *Let $\mathcal{F}$ be an open filter on $e[X]$, then there is an unique element $\alpha_{\mathcal{F}} \in \rho X$ such that $\mathcal{G}(\alpha_{\mathcal{F}}) = e(\mathcal{F})$.*

Proof: Let $\mathcal{F}$ be an open filter on X and U be open in X. For $f \in C^+(X)$, define $\alpha_{\mathcal{F}}(f) = \inf(c(f(\mathcal{F})))$. First we show that $\alpha_{\mathcal{F}} \in \beta^+ X$. Suppose $\alpha_{\mathcal{F}} \in T = \bigcap\{\pi_{f_i}^{\leftarrow}[[0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)] : 1 \leq i \leq n\}$. Then, for $1 \leq i \leq n$, $\pi_{f_i}(\alpha_{\mathcal{F}}) \in [0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)$. Now, $\pi_{f_i}(\alpha_{\mathcal{F}}) = \alpha_{\mathcal{F}}(f_i)$ and $f_i(\mathcal{F})$ converges to $\alpha_{\mathcal{F}}(f_i) \in [0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)$. Thus, there is a $F_i \in \mathcal{F}$ such that $f_i[F_i] \subseteq [0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)$. Let $F = \cap\{F_i : 1 \leq i \leq n\}$ and $x \in F$. Then $f_i(x) \in [0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)$ for every i such that $1 \leq i \leq n$. But, $f_i(x) = \pi_{f_i}(e(x))$, so, $e(x) \in \pi_{f_i}^{\leftarrow}[[0, \alpha_{\mathcal{F}}(f_i) + 1/n_i)]$. Thus, $e(x) \in T$ and we have shown that $\alpha_{\mathcal{F}} \in \beta^+ X$. Next, we prove that $\mathcal{G}(\alpha_{\mathcal{F}}) = e(\mathcal{F})$. Let U be open in X, $f \in C^+(X)$ and $n \in \mathbf{N}$ such that $e[U] = \pi_f^{\leftarrow}[[0, \alpha_{\mathcal{F}}(f) + 1/n)] \cap e[X]$. As $\alpha_{\mathcal{F}}(f) = \inf(c(f(\mathcal{F})))$, there is a $F \in \mathcal{F}$ such that $f[F] \subseteq [0, \alpha_{\mathcal{F}}(f) + 1/n)$. But, since $\pi_g \circ e = g$ for $g \in C^+(X)$, we have that $f[F] = \pi_f \circ e[F]$. In particular, $\pi_f \circ e[F] \subseteq [0, \alpha_{\mathcal{F}}(f) + 1/n)$ from which it follows that $e[F] \subseteq \pi_f^{\leftarrow}[[0, \alpha_{\mathcal{F}}(f) + 1/n)] \cap e[X] = e[U]$. Then, $F \subseteq U$

and $U \in \mathcal{F}$. This proves that $\mathcal{G}(\alpha_{\mathcal{F}}) \subseteq e(\mathcal{F})$. Now to show that $e(\mathcal{F}) \subseteq \mathcal{G}(\alpha_{\mathcal{F}})$, let $F \in \mathcal{F}$. Since $\pi_{\chi_F}^{\leftarrow}[[0,1/2)] \cap e[X] = e[F]$, it suffices to show that $\pi_{\chi_F}(\alpha_{\mathcal{F}}) = 0$. An open neighborhood of 0 in I^+ is of the form $[0,a)$, where, $0 < a \leq 1$. Note that $\chi_F[F] = 0 \in [0,a)$. Thus, $\chi_F(\mathcal{F})$ converges to 0 and we have that $\inf(c(\chi_F(\mathcal{F}))) = 0$, i.e., $\alpha_{\mathcal{F}}(\chi_F) = 0$. Hence, $e(\mathcal{F}) = \mathcal{G}(\alpha_{\mathcal{F}})$. Also, note that $\alpha^{\star}_{\mathcal{F}}(f) = \inf(c(\pi_f[\langle\mathcal{G}(\alpha)\rangle])) = \inf(c(\pi_f[e(\mathcal{F})])) = \inf(c(f(\mathcal{F}))) = \alpha_{\mathcal{F}}(f)$ i.e., $\alpha_{\mathcal{F}} \in \rho X$. The uniqueness of $\alpha_{\mathcal{F}}$ is an immediate consequence of Proposition 3.9. □

Let $\mathcal{OF}(X)$ denote $\{\mathcal{F} : \mathcal{F}$ is an open filter on $X\}$. The set $\mathcal{OF}(X)$ is partially ordered by inclusion.

Theorem 5.2. *$\phi : \rho X \to \mathcal{OF}(X)$ defined by $\phi(\alpha) = \mathcal{G}(\alpha)$ is a reverse-order isomorphism.*

Proof: First, we show that ϕ is one-one. Let $\alpha, \beta \in \rho X$. Since $\alpha^\star = \alpha \neq \beta = \beta^\star$, by Proposition 3.9 $\mathcal{G}(\alpha^\star) \neq \mathcal{G}(\beta^\star)$. By Proposition 5.1, ϕ is onto. Thus it only remains to show that ϕ is a reverse-order homomorphism. Let α, $\beta \in \rho X$. Suppose, $\alpha \leq \beta$. By Proposition 3.10, $\phi(\alpha) = \mathcal{G}(\alpha) \supseteq \mathcal{G}(\beta) = \phi(\beta)$. Conversely, suppose that $\phi(\beta) \leq \phi(\alpha)$. Then, $\mathcal{G}(\alpha) \supseteq \mathcal{G}(\beta)$. By 3.13, $\alpha = \alpha^\star \leq \beta^\star = \beta$ □

We now define a topology on $\mathcal{OF}(X)$ and show that in fact ρX and $\mathcal{OF}(X)$ are homeomorphic.

Definition 5.3. *For U open in X let $OU = \{\mathcal{F} \in \mathcal{OF}(X) : U \in \mathcal{F}\}$.*

Proposition 5.4. *Let U, V be open in X.*

(a) *If $U \subseteq V$, then $OU \subseteq OV$.*
(b) *$OX = \mathcal{OF}(X)$ and $O\emptyset = \emptyset$.*
(c) *$OU \cap OV = O(U \cap V)$.*
(d) *$OU \cup OV \subseteq O(U \cup V)$.*

Proof: The proof to this is similar to Proposition 7.1(c) in [6]. □

Proposition 5.5. *$\{OU : U$ open in $X\}$ forms a base for a T_0 topology on $\mathcal{OF}(X)$.*

Proof: By Proposition 5.4, $\{OU : U \in \tau(X)\}$ is a base for a topology on $\mathcal{OF}(X)$. Let $\mathcal{F}$ and $\mathcal{G}$ be distinct open filters on X. There is an open set U in X such that $U \in \mathcal{F}\backslash\mathcal{G}$ (or $\mathcal{G}\backslash\mathcal{F}$). Then, $\mathcal{F} \in OU$ and $\mathcal{G} \notin OU$ (or $\mathcal{G} \in OU$, $\mathcal{F} \notin OU$). □

Recall by Proposition 3.7 that a basic open set in ρX is a set of the form $\pi_f^{\leftarrow}[[0,b)] \cap \rho X$, where $0 < b < 1$ and $f \in C^+(X)$. The next fact shows that we can reformulate the basic open sets in terms of the characteristic functions of open sets in X.

Proposition 5.6. *$\{\pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X : U$ is open in $X\}$ forms a base for ρX.*

Proof: First note that for $U \in \tau(X)$, $\pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X = \pi_{\chi_U}^{\leftarrow}[[0,1/2)] \cap \rho X$ is open in ρX. Let $f \in C^+(X)$ and $\alpha \in \pi_f^{\leftarrow}[[0,b)] \cap \rho X$. Then $\pi_f(\alpha) < b$. Let $0 < a < 1$ such that $\pi_f(\alpha) < a < b$. Now, $\alpha \in \pi_f^{\leftarrow}[[0,a)] \cap \rho X$ and $\alpha = \alpha^\star$. Thus, there is an open set U in X such that $e[U] \in \mathcal{G}(\alpha)$ and $\pi_f[e[U]] \subseteq [0,a)$. By Proposition 3.8, $\pi_{\chi_U}(\alpha) = 0$. Hence, $\alpha \in \pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X$. To complete the proof of this result, it suffices to show that $\pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X \subseteq \pi_f^{\leftarrow}[[0,b)] \cap \rho X$. Let $\beta \in \pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X$. Then $\pi_{\chi_U}(\beta) = 0$ and by Proposition 3.8, $e[U] \in \mathcal{G}(\beta)$. But $\pi_f[e[U]] \subseteq [0,a)$ implies $\pi_f[\langle\mathcal{G}(\beta)\rangle]$ converges to a and since, $\beta(f) = \beta^\star(f) = \inf c(\pi_f[\langle\mathcal{G}(\beta)\rangle] \leq a < b$, then $\beta \in \pi_f^{\leftarrow}[[0,b)] \cap \rho X$. Thus, $\alpha \in \pi_{\chi_U}^{\leftarrow}[\{0\}] \cap \rho X \subseteq \pi_f^{\leftarrow}[[0,b)] \cap \rho X$. □

Theorem 5.7. *ρX is homeomorphic to $\mathcal{OF}(X)$.*

Proof: Define $\phi : \rho X \to \mathcal{OF}(X)$ as in Theorem 5.2. By Theorem 5.2, ϕ is a bijection, so all that remains to be proven is that ϕ is continuous and open. In proving that ϕ is continuous we show that the pre-image of basic open sets in $\mathcal{OF}(X)$ is in fact basic open in ρX and this fact along with Proposition 5.6 shows that ϕ is open, since ϕ is a bijection.

Let U be open in X. Then $\phi^{\leftarrow}[OU] = \{\alpha_{\mathcal{F}} : \mathcal{F} \in OU\} = \{\alpha_{\mathcal{F}} : U \in \mathcal{F}\}$. But, $e[\mathcal{F}] = \mathcal{G}(\alpha_{\mathcal{F}})$ by Proposition 5.1, so by Proposition 3.8, $U \in \mathcal{F}$ iff $\pi_{\chi_U}(\alpha_{\mathcal{F}}) = 0$. Hence, $\phi^{\leftarrow}[OU] = \{\alpha_{\mathcal{F}} : \pi_{\chi_U}(\alpha_{\mathcal{F}}) = 0\} = \{\alpha \in \rho X : \pi_{\chi_U}(\alpha) = 0\} = \pi_{\chi_U}^{\leftarrow}[\{0\}] \cap$

ρX, which is open in ρX. Thus, we have shown ϕ to be continuous and hence, ρX is homeomorphic to $\mathcal{OF}(X)$. $\square$

6. ρX and Strict T_0 Extensions

The T_0 compactification $\beta^+ X$ of $e[X]$ has now been pruned to ρX, so that the number of extensions of $e[X]$ is reduced. The construction of ρX and its being homeomorphic to $\mathcal{OF}(X)$ enables us to completely characterize all the extensions of $e[X]$ contained in ρX. ρX has been formed by selecting the minimal elements $\alpha^\star$ from $\prod_{C^+(X)} I^+$ based on the open filter subbases $\mathcal{G}(\alpha)$.

Lemma 6.1. *Let $Y \subseteq \rho X$ be an extension of $e[X]$.*

(a) *For $\alpha \in Y$, $\mathcal{G}(\alpha) = O_Y^\alpha$.*
(b) *For $U \in \tau(X)$, $\pi_{\chi_U}^{\leftarrow}[\{0\}] \cap Y = o_Y e[U]$.*

Proof: To prove (a), let $U \in \mathcal{G}(\alpha)$. By Proposition 3.7, there are $f \in C^+(X)$ and $n \in \mathbf{N}$ such that $U = \pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap e[X]$. Since $\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n)] \cap Y$ is an open set containing α, we have that $U \in O_Y^\alpha$. Conversely, let $V \in O_Y^\alpha$. There is an open set W in $\prod_{C^+(X)} I^+$ such that $\alpha \in W$ and $(W \cap Y) \cap e[X] = V$. There is a finite set $F \subseteq C^+(X)$ and $\{n_f : f \in F\} \subseteq \mathbf{N}$ such that $\alpha \in \bigcap\{\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n_f)] : f \in F\} \subseteq W$. Let $T = \bigcap\{\pi_f^{\leftarrow}[[0, \alpha(f) + 1/n_f)] \cap e[X] : f \in F\}$. Then $T \subseteq W \cap e[X] = V$. As $T \in \mathcal{G}(\alpha)$, $V \in \mathcal{G}(\alpha)$.
To prove (b), note that

$\alpha \in \pi_{\chi_U}^{\leftarrow}[\{0\}] \cap Y$ iff $\alpha \in Y$ and $\pi_{\chi_U}(\alpha) = 0$
iff $\alpha \in Y$ and $\alpha(\chi_U) = 0$
iff (using Proposition 3.8) $\alpha \in Y$ and $e[U] \in \mathcal{G}(\alpha)$
iff (using Proposition 6.1 (a)) $\alpha \in Y$ and $e[U] \in O_Y^\alpha$
iff $\alpha \in o_Y e[U]$.

Thus, $\pi_{\chi_U}^{\leftarrow}[\{0\}] \cap Y = o_Y e[U]$. $\square$

Theorem 6.2. *The space ρX is a strict T_0 compactification of $e[X]$. In particular, if $Y \subseteq \rho X$ is an extension of $e[X]$, then Y is a strict T_0 extension of $e[X]$.*

Proof: Since $\prod_{C^+(X)} I^+$ is a T_0 space, any subspace is T_0; so ρX is T_0. By Theorem 3.15, ρX is a compactification of $e[X]$. By Proposition 5.6 and Proposition 6.1 (b), if $Y \subseteq \rho X$ is an extension of $e[X]$, Y is a strict extension of $e[X]$. $\square$

Lemma 6.3. *Let Y, $Z \subseteq \rho X$ be extensions of $e[X]$. If $\alpha \in Y$ and $\beta \in Z$ such that $O_Y^\alpha = O_Z^\alpha$, then $\alpha = \beta$.*

Proof: If $O_Y^\alpha = O_Z^\beta$, then $\mathcal{G}(\alpha) = \mathcal{G}(\beta)$ by Lemma 6.1. By Propositions 3.7 and 3.8, $\alpha = \beta$. $\square$

Theorem 6.4. *If X is T_0, ρX contains all the strict T_0 extensions of X and no others. Also, ρX contains only one copy of each strict T_0 extension.*

Proof: By Theorem 6.2, ρX only contains strict T_0 extensions of X, and by Lemma 6.3, ρX contains a copy of every strict T_0 extension of X. Suppose Y and Z are extensions of $e[X]$, and $Y \cup Z \subseteq \rho X$, and $Y \equiv_{e[X]} Z$. Then there is a homeomorphism $f : Y \to Z$ such that $f(e(x)) = e(x)$ for all $x \in X$. If $\alpha \in Y$, then $O_Y^\alpha = O_Z^{f(\alpha)}$. In particular, $\alpha = f(\alpha)$ by Lemma 6.3. Hence, $Y = Z$ as subsets of ρX. So, ρX contains only one copy of each extension of $e[X]$. $\square$

One of the primary research goals of this article was to determine which H-closed extension of $e[X]$, when X is Hausdorff, are contained in ρX. As a consequence of 6.4, we have the next result.

Corollary 6.5. *Let X be a Hausdorff space.*

(a) *If $Y \subseteq \rho X$ is an H-closed extension of $e[X]$, then ρX contains no other copy of Y.*
(b) *ρX contains exactly one element from each S-equivalence class of H-closed extensions.*

References

[1] Z.Frolik and C.T.Liu, *An Embedding Characterization Of Almost Realcompact Spaces*, Proc. Amer. Math. Soc., **32** (1972), 294-298.

[2] H.Herrlich, *Wann sind alle stetigen Abbildungen in Y konstant?*, Math. Z., **90** (1965), 152-154.

[3] Nielsen and Sloyer, *On Embedding in Quasi Cubes*, Amer. Math. Monthly, (1968), 514-515.

[4] J.R.Porter, *Categorical problems in minimal spaces*, Proc. Mannheim Categorical Top. Conf., Lecture Notes in Math., (1976), 482-500.

[5] J.Porter, *Hausdorff-closed Extensions and Tychonoff*, Proc. 1992 International Conf. General Topology and its Applications, in memory of Felix Hausdorff, Mathematical Research, **67** (1992), 261-267.

[6] J.R.Porter and R.Grant Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer Verlag, 1987.

[7] V. Prabhu, *Doctoral Dissertation*, University of Kansas,(1993).

[8] V. Prabhu, *Embeddings in the upper Stone-Čech compactification*, Topology and its Applications **73** (1996), 217-226.

[9] S.Salbany. *An Embedding Characterization Of Almost Compact Spaces*, Trans. Amer. Math. Soc., **275** (1983), 611-621.

[10] W.J.Thron and S.J.Zimmerman, *A characterization of order topologies by means of minimal T_0-topologies*, Proc. Amer. Math. Soc., **27** (1971), 161- 167.

[11] M.Tikoo, *Remainders of H-closed Extensions*, Doctoral Dissertation, University of Kansas 1984.

[13] M.Tikoo, *The Banaschewski-Fomin-Shanin Extension μX*, Topology Proc., **10** (1985), 187-206.

WILLIAM WOODS UNIVERSITY, FULTON, MO. 65251-1098