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THE COMBINATORICS OF SUB-OSTASZEWSKI SPACES

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ABSTRACT. The notion of sub-Ostaszewski is shown to be equivalent to a statement about ideals which is reminiscent of HFD's

0. INTRODUCTION

The purpose of this paper is to find a combinatorial equivalent to the notion of sub-Ostaszewski spaces.

Definition 1. A non-compact space is sub-Ostaszewski iff every closed subset is either countable or co-countable.

An Ostaszewski space is a countably compact sub-Ostaszewski space.

Ostaszewski spaces have been quite useful for generating counterexamples. But their construction needs some sort of reflection principle, such as $\diamondsuit$. In particular, "CH + there are no Ostaszewski spaces" is consistent [Eisworth, Roitman]. What about sub-Ostaszewski spaces?

HFD's (see below) are sub-Ostaszewski, so sub-Ostaszewski spaces exist under CH. But any locally compact sub-Ostaszewski space can be forced to have an uncountable discrete subspace by a forcing that does not add reals [Eisworth, Roitman]. We

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don't know how to iterate this forcing without adding reals. That is where countable compactness is used in the Eisworth-Roitman argument. So the interesting question is

Question 2. Is "CH + there are no locally compact sub-Ostaszewski spaces" consistent?

Countable compactness turns out to be a very strong property in this sort of context — there are several theorems proving the consistency of statements of the form "CH + there are no countably compact spaces with property _____". [Eisworth], [Eisworth, Nyikos], [Eisworth, Nyikos, Shelah].

But what happens without countable compactness? This is why the combinatorics of sub-Ostaszewski spaces is worth looking at.

1. What do sub-Ostaszewski spaces look like?

Fact 3. *Every sub-Ostaszewski space has size ω_1, is hereditarily separable, and has at most one point with no countable neighborhood.*

In thinking of the combinatorics of sub-Ostaszewski spaces, we generally ignore the point, if any, with no countable neighborhood. I.e., modulo at most one point, sub-Ostaszewski spaces are locally countable, hence 0-dimensional.

We will characterize sub-Ostaszewski spaces in terms of the combinatorics of the ideal of countable clopen sets. This characterization will be a variation on the notion (due to Hajnal and Juhasz) of HFD (to be defined below). This may not be surprising in view of Juhasz' theorem that every locally countable weak HFD of size ω_1 is a sub-Ostaszewski space.

Attempts to reverse Juhasz' theorem, however, are doomed: Ostaszewski spaces are locally compact scattered, hence their clopen algebras have no infinite free subalgebras; weak HFD's have uncountable free subalgebras. But there is a Boolean version of weak HFD, and that version is equivalent to sub-Ostaszewski.

Let's define the notions of HFD and weak HFD.

Definition 4. A set $E \subset 2^{\omega_1}$ is finally dense iff $\exists\alpha\forall\sigma$ a finite function from $\omega_1 \setminus \alpha$ into 2 infinitely many $x \in E$ extend σ. $X \subset 2^{\omega_1}$ is an HFD (for hereditarily finally dense) iff every infinite subset of X is finally dense. It is a weak HFD iff every $Y \in [X]^{\omega_1}$ has an infinite finally dense subset.

Note that an HFD can't be destroyed without adding reals: Being an HFD is hereditary, so we can assume the HFD we want to destroy has size ω_1. Destroying it means adding a countable subset of ω_1 with the right (i.e., wrong) property. Since $\omega_1 \leq 2^\omega$, $[\omega_1]^\omega$ can be coded by $\mathbb{R}$[1]; adding a countable subset of ω_1 adds a real.

Question 5. Is there a weak HFD which can be destroyed without adding reals?

Now for HDT's and weak HDT's.

Definition 6. Let $\mathcal{J} \subset \wp(\omega_1)$, and let $E \subset \omega_1$. We say that E is dense on a tail of $\mathcal{J}$ iff there is some $\alpha < \omega_1$ so that if $J \in \mathcal{J}$ and $J \setminus \alpha \neq \emptyset$ then $E \cap J$ is infinite. $\mathcal{J} \subset \wp(\omega_1)$ is an HDT (for hereditarily dense on a tail) iff every infinite $E \subset \omega_1$ is dense on a tail of $\mathcal{J}$; it is a weak HDT iff $\forall E \in [\omega_1]^{\omega_1} \exists F \in [E]^\omega$, F is dense on a tail of $\mathcal{J}$.

Again, HDT's can be destroyed without adding reals.

Question 7. Assuming CH, is there a weak HDT that can be destroyed without adding reals?

A corollary of theorem 8 will be that a weak HDT corresponding to a locally compact sub-Ostaszewski space can be destroyed without adding reals. But it is not known if such a space exists under CH.

To end the sequence of definitions, note that the collection of countable clopen subsets of any topological space is an ideal (in the algebra of clopen sets) closed under finite unions, finite

[1]first code ω_1 as a subset of $\mathbb{R}$ and then use standard coding techniques

intersections and relative complements. We will call such an ideal a *pre − algebra.*

Theorem 8. *The following are equivalent:*

I. There is a sub-Ostaszewski space.

II. There is a weak HDT pre-algebra $\mathcal{J} \subset [\omega_1]^\omega$.

Proof: Assume X is sub-Ostaszewski. Without loss of generality we may assume it is locally countable, 0-dimensional hereditarily separable with underlying set ω_1. Let $\mathcal{J} = \{u \subset X : u \text{ is countable and clopen}\}$. $\mathcal{J}$ is a pre-algebra and a base for X. We will show that $\mathcal{J}$ is a weak HDT.

Let $E \in [X]^{\omega_1}$. By sub-Ostaszewski, there is some $\alpha \in \omega_1$ so $\mathrm{cl}E \supset X \setminus \alpha$. By hereditary separability, E has a countable dense set F. So if $J \in \mathcal{J}$ and $J \setminus \alpha \neq \emptyset$, $F \cap J$ is infinite.

For the other direction, let $\mathcal{J} \subset [\omega_1]^\omega$ be a weak HDT pre-algebra. For $\alpha \in \omega_1$ we let $\mu_\alpha = \{J \in \mathcal{J} : \alpha \in J\}$. For each α, $\{\beta : \mu_\alpha = \mu_\beta\}$ is countable, so without loss of generality we may assume $\omega_1 \subset \bigcup \mathcal{J}$, and if $\alpha \neq \beta$ then $\mu_\alpha \neq \mu_\beta$. Let X be the space whose underlying set is ω_1 with $\mathcal{J}$ as a base. X is 0-dimensional and Hausdorff. Since $\mathcal{J}$ is HDT, if $E \in [\omega_1]^{\omega_1}$ there is $F \in [E]^\omega$ and $\alpha < \omega_1$ with F dense in $X \setminus \alpha$. So cl E is co-countable. □

In light of question 2, we have the following addition (due to the referee) to theorem 8:

Theorem 9. *The following are equivalent:*

I. There is a locally compact sub-Ostaszewski space.

II. There is a weak HDT pre-algebra $\mathcal{J} \subset [\omega_1]^\omega$ so that no $J \in \mathcal{J}$ is the union of a chain $\mathcal{C}$ of elements of $\mathcal{J}$ with $J \notin \mathcal{C}$.

Proof: Again, I $\Rightarrow$ II follows from letting $\mathcal{J}$ be the pre-algebra of compact clopen sets.

For II $\Rightarrow$ I, suppose $\mathcal{J}$ satisfies II and let B be the algebra generated by $\mathcal{J}$. Assume as before that the μ_α's are distinct, and let μ be an arbitrary ultrafilter on B with $\mu \cap \mathcal{J} \neq \emptyset$. It suffices to show that μ is principal. If μ is not principal, fix $J \in \mu$ and note that $\bigcap\{J \setminus K : K \in \mathcal{J} \setminus \mu\} = \emptyset$. So

$J = \bigcup\{K \cap J : K \in \mathcal{J} \setminus \mu\}$. Since J is countable, we can find an increasing chain $\{K_n : n < \omega\} \subset \mathcal{J} \setminus \mu$ with $J = \bigcup_n \{K_n : n < \omega\}$. □

2. Preimages of weak HFD's

Now let's consider a stronger anti-converse to Juhasz' theorem that weak HFD's are sub-Ostaszewski: locally compact sub-Ostaszewski spaces (hence Ostaszewski spaces) are not even continuous preimages of a weak HFD.

Definition 10. A subset D of a Boolean algebra is ideal-independent iff every Boolean combination of the form $a - \vee E$ is non-zero, where $a \in D, a \notin E \in [D]^{<\omega}$.

Here, $\vee$ is the Boolean sup.

It's easy to see that a continuous pre-image of a weak HFD has an uncountable ideal-independent family of clopen sets: for some α, $\{f^{-1}(\beta, 1) : \beta > \alpha\}$ is ideal-independent, where f is the continuous function and $(\beta, 1) = \{x \in 2^{\omega_1} : x(\beta) = 1\}$.

But an ideal-independent family of size ω_1 gives rise, in the Stone space, to an uncountable discrete subspace: if D is ideal independent, for each $d \in D$ let x_d be an ultrafilter so that $d \in x_d$ and, for each $e \in D \setminus \{d\}$, $e \notin x_d$.

Proposition 11. *A locally compact sub-Ostaszewski space X is not a continuous pre-image of a weak HFD.*

Proof: By hereditary separability X has no uncountable discrete subspace. By local compactness, the Stone space αX is the one-point compactification. So the Stone space has no uncountable discrete subspace. □

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