

Topology Proceedings



Web: <http://topology.auburn.edu/tp/>
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

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**POINT-COUNTABLE k -NETWORKS,
 cs^* -NETWORKS AND α_4 -SPACES¹**PENGFEI YAN AND SHOU LIN²

ABSTRACT. In this paper the authors give the relations between point-countable k -networks and cs^* -networks by virtue of α_4 -spaces, and prove some mapping theorems on spaces with point-countable p - k -networks.

Since Burke, Gruenhage, Michael and Tanaka[3,4,7,20] established the fundamental theory on point-countable covers in generalized metric spaces, many topologists have discussed the point-countable covers with various characters. Meanwhile, the conceptions of k -networks, cs^* -networks and p - k -networks were introduced. The study for relations among certain point-countable covers has become one of the most important subjects in general topology. In this paper, we shall consider the connections between point-countable k -networks and cs^* -networks, prove that α_4 -spaces with point-countable k -networks have point-countable cs^* -networks, and obtain some closed mapping theorems about spaces with point-countable p - k -networks, which generalized some results in [7,13,14].

1991 *Mathematics Subject Classification.* 54E99, 54C10, 54D55.

Key words and phrases. k -networks, cs^* -networks, α_4 -spaces, closed mappings, p - k -networks.

¹ The project supported by NSFC(No. 19501023, 19971048) and Fujian Provincial NSFC(A97025).

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In this paper, all spaces are assumed to be regular and T_1 , and all mappings are continuous and surjective. We recall some basic definitions.

Definition 1. Let X be a space, and \mathcal{P} a cover of X . Put

$$\mathcal{P}^{<\omega} = \{\mathcal{P}' \subset \mathcal{P} : |\mathcal{P}'| < \omega\}.$$

- (1) \mathcal{P} is a k -network [15] if, whenever $K \subset U$ with K compact and U open in X , then $K \subset \cup \mathcal{F} \subset U$ for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.
- (2) \mathcal{P} is a p - k -network [7] if, whenever $K \subset X \setminus \{y\}$ with K compact in X , then $K \subset \cup \mathcal{F} \subset X \setminus \{y\}$ for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.
- (3) \mathcal{P} is a cs^* -network [6] if $\{x_n\}$ is a sequence converging to $x \in X$ and U is a neighborhood of x , there exists a $P \in \mathcal{P}$ such that $\{x\} \cup \{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.
- (4) \mathcal{P} is a wcs^* -network [11] if $\{x_n\}$ is a sequence converging to $x \in X$ and U is a neighborhood of x , there exists a $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.
- (5) \mathcal{P} is a p - wcs^* -network if $\{x_n\}$ is a sequence converging to $x \in X$ and $y \neq x$, there exists a $P \in \mathcal{P}$ such that $\{x_{n_i} : i \in N\} \subset P \subset X \setminus \{y\}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$:

p - k -networks were studied in [7], where they were labeled $(1.4)_p$. cs^* -networks, wcs^* -networks, p - k -networks and p - wcs^* -networks were studied in [20], where they were labeled (C_1) , (C_2) , (C_3) and $(C_3)'$ respectively.

Definition 2. [9] Call a subspace of a space a fan (at a point x) if it consists of a point x , and a countably infinite family of disjoint sequences converging to x . Call a subset of a fan a diagonal if it is a convergent sequence meeting infinitely many of the sequences converging to x and converges to some point in the fan.

- (1) A space X is an α_4 -space [2] if every fan at x of X has a diagonal converging to x .
- (2) S_ω is a fan without a diagonal.

Definition 3. [5] For a space X and $x \in P \subset X$, P is a sequential neighborhood at x in X if, whenever $\{x_n\}$ is a sequence converging to x in X , then $x_n \in P$ for all but finitely many $n \in \mathbb{N}$.

Definition 4. Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of X which satisfies that for each $x \in X$,

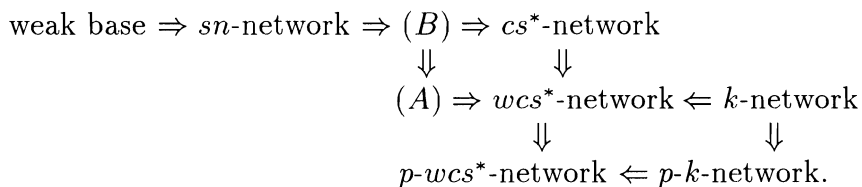
- (1) \mathcal{P}_x is a network of x in X ,
- (2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

\mathcal{P} is an sn -network[19] for X if each element of \mathcal{P}_x is a sequential neighborhood of x in X . \mathcal{P} is a weak base [1] for X if, whenever G is subset of X such that for each $x \in G$, $P \subset G$ for some $P \in \mathcal{P}_x$, then G is open in X . A space X is an snf -countable space if X has an sn -network \mathcal{P} such that each \mathcal{P}_x is countable. A space X is a gf -countable space if X has a weak base \mathcal{P} such that each \mathcal{P}_x is countable.

We introduce the following symbols. For a subset collection \mathcal{F} of a space X , let $Int_s(\mathcal{F}) = \{x : \cup\mathcal{F} \text{ is a sequential neighborhood at } x \text{ in } X\}$. A subset collection \mathcal{F} of X is an sn -cover of A if $A \subset Int_s(\cup\mathcal{F})$. For a cover \mathcal{P} of a space X , put

- (A): If $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in Int_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$.
- (B): If $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in Int_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$, and $x \in \cap\mathcal{F}$.

It is clear that [9] gf -countable spaces $\Rightarrow snf$ -countable spaces $\Rightarrow \alpha_4$ -spaces \Rightarrow spaces without any closed copy of S_ω , and that



It is well known that for a space X ,

- (1) X has a point-countable weak base $\Leftrightarrow X$ is a gf -countable space with a point-countable sn -network $\Rightarrow X$ has a point-countable k -network [11].
- (2) X has a point-countable sn -network $\not\Rightarrow X$ has a point-countable p - k -network, for example, the Stone-Ćech compactification βN .
- (3) X has a point-countable cs^* -network $\not\Rightarrow X$ has a point-countable cover satisfying (A), for example, S_ω .
- (4) X has a point-countable cs^* -network $\not\Rightarrow X$ has a point-countable cs^* -network, or X has a point-countable cover satisfying (A), for example, S_{ω_1} [20].
- (5) X has a point-countable p - k -network $\not\Rightarrow X$ has a point-countable wcs^* -network, for example, let ψ^* be the well known Mrowka's space. Then ψ^* is a Moore space without any point-countable base. A Moore space has a point-countable p - k -network by Corollary 3.8 in [7].

In this paper we shall further study the relations among point-countable covers satisfying the conditions above.

Lemma 5. *Let X be an α_4 -space, and \mathcal{P} a point-countable wcs^* -network, then \mathcal{P} has (A).*

Proof: Suppose the conclusion is false for some $x \in U \in \tau(X)$. For each countable $C \subset U$, let $\mathcal{P}(C) = \{P \in \mathcal{P} : P \cap C \neq \emptyset, P \subset U\} = \{P_i(C) : i \in N\}$. Put $C_0 = \{x\}$, then $P_1(C_0)$ is not a sequential neighborhood at x . There exists a sequence $\{x_{1n}\}$ in $U \setminus P_1(C_0)$ converging to x . Let $C_1 = \{x_{1n} : n \in N\}$, as \mathcal{P} is a wcs^* -network of X , there is an $n_1 \in N$ and an infinite subset C_1' of C_1 such that $C_1' \subset \cup\{P_i(C_j) : 1 \leq i \leq n_1, 0 \leq j \leq 1\}$. Without losing generality, we can suppose $C_1' = C_1$, then $\cup\{P_i(C_j) : 1 \leq i \leq n_1, 0 \leq j \leq 1\}$ is not a sequential neighborhood at x . Repeating the process above, we can inductively choose $C_m = \{x_{nm} : n \in N\} \subset U$ such that $\{x_{nm}\}$ converges to x for each $m \in N$, and an increasing sequence $\{n_m\}$ with $C_m \subset \cup\{P_i(C_j) : 1 \leq i \leq n_m, 1 \leq j \leq m\} \setminus \cup\{P_i(C_j) : 1 \leq i \leq n_{m-1}, 0 \leq j \leq m-1\}$. The last

condition implies that each $P \in \mathcal{P}$ meets only finitely many C_m .

Let $S = \{x_{nm} : m \in N, n \in N\} \cup \{x\}$, then S is a fan at the point x . Because X is an α_4 -space, S has a diagonal converging to x , thus we can find $P \in \mathcal{P}$ and $P \subset U$ such that P meets infinitely many C_m , a contradiction.

The following Lemma is similar to Miščenko's Lemma [16].

Lemma 6. *If \mathcal{P} is a point-countable cover of a space X , then every $A \subset X$ has only countably many minimal finite sn -covers by elements of \mathcal{P} .*

Proof: For every $\mathcal{F} \in \mathcal{P}^{<\omega}$, let $\mathcal{H}(\mathcal{F}) = \{H \subset X : \mathcal{F} \text{ is a minimal } sn\text{-cover of } H\}$. If there are uncountable many $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $A \in \mathcal{H}(\mathcal{F})$, then we can choose an $m \in N$ and an uncountable subset Λ of $\mathcal{P}^{<\omega}$ such that $|\mathcal{F}| = m$ and $A \in \mathcal{H}(\mathcal{F})$ for every $\mathcal{F} \in \Lambda$.

Suppose \mathcal{R} is a maximal subset of \mathcal{P} satisfying $\mathcal{R} \subset \mathcal{F}$ for uncountably many $\mathcal{F} \in \Lambda$, then $0 \leq |\mathcal{R}| < m$. Let $\Gamma = \{\mathcal{F} \in \Lambda : \mathcal{R} \subset \mathcal{F}\}$. If $\mathcal{F} \in \Gamma$, $A \notin \text{Int}_s(\cup \mathcal{R})$. Choosing $x \in A \setminus \text{Int}_s(\cup \mathcal{R})$, then there exists a sequence $\{x_n\}$ converging to x such that all $x_n \notin \cup \mathcal{R}$. Let $L = \{x_n : n \in N\}$, then L meets an element of \mathcal{F} . Because \mathcal{P} is point-countable and Γ is uncountable, we can obtain $P \in \mathcal{P}$ such that $L \cap P \neq \phi$ and uncountably many elements of Γ contain P , so $P \notin \mathcal{R}$ and uncountably many elements of Γ contain $\mathcal{R} \cup \{P\}$. The last condition implies a contradiction.

Theorem 7. *The following are equivalent for a space X :*

- (1) X has a point-countable cover satisfying (A).
- (2) X has a point-countable cover satisfying (B).
- (3) X is an α_4 -space (or an snf -countable space) with a point-countable cs^* -network.
- (4) X is an α_4 -space (or an snf -countable space) with a point-countable wcs^* -network.

Proof: (2) \Rightarrow (3) \Rightarrow (4) is obvious, and (4) \Rightarrow (1) holds by Lemma 5, We only need to show that (1) \Rightarrow (2). Let \mathcal{P} be a

point-countable cover satisfying (A). For each $\mathcal{F} \in \mathcal{P}^{<\omega}$, put $M(\mathcal{F}) = \{x \in X : \cup \mathcal{F} \text{ is a minimal sn-cover of } \{x\}\}$. It follows from Lemma 6 that if $x \in X$, then $x \in M(\mathcal{F})$ for only countable many $\mathcal{F} \in \mathcal{P}^{<\omega}$. For each $P \in \mathcal{P}$, let

$$P' = P \cup (\cup \{M(\mathcal{F}) : \mathcal{F} \in \mathcal{P}^{<\omega}, P \in \mathcal{F}\}).$$

Then $P' \subset \overline{P}$. In fact, for every $x \in M(\mathcal{F})$ and $P \in \mathcal{F}$, then $x \in \text{Int}_s(\cup \mathcal{F})$, while $x \notin \text{Int}_s(\cup(\mathcal{F} \setminus \{P\}))$, so there exists a sequence in P converging to x . Thus $x \in \overline{P}$, then $P' \subset \overline{P}$.

Let $\mathcal{P}' = \{P' : P \in \mathcal{P}\}$. For every $x \in X$, put $\mathcal{A}_x = \cup \{\mathcal{F} \in \mathcal{P}^{<\omega} : x \in M(\mathcal{F})\}$, then \mathcal{A}_x is countable. Since $x \in P' \Leftrightarrow x \in P$ or $x \in M(\mathcal{F})$ with $P \in \mathcal{F}$, and $P \in \mathcal{A}_x$, \mathcal{P}' is point-countable

For every $x \in W \in \tau(X)$, there exists $U \in \tau(X)$ such that $x \in U \subset \overline{U} \subset W$. Choosing $\mathcal{F}_0 \in \mathcal{P}^{<\omega}$ such that $x \in \text{Int}_s(\cup \mathcal{F}_0) \subset \mathcal{F}_0 \subset U$. We can suppose $x \in M(\mathcal{F}_0)$. Let $\mathcal{F}'_0 = \{P' : P \in \mathcal{F}_0\}$, then $x \in P'$ for every $P' \in \mathcal{F}'_0$. On the other hand, $x \in \cup \mathcal{F}'_0 \subset \overline{\cup \mathcal{F}'_0} \subset \overline{U} \subset W$. \mathcal{P}' satisfies (B).

Now, we give two interesting corollaries about Theorem 7. In view of Proposition 1.2 and Remark 1.3 in [20], we have

Lemma 8. *Let \mathcal{P} be a point-countable cover of X , then \mathcal{P} is a k -network (or a p - k -network) if and only if it is a wcs^* -network (or a p - wcs^* -network), and every compact subset of X is sequentially compact.*

Corollary 9. *For an α_4 -space X , the following are equivalent:*

- (1) X has a point-countable k -network.
- (2) X has a point-countable cs^* -network, and each compact subset of X is sequentially compact.

Corollary 10. *The following are equivalent for a k -space X , with a point-countable k -network:*

- (1) X is an α_4 -space
- (2) X contains no closed copy of S_ω .
- (3) X is an snf -countable space.
- (4) X is a gf -countable space.

Proof: (1) \Leftrightarrow (4) \Leftrightarrow (3) by Theorem 3.13 in [9] and Theorem 7, and (1) \Leftrightarrow (2) by Corollary 3.9 in [9].

Remark 11. Burke and Michael [3,4] proved that for a space X the following are equivalent:

- (1) X has a point-countable base.
- (2) X has a point-countable cover \mathcal{P} such that, if $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup \mathcal{F})^0 \subset \cup \mathcal{F} \subset U$, and $x \in \cap \mathcal{F}$.
- (3) X is a k -space with a point-countable cover \mathcal{P} such that, if $x \in U \in \tau(X)$, there exists $\mathcal{F} \in \mathcal{P}^{<\omega}$ such that $x \in (\cup \mathcal{F})^0 \subset \cup \mathcal{F} \subset U$.

But a space with a point-countable cover satisfying (B) can not be a space with a point-countable sn -network. By Remark 14(2) in [11], a gf -countable space X with a point-countable k -network (hence a space with a point-countable cover satisfying (B)) $\not\Rightarrow X$ has a point-countable sn -network. By Half-Disk Topology in [19, Example 78], there is a first countable and T_2 -space X with a point-countable cover satisfying (A) such that X has not any point-countable cs^* -network, thus it shows that the regularity of the space in Theorem 7 is essential. Corollary 9 partly answers the following question posed by Chuan Liu[12]: Is a Fréchet space with a point-countable k -network which contains no closed copy of S_{ω_1} a space with a point-countable cs^* -network? Corollary 10 improves Theorem 1 in [13] and Theorem 20 in [14], and affirmatively answers the following question posed by Liu and Tanaka in [14]: Is a k -space with a σ -compact-finite k -network a gf -countable space if it contains no closed copy of S_ω ?

Finally, we shall prove some closed mapping theorems on spaces with point-countable p - k -networks. It is well known that spaces with point-countable p - k -networks are preserved by perfect maps[7].

Theorem 12. *Let $f : X \rightarrow Y$ be a closed map such that X has a point-countable p - k -network, if one of the following holds, then Y has a point-countable p - k -network.*

- (1) X is a k -space.
- (2) Each point of X is a G_δ -set.
- (3) X is a normal, isocompact space.
- (4) Each $Bf^{-1}(y)$ is Lindelöf.

Proof: We only prove the result for (1), the other proof is similar to [11]. We note that every compact subset of X is metrizable in view of Lemma 8. If (1) holds, then X is a sequential space. By Lemma 2 in [11], each compact subset of Y is sequentially compact. So it suffices to show that Y has a point-countable p - wcs^* -network by Lemma 8. Let \mathcal{P} be a point-countable p - k -network of X . For each $y \in Y$, choose $x_y \in f^{-1}(y)$ and put $A = \{x_y : y \in Y\}$, $\mathcal{P}^* = \{f(A \cap P) : P \in \mathcal{P}\}$, then \mathcal{P}^* is a point-countable cover of Y . Let $S = \{y_n : n \in N\}$ be a sequence converging to $y \in Y$, $z \neq y$ and all $y_n \neq z$. Choosing $x_n \in f^{-1}(y_n) \cap A$, $x' \in f^{-1}(z) \cap A$, and $x \in f^{-1}(y) \cap A$. Then there exists a convergent subsequence T of $\{x_n\}$ in $X \setminus \{x'\}$ [11]. \mathcal{P} is a p - k -network for X , hence we can find out a $P \in \mathcal{P}$ and a subsequence Z of T such that $Z \subset P \subset X \setminus \{x'\}$, and $f(Z) \subset f(P \cap A) \subset Y \setminus \{z\}$, so this shows that \mathcal{P}^* is a point-countable p - wcs^* -network for Y .

Corollary 13. *Let $f : X \rightarrow Y$ be a closed map with Lindelöf fibres such that X has a point-countable p - k -network, then Y has a point-countable p - k -network.*

Remark 14. M. Sakai[17] showed that there is a closed map from a space X in which every compact subset is finite onto the one-point compactification Y of ω_1 . That X has a point-countable k -network, and Y has not a point-countable p - k -network. Let $f : X \rightarrow Y$ be a map. f is a compact-covering map if each compact subset of Y is the image of some compact subset of X . Let $f : X \rightarrow Y$ be a closed map. If X is a normal, isocompact space or each $Bf^{-1}(y)$ is Lindelöf, then f is a compact-covering map [11]. If X has a point-countable

k -network, and X is a k -space or each point of X is a G_δ -set, then f is also a compact-covering map [10,18]. But, if X has a point-countable p - k -network, and X is a k -space or each point of X is a G_δ -set, then f can not be a compact-covering map. In fact, let ψ^* be the Mrowka's space. Then taking $f : \psi^* \rightarrow S$ where f maps all the nonisolated points of ψ^* into a single point one obtains a closed mapping of a Moore space onto a convergent sequence which is not compact-covering [10].

REFERENCES

- [1] A. V. Arhangel'skii, *Mappings and spaces*, Uspechi Mat. Nauk., **21** (1966)(4), 133-184.
- [2] A. V. Arhangel'skii, *The frequency spectrum of a topological space and the clasification of spaces*, Soviet Math. Dokl., **13** (1972), 265-268.
- [3] D. K. Burke and E. Michael, *On a theorem of V. V. Filippov*, Israel J. Math., **11** (1972), 394-397.
- [4] D. K. Burke and E. Michael, *On certain point-countable covers*, Pacific J. Math., **64** (1976), 79-92.
- [5] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math., **57** (1965), 107-115.
- [6] Zhimin Gao, *\aleph -space is invariant under perfect mappings*, Questions Answers General Topology, **5** (1987), 271-279.
- [7] G. Gruenhage, E. Michael and Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math., **113** (1984), 303-332.
- [8] Shou Lin, *Generalized Metric Spaces and Mappings*, Beijing: Chinese Science Press, 1995.
- [9] Shou Lin, *A note on the Arens' space and sequential fan*, Topology Appl., **81** (1997), 185-196.
- [10] Shou Lin and Chuan Liu, *On spaces with point-countable cs -networks*, Topology Appl., **74** (1996), 51-60.
- [11] Shou Lin and Y. Tanaka, *Point-countable k -networks, closed maps and related results*, Topology Appl., **59** (1994), 79-86.
- [12] Shou Lin and Pengfei Yan, *On point-countable covers*, J. Ningde Teachers' College(Natural Science), **10** (1998), 247-255.
- [13] Chuan Liu and Mumin Dai, *g -metrizability and S_ω* , Topology Appl., **60** (1994), 185-189.
- [14] Chuan Liu and Y. Tanaka, *Spaces having σ -compact-finite k -networks and related matters*, Topology Proc., **21** (1996), 173-200.

- [15] P. O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc., **29** (1971), 183-189.
- [16] A. S. Miščenko, *Spaces with a pointwise denumerable bases*, Dokl. Akad. Nauk SSSR, **145** (1962), 985-988(=Soviet Math. Dokl., **3** (1962), 855-858).
- [17] M. Sakai, *Remarks on spaces with special type of k -networks*, Tsukuba J. Math., **21** (1997), 443-448.
- [18] A. Shibakov, *Closed mapping theorems on k -spaces with point-countable k -networks*, Comment. Math. Univ. Carolinae, **36** (1995), 77-87.
- [19] L. A. Steen and J. A. Seebach Jr. *Counterexamples in Topology*(Second Edition), New York: Springer-Verlag, 1978.
- [20] Y. Tanaka, *Point-countable covers and k -networks*, Topology Proc., **12** (1987), 327-349.

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