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ON NON-ARCHIMEDEAN SUBADDITIVE SEPARATING MAPS

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Abstract

Let X and Y denote compact Tihonov spaces. Let $C(X)$ and $C(Y)$ denote the spaces of continuous functions on X and Y, respectively, taking values in a non-Archimedean, nontrivially valued field $(K, | |)$. A map $H : C(X) \to C(Y)$ is *separating* if $f(x)g(x) = 0$ for all $x \in X$ implies that $Hf(y)Hg(y) = 0$ for all $y \in Y$. If $f(x)g(x) = 0$ for all $x \in X$ if and only if $Hf(y)Hg(y) = 0$ for all $y \in Y$, we say that H is *biseparating*. Results about automatic continuity and the form of additive and linear separating maps have been developed in [ABN1]–[ABN4], [BN1], [BNT], [HBN], and [NBA]. Under certain circumstances, additive separating maps induce a homeomorphism $h : Y \to X$ and are of the form $Hf(y) = H(f(h(y))\mathbf{1})(y)$; maps of this form are called *pseudocomposition maps*. In this article we investigate *subadditive* separating maps (Def. 2.2). We show (Theorem 3.15) that a bijective, biseparating subadditive map H must be a pseudocomposition map; in addition, such maps are "norm bounded" (Def. 2.7). We present a number of examples in Sec. 4 to demonstrate the necessity of certain hypotheses.

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1. Notation

- X and Y are compact 0-dimensional Hausdorff spaces.
- $(K, | \, |)$ is a non-Archimedean, nontrivially valued field, i.e., $| \, |$ satisfies the strong triangle inequality:
$$|a+b| \leq \max(|a|, |b|) \text{ for all } a, b \in K.$$
- For $U \subset X$ or Y, k_U denotes the K-valued characteristic function of a set U.
- $C(X)$ and $C(Y)$ denote the sup-normed Banach algebras of K-valued continuous functions on X and Y, respectively.
- $\mathbf{1}$ and $\mathbf{0}$ denote the functions that maps every $x \in X$ into $1 \in K$ and $0 \in K$, respectively.
- $H : C(X) \to C(Y)$ is a separating map, i.e., $fg = \mathbf{0}$ implies that $HfHg = \mathbf{0}$, and $H\mathbf{1}(y) \neq 0$ for all $y \in Y$. If $fg = \mathbf{0}$ if and only if $HfHg = \mathbf{0}$, we say that H is BISEPARATING.
- $\operatorname{cl} U$ denotes the topological closure of the set U.
- For any function $f : X \to K$, $\operatorname{coz} f$ denotes the cozero set $\{x \in X : f(x) \neq 0\}$ of f.
- $\mathbf{C}U$ denotes the complement of the set U.
- For $y \in Y$, and $g \in C(Y)$, $y\hat{} \circ g$ is the evaluation of g at y. Thus, $y\hat{} \circ H : C(X) \to K$, denotes the map $f \mapsto Hf(y)$.
- E and F denote normed linear spaces over K.

Given the separating map $H : C(X) \to C(Y)$, since $\mathbf{1} \cdot \mathbf{0} = \mathbf{0}$, it follows that $H\mathbf{1}H\mathbf{0} = \mathbf{0}$. Since $H\mathbf{1}$ never vanishes, it follows that $H\mathbf{0} = \mathbf{0}$.

2. Subadditive Operators - Properties and Examples

The weighted compositions of Definition 2.1 are continuous, linear separating maps.

Definition 2.1. *Consider a* WEIGHT FUNCTION $w \in C(Y)$ *and let* $g : Y \to X$ *be continuous. A* WEIGHTED COMPOSITION *is a map of the form*

$$\begin{array}{rcl} H: C(X) & \longrightarrow & C(Y) \\ f & \longmapsto & w \cdot (f \circ g) \end{array}$$

Continuous linear separating maps are weighted compositions ([BNT], 2.2). The more difficult question to answer is: Under what conditions must a separating linear map be continuous and therefore a weighted composition? Some results of this nature have been obtained in [BNT].

Definition 2.2. *E and F denote non-Archimedean normed linear spaces over K (i.e., $\|e + e'\| \leq \max\{\|e\|, \|e'\|\}$ for all $e, e' \in E$), and A maps E into F. We say that A is* SUBADDITIVE *if there exists $M > 0$ such that for all $e, e' \in E$, $\|A(e + e')\| \leq M(\|Ae\| + \|Ae'\|)$ or, equivalently,*

$$\|A(e + e')\| \leq M \max\{\|Ae\|, \|Ae'\|\}.$$

Definition 2.3. *E and F denote non-Archimedean normed linear spaces over K, and let A be subadditive. If for each $e \in E$ there exists $\delta > 0$ and $P_e > 0$ such that for all $e' \in E$ such that $\|e - e'\| \leq \delta$, then $\|Ae - Ae'\| \leq P_e \|A(e - e')\|$, then A is called* STRONGLY SUBADDITIVE.

Theorem 2.4. *E and F denote non-Archimedean normed linear spaces over K. A strongly subadditive map $A : E \to F$ is discontinuous [continuous] if and only if A is discontinuous [continuous] at 0.*

Proof. Suppose that there exists $e \in E$ such that for some net $\{e_a \in E : a \in A\}$, $e_a \to e$, but $Ae_a \nrightarrow Ae$. Let $\delta > 0$ be as in Def. 2.3. Since $\|e_a - e\| < \delta$ for sufficiently large a, $\|Ae - Ae_a\| \leq P_e \|A(e - e_a)\|$. Thus $e - e_a \to 0$, but $A(e - e_a)$ does not converge to 0. We leave the continuity argument to the reader. □

We next consider a collection of subadditive non-Archimedean isometries which are not additive. We begin with a few preliminary results.

Definition 2.5. *Let E and F be non-Archimedean normed linear spaces over K. An isometry $w : E \to F$ such that $w(0) = 0$ is called a* 0-ISOMETRY.

Lemma 2.6. *Let E and F be non-Archimedean normed spaces over K. A 0-isometry $w : E \to F$ is strongly subadditive.*

Proof. Since w is a 0-isometry and K is non-Archimedean, then for all $e, e' \in E$,

$$\|w(e+e')\| = \|e+e'\| \leq \max\{\|e\|, \|e'\|\} = \max\{\|w(e)\|, \|w(e')\|\}$$

and w is seen to be subadditive. Since w is a 0-isometry, $\|w(e-e')\| = \|e-e'\| = \|w(e) - w(e')\|$, w is strongly subadditive. □

Definition 2.7. *Let A map $C(X)$ into $C(Y)$.*

(a) *If there exists $D > 0$ such that $\|A(a\mathbf{1})\| \leq D|a|\,\|A\mathbf{1}\|$ for all $a \in K$, then A is* ***1***-BOUNDED.

(b) *If there exists $D > 0$ such that $\|Af\| \leq D\|f\|$ for all $f \in C(X)$, then A is* NORM BOUNDED.

Unlike the situation for linear or additive maps, norm-boundedness does not guarantee continuity of subadditive maps (Example 4.4), but:

Theorem 2.8. *If $A : C(X) \to C(Y)$ is $\mathbf{1}$-bounded and strongly subadditive on $[\mathbf{1}]$, then A is continuous on $[\mathbf{1}]$.*

Proof. Let (c_a) be a net from K that converges to $c \in K$. Since A is strongly subadditive and $\mathbf{1}$- bounded, there exist $P_c > 0$ and $D > 0$ such that for sufficiently large indices a

$$\|A(c_a\mathbf{1}) - A(c\mathbf{1})\| \leq P_c \|A((c_a - c)\mathbf{1})\| \leq P_c D |c_a - c| \|A\mathbf{1}\|. \square$$

Proposition 2.9. *Let E and F denote non-Archimedean normed linear spaces over K and let A map E into F.*

(a) *If A is norm preserving ($\|Ae\| = \|e\|$ for all $e \in E$), then A is subadditive.*

(b) *If A is strongly subadditive and norm bounded, then A is continuous.*

Proof. (a) Let $e, e' \in E$. Then $\|A(e+e')\| = \|e+e'\| \leq \max\{\|e\|, \|e'\|\} = \max\{\|Ae\|, \|Ae'\|\}$

(b) Since A is strongly subadditive, for each $e \in E$ there exists $P_e > 0$ and $\delta > 0$ such that for $\|e - e'\| < \delta$, then $\|Ae - Ae'\| \leq P_e \|A(e - e')\| \leq P_e D \|e - e'\| \to 0$. $\square$

Proposition 2.10. *Let E and F denote non-Archimedean normed linear spaces over K and let $A : E \to F$ be injective. If A and A^{-1} are norm bounded, then A is subadditive.*

Proof. Since A^{-1} is norm bounded, there exists $T > 0$ such that $\|e\| \leq T \|Ae\|$ for all $e \in E$. Since A is norm bounded, there is a $D > 0$ such that

$$\begin{aligned} \|A(e+e')\| &\leq D\|e+e'\| \\ &\leq D\max\{\|e\|, \|e'\|\} \\ &= D\max\{\|A^{-1}Ae\|, \|A^{-1}Ae'\|\} \\ &\leq DT\max\{\|Ae\|, \|Ae'\|\} \end{aligned} \qquad \square$$

Proposition 2.11. *Let E and F denote non-Archimedean normed linear spaces over K and let $A : E \to F$.*

(a) *If A is additive, then A is strongly subadditive.*

Proof. Take the M and P_e of Defs. 2.2 and 2.3, respectively, to be 1. $\square$

The following result (cf. [BNS], Example 3.1) is useful in Prop. 2.13.

Theorem 2.12. *There exist nondenumerably many non-additive surjective* 0*-isometries* $w : K \to K$*.*

Proof. Since K is nontrivially valued, choose $a \in K$, $0 < |a| < 1$. For each $n \in \mathbf{Z}$, let $S_n = \{n, n+1, \ldots\}$ and let $S = \prod_{n \in \mathbf{Z}} S_n$. For each $s = (s_n) \in S$ define $w_s : K \to K$ by taking $w_s(0) = 0$ and $w_s(x) = x + a^{s_n}$ for $|a|^{n-1} \leq |x| < |a|^{n-2}$. It is routine to check that each w_s is a bijective 0-isometry. To see that w_s is not additive, for any $s \in S$, note that since $|1 + a| = 1$ we have $w_s(1 + a) = 1 + a + a^{s_1}$ whereas $w_s(1) = 1 + a^{s_1}$ and $w_s(a) = a + a^{s_2}$. Hence $w_s(1 + a) \neq w_s(1) + w_s(a)$. Since S is uncountable so is $\{w_s : s \in S\}$.

Proposition 2.13. *There exist uncountably many continuous, bijective, strongly subadditive maps* $g : K \to K$*.*

Proof. Apply Lemma 2.6 to Theorem 2.12. □

Example 2.14. (cf. [BNS2].) *Let* K *be complete and discretely valued. An example of a continuous, biseparating, strongly subadditive map* $w : K \to K$ *which is neither an isometry nor a* VALUATION MULTIPLIER *(*$|w(a)| = c|a|$*, for some* $c > 0$*, all* $a \in K$*). The map* w *is not surjective, and not injective.*

Proof. Let $a \in K$, $|a| > 1$, be such that $|a|$ generates the value group $|K^*| = \{|b| : b \in K \setminus \{0\}\}$ of K. Define $w : K \to K$ by taking $w(0) = 0$ and, for $b \neq 0$

$$w(b) = \begin{cases} a^{10}b, & |b| = |a|^n,\ n = 10^k \text{ for some } k \in \mathbf{Z} \\ ab, & \text{all other } n \end{cases}$$

Then w is strongly subadditive, not valuation preserving, continuous, and not additive. By [BNS2], for all $c, d \in K$, $|w(c + d)| \leq |a|^9 \max\{|w(c)|, |w(d)|\}$ and $|w(c) - w(d)| \leq |a|^9 |w(c - d)|$. □

Example 2.15. *Let K be complete and discretely valued. An example of a continuous, bijective, biseparating, strongly subadditive map $w : K \to K$ which is neither an isometry, nor a valuation multiplier.*

Proof. Let $|a| > 1$ generate the value group $|K^*|$ of K. Let $w : K \to K$ be defined by $w(0) = 0$; for nonzero b,

$$w(b) = \begin{cases} a^{10}b, & |b| = |a|^n, \ n = 10^k, \ k \in \mathbf{Z} \\ ab & n \neq 10^k - 1, \ k \in \mathbf{Z} \\ a^2 b & n = 10^k - 1, \ k \in \mathbf{Z} \end{cases}.$$

By [BNS2], for all $c, d \in K$, $|w(c+b)| \leq |a|^9 \max\{|w(c)|, |w(d)|\}$ and $|w(c) - w(d)| \leq |a|^9 |w(c-d)|$. □

Example 2.16. *A bijective, continuous, strongly subadditive map $w : K \to K$.*

Proof. Let $a, a' \in K$ with $|a| = |a'| > 1$, and $|a - a'| = |a|$. For any $b \in K$

$$w(b) = \begin{cases} ab & |b| \leq 1 \\ a'b & |b| > 1 \end{cases}$$

Then by [BNS2], for $b, c \in K$,$|w(b+c)| \leq |a| \max\{|w(b)|, |w(c)|\}$ and $|w(b) - w(c)| \leq |w(b-c)|$. □

3. Subadditive Separating Operators

We next consider continuity and form of subadditive operators.

Definition 3.1. *The separating map $H : C(X) \to C(Y)$ is* POINTWISE SUBADDITIVE *if for each $y \in Y$ there exists $M_y > 0$ such that for every $y \in Y$ and every $f, g \in C(X)$,*

$$|\hat{y} \circ H(f+g)| = |H(f+g)(y)| \leq M_y (|Hf(y)| + |Hg(y)|).$$

H is ULTRAPOINTWISE SUBADDITIVE *if there exists $M_y > 0$ for every $y \in Y$ such that for all $f, g \in C(X)$,*

$$|\hat{y} \circ H(f+g)| = |H(f+g)(y)| \leq M \max\{|Hf(y)|, |Hg(y)|\}.$$

We say that the pointwise subadditive map H is POINTWISE EXPANDABLE *if for any $0 < c < d$ and $y \in Y$ and $f \in C(X)$ such that $c \leq |Hf(y)| < d$, there exists $k_{c,d} > 0$ such that if $a \in K$ with $|a| \geq k_{c,d}$, then $|H(af)(y)| \geq d$.*

As we show in Theorem 3.11, the support map h of a separating map can be defined by means of VANISHING SETS, defined below.

Definition 3.2. *Let $H : C(X) \to C(Y)$ be separating. For $y \in Y$, a* VANISHING SET *for $y\hat{} \circ H$ is a clopen set $U \subset X$ such that, for any $f \in C(X)$, $\operatorname{coz} f \subset U$ implies that $y\hat{} \circ H(f) = Hf(y) = 0$.*

Definition 3.3. *The separating map $H : C(X) \to C(Y)$ is* DETACHING *if for all pairs of distinct elements $y, y\prime \in Y$, there exist $f, g \in C(X)$ such that $\operatorname{cl} \operatorname{coz} f \cap \operatorname{cl} \operatorname{coz} g = \emptyset$ and $Hf(y)Hg(y') \neq 0$.*

Definition 3.4. *The* CONTINUITY SET *of the separating map $H : C(X) \to C(Y)$ is the set $Y_c = \{y \in Y : y\hat{} \circ H \text{ is continuous}\}$.*

Definition 3.5. *If for some continuous map $g : Y \to X$, $Hf(y) = H[f(g(y))\mathbf{1}](y)$ for all $y \in Y$ and $f \in C(X)$, then H is called a* PSEUDOCOMPOSITION MAP.

Remark 1. *Pseudocomposition maps are separating with support map g.*

Theorem 3.6. *If the separating map $H : C(X) \to C(Y)$ is a **1**-bounded pseudocomposition map, then H is norm bounded.*

Proof. Let H and g be as in Def. 3.5 so that for each $y \in Y$, $|Hf(y)| = |H[f(g(y))\mathbf{1}](y)|$. For any $y \in Y$,

$$\begin{aligned} |H[f(g(y))\mathbf{1}](y)| &\leq \max_{z\in Y} |H[f(g(y))\mathbf{1}](z)| \\ &= \|H[f(g(y))\mathbf{1}]\|. \end{aligned}$$

By Def. 2.7 there exists $D>0$ such that for all $y\in Y$ and $f\in C(X)$, $\|H[f(g(y))\mathbf{1}]\|\leq D|f(g(y))|\,\|H\mathbf{1}\|$. It follows that

$$\max_{y\in Y}|H[f(g(y))\mathbf{1}](y)|=\|Hf\|\leq D\|f\|\,\|H\mathbf{1}\|. \qquad \square$$

Proofs of certain basic results for linear separating operators—Lemmas 2.1, 2.2, and Proposition 2.1 of [BNT]—can be modified to prove identical or similar results for subadditive separating maps. We present a few of the arguments in Theorem 3.11. The notion of CONTINUOUS DECOMPOSITION OF THE IDENTITY is useful in these arguments.

Definition 3.7. *Let* $\{U_i : i=1,2,\dots,n\}$ *be an open cover of a topological space* T. *A* CONTINUOUS DECOMPOSITION OF THE IDENTITY SUBORDINATE TO
$\{U_i : i=1,2,\dots,n\}$ *is a set of functions* $e_i : T\to K$, $i=1,2,\dots,n$, *such that* $\operatorname{coz} e_i\subset U_i$ *for each* i *and* $\sum_{i=1}^{n} e_i=\mathbf{1}$.

Remark 2. *For any clopen cover* $\{U_i : i=1,2,\dots,n\}$ *of* X *there exists a continuous decomposition of the identity subordinate to* $\{U_i : i=1,...,n\}$*: for each* i*, let* $V_i=U_i-\bigcup_{j\neq i,j=1}^{n}U_i$*; then consider* $e_i=k_{V_i}$, $i=1,...,n$.

In previous articles the notion of "support" of a linear separating map has been considered. We investigate the support of a subadditive separating map here (Def. 3.8(b)). Although the properties of the support h of a subadditive separating map H are similar to those of a linear separating map (Theorems 3.11, 3.15), there are differences (Examples 4.1 and 4.4).

Definition 3.8. *Let* $H : C(X)\longrightarrow C(Y)$ *be separating.*

The SUPPORT $\operatorname{supp} y\hat{}\circ H$ *of* $y\hat{}\circ H$ *is the set.* $\operatorname{supp} y\hat{}\circ H=\bigcap_{Hf(y)\neq 0}\operatorname{cl}\operatorname{coz} f$.

Theorem 3.9. *Let* $H : C(X)\to C(Y)$ *be separating and pointwise subadditive. For all*
$y\in Y$, $\mathbf{C}\bigcup\{U : U$ *is a vanishing set for* $y\hat{}\circ H\}$
is a singleton which we denote by $h(y)$.

Proof. Let $y \in Y$. Observe that if $\{U_a : a \in A\}$ is the collection of vanishing sets for $y\hat{}\circ H$, then $\mathbf{C}\left(\bigcup_{a\in A} U_a\right) \neq \emptyset$ by the following argument. For $X = \bigcup_{a\in A} U_a$, choose U_{a_i}, $i = 1, ..., n$, such that $X = \bigcup_{i=1}^{n} U_{a_i}$. Let e_i, $i = 1, ..., n$, be a continuous decomposition of the identity subordinate to $\{U_{a_i} : i = 1, 2, \ldots, n\}$. Thus $f = \sum_{i=1}^{n} fe_i$ for all $f \in C(X)$. Since

$$|Hf(y)| \leq M_y \sum_{i=1}^{n} |Hfe_i(y)| = 0,$$

it follows that $y\hat{}\circ H(f) = 0$ for all $f \in C(X)$. This contradicts our standing assumption that $H\mathbf{1}(y) \neq 0$ for all $y \in Y$. Hence $X \neq \bigcup_{a\in A} U_a$ and$\{h(y)\} \neq \emptyset$. Suppose that x and $x' \in$ are distinct elements of $\mathbf{C}\left(\bigcup_{a\in A} U_a\right)$. Let U and V be disjoint clopen neighborhoods of x and x', respectively. From the definition of vanishing set (Def. 3.2), neither U or V can be a vanishing set for $y\hat{}\circ H$. Thus there exist $f, g \in C(X)$ such that $\operatorname{coz} f \subset U$, $\operatorname{coz} g \subset V$, and $Hf(y)\, Hg(y) \neq 0$. This contradicts the fact that H is separating. Consequently $\mathbf{C}\left(\bigcup_{a\in A} U_a\right)$ is a singleton. □

Definition 3.10. *The pointwise subadditive map H is* POINTWISE STRONGLY SUBADDITIVE *if for each $f \in C(X)$ and $y \in Y$ there exists $\delta_{f,y} > 0$ and $P_y > 0$ such that for all $g \in C(X)$ with $|f(h(y)) - g(h(y))| \leq \delta_{f,y}$, then*

$$|Hf(y) - Hg(y)| \leq P_y\, |H(f-g)(y)|. \tag{1}$$

If $a \in K$ and $f = a\mathbf{1}$, there exists $\delta_a > 0$ such that if $b \in K$ and $|a - b| < \delta_a$ then

$$|H(b\mathbf{1})(y) - H(a\mathbf{1})(y)| \leq P_y\, |H[(b-a)\mathbf{1}](y)|$$

for all $y \in Y$.

Theorem 3.11. *Let $H : C(X) \to C(Y)$ be separating and pointwise strongly subadditive.*

(a) *For any $f, g \in C(X)$, if $f = g$ on the clopen set $U \subset X$, then $Hf = Hg$ on $h^{-1}(U)$.*

(b) *The* SUPPORT MAP $h : Y \to X$, $y \mapsto h(y)$ *of H is continuous.*

(c) *For any $f \in C(X)$, $h(\operatorname{coz} Hf) \subset \operatorname{cl}\operatorname{coz} f$.*

(d) *If H is injective, then h is surjective.*

(e) *For all $y \in Y$, $h(y) = \operatorname{supp} y\hat{} \circ H$.*

(f) *The support map h is injective if and only if H is detaching.*

Proof. (a) The proof that $f = \mathbf{0}$ on U implies $Hf = \mathbf{0}$ on $h^{-1}(U)$ proceeds as in Lemma 2.2(a) of [BNT]. Suppose that $f, g \in C(X)$ and that $f = g$ on U. Then $H(f-g) \equiv \mathbf{0}$ on $h^{-1}(U)$. Let $y \in Y$ and let P_y be as in Def. 3.10. Then $|Hf(y) - Hg(y)| \leq P_y |H(f-g)(y)| = 0$ for $y \in h^{-1}(U)$.

(b) Suppose that $V \neq X$ is a clopen neighborhood of $h(y)$, $y \in Y$, in X. Then the K-valued characteristic function k_V of V belongs to $C(X)$. By (a) $Hk_V = H\mathbf{1}$ on $h^{-1}(V)$. Since $H\mathbf{1}$ never vanishes, $Hk_V(y) = H\mathbf{1}(y) \neq 0$; therefore $\operatorname{coz} Hk_V$ is a neighborhood of y. Suppose that $y' \in h^{-1}(\mathbf{C}V)$. Since $h(y') \in \mathbf{C}V$ and $k_V = \mathbf{0}$ on $\mathbf{C}V$, $Hk_V = H\mathbf{0}$ on $h^{-1}(\mathbf{C}V)$ by Theorem 3.9. Therefore $Hk_V(y') = 0$. Since $\operatorname{coz} Hk_V \subset h^{-1}(V)$, $h(\operatorname{coz} Hk_V) \subset V$. Therefore h is continuous.

We omit the proofs of (c)–(e) as they are virtually identical to the analogous statements in [BNT].

(f) Choose be disjoint clopen neighborhoods U and V of the distinct points $y, y' \in Y$, respectively, such that $Hk_U(y)\, Hk_V(y') \neq 0$. By the way h is defined, $h(y)$ and $h(y')$ belong to the disjoint sets U and V, respectively. The converse is clear.

Theorem 3.12. *If H is bijective, biseparating, and H and H^{-1} are pointwise strongly subadditive, then the support map h of H is a homeomorphism and the support map of H^{-1} is h^{-1}.*

Proof. If h is not 1-1, choose distinct elements $y, z \in Y$ such that $h(y) = h(z)$. Let $f, g \in C(X)$ be such that $\mathrm{cl}\,\mathrm{coz}\, Hf \cap \mathrm{cl}\,\mathrm{coz}\, Hg = \emptyset$, $Hf(y) \neq 0$, and $Hg(z) \neq 0$. As H is biseparating, $\mathrm{coz}\, f \cap \mathrm{coz}\, g = \emptyset$. By Theorem 3.11(c), $h(y) = h(z) \in \mathrm{cl}\,\mathrm{coz}\, f \cap \mathrm{cl}\,\mathrm{coz}\, g$. Let w denote the support map of H^{-1}. By Theorem 3.11(c), $w(H^{-1}(H\mathrm{coz}\, f)) = w(\mathrm{coz}\, f) \subset \mathrm{cl}\,\mathrm{coz}\, H(f)$ and $w(H^{-1}(H\mathrm{coz}\, g)) = w(\mathrm{coz}\, g) \subset \mathrm{cl}\,\mathrm{coz}\, H(g)$. Since w is continuous and $h(y) = h(z) \in \mathrm{cl}\,\mathrm{coz}\, f \cap \mathrm{cl}\,\mathrm{coz}\, g$,

$$w(h(y)) = w(h(z)) \in \mathrm{cl}\,\mathrm{coz}\, Hf \cap \mathrm{cl}\,\mathrm{coz}\, Hg$$

which is a contradiction ($\mathrm{cl}\,\mathrm{coz}\, Hf \cap \mathrm{cl}\,\mathrm{coz}\, Hg = \emptyset$). Since X and Y are compact and H is bijective, h is a homeomorphism from Y onto X by Theorem 3.11(d). By a similar argument, w is 1-1.

To prove that $w = h^{-1}$, let $x = h(y) \in X$. Let $\{U_a : a \in A\}$ denote the neighborhood filter of y. For each $a \in A$, since H is bijective, choose $f_a \in C(X)$ such that $H(f_a)(y) \neq 0$ with $\mathrm{coz}\, H(f_a) \subset U_a$. Since $f_a = H^{-1}(H(f_a))$ and H^{-1} is separating, $w(x) \in \mathrm{cl}\,\mathrm{coz}\, H(f_a)$ for all $a \in A$. As $\bigcap_{a\in A} U_a = \{y\}$, it follows that $w(x) = y$. Since $x = h(y)$, $w(x) = w(h(y)) = y$ and $w = h^{-1}$. □

For certain H, there is a relationship between subadditivity and pointwise subadditivity as well as a relationship between strong subadditivity and pointwise strong subadditivity.

Theorem 3.13. *Let H be bijective, biseparating, and both H and H^{-1} pointwise strongly subadditive. Then:*

(a) *There exists a bounded set of scalars $\{M_y : y \in Y\}$ satisfying*

$$|H(f+g)(y)| \leq M_y (|Hf(y)| + |Hg(y)|) \text{ for all } f, g \in C(X)$$

for all $y \in Y$ if and only if H is subadditive and $\|H(f+g)\| \leq M(\| Hf \| + \| Hg \|)$ for all $f, g \in C(X)$ and $M = \sup\{M_y : y \in Y\}$.

(b) *There exists a bounded set of scalars* $\{P_y : y \in Y\}$ *such that for each* $y \in Y$,

$$|Hf(y) - Hg(y)| \leq P_y |H(f-g)(y)|,$$

for all $f, g \in C(X)$ *if and only if* H *is strongly subadditive and* $\|Hf - Hg\| \leq \sup\{P_y : y \in Y\}] \|H(f-g)\|$ *for all* $f, g \in C(X)$.

Proof. (a) It is clear that if H is pointwise subadditive and $\{M_y : y \in Y\}$ is a bounded set, then H is subadditive with

$$\|H(f+g)\| \leq \sup\{M_y : y \in Y\} (\|H(f)\| + \|H(g)\|).$$

Conversely, suppose H is subadditive with

$$\|H(f+g)\| \leq M(\|H(f)\| + \|H(g)\|)$$

for all $f, g \in C(Y)$. To prove that H is pointwise subadditive by contradiction, suppose that there exists $y \in Y$, $\varepsilon > 0$ and $f, g \in C(X)$ such that $|H(f+g)(y)| - M(|Hf(y)| + |Hg(y)|) > \varepsilon$. Let U be a clopen neighborhood of y chosen such that the inequalities $|Hf(y) - Hf(z)|, |Hg(y) - Hg(z)|$ for are small enough for the inequalities

$$|H(f+g)(y)| - M(|Hf(z)| + |Hg(z)|) > \varepsilon, z \in U$$

are true for all $z \in U$. Let $k_U \in C(X)$ be the characteristic function of U. By the inequality above it follows that $|H(f+g)(y)| > M(\|k_U Hf\| + \|k_U Hg\|)$. As H is bijective, there exists $w_1, w_2 \in C(X)$ such that $Hw_1 = k_U Hf$ and $Hw_2 = k_U Hg$. Since $k_U \equiv 1$ on U, by Theorem 3.12, $w_1 = f$ and $w_2 = g$ on $l^{-1}(U) = (h^{-1})^{-1} U = h(U)$ where $l = h^{-1}$ is the support map of H^{-1}. Thus $H^{-1}(k_U H(f+g))) = H^{-1}H(f+g) = f + g = w_1 + w_2$ on $h(U)$. By Theorem 3.12, $H(w_1 + w_2) = H(f+g)$ on $h^{-1}(h(U)) = U$. Since

$$\begin{aligned} \|H(w_1 + w_2)\| &\geq |H(w_1 + w_2)(y)| \\ &= |H(f+g)(y)| \\ &> M(\|k_U Hf\| + \|k_U Hg\|) \\ &= M(\|Hw_1\| + \|Hw_2\|) \end{aligned}$$

we have obtained a contradiction of the relationship $\|H(f+g)\| \leq M(\|H(f)\| + \|H(g)\|)$ for all $f, g \in C(X)$.

(b) If $|Hf(y) - Hg(y)| \leq P_y |H(f-g)(y)|$ for all $y \in Y$, then clearly

$$|Hf(y) - Hg(y)| \leq \sup\{P_y : y \in Y\} |H(f-g)(y)|$$

and therefore $\|Hf - Hg\| \leq \sup\{P_y : y \in Y\} \|H(f-g)\|$. The converse can be proved utilizing an argument quite similar to the second part of the argument in (a). □

Theorem 3.14. *Let H be pointwise strongly subadditive.*

(a) *If $y \in Y_c$, then*

$$Hf(y) = H(f(h(y))\mathbf{1})(y) \text{ for all } f \in C(X).$$

(b) *If H is $\mathbf{1}$-bounded, then for any $y \in Y$, $y \in Y_c$ if and only if $Hf(y) = H(f(h(y))\mathbf{1})(y)$ for all $f \in C(Y)$.*

Proof. (a) Let $x = h(y)$ and $\varepsilon > 0$. Consider the clopen neighborhood $U_\varepsilon = \{z \in X : |f(z) - f(x)| < \varepsilon\}$ of x. Let $k_{U_\varepsilon} \in C(X)$ be the characteristic function of U_ε. Then as $\varepsilon \to 0$, $k_{U_\varepsilon}(f - f(h(y))\mathbf{1}) \to 0$; since $y \in Y_c$, $H(k_{U_\varepsilon}(f - f(h(y))\mathbf{1}))(y) \to 0$. But since H is pointwise strongly subadditive, there exists $P_y > 0$ such that

$$\begin{aligned} &|H(k_{U_\varepsilon} f)(y) - H(k_{U_\varepsilon} f(h(y))\mathbf{1})(y)| \\ &\qquad \leq P_y |H(k_{U_\varepsilon}(f - f(h(y)\mathbf{1})))(y)| \\ &\qquad \to 0 \text{ as } \varepsilon \to 0 \end{aligned}$$

Since $fk_{U_\varepsilon} = f$ and $k_{U_\varepsilon} f(h(y))\mathbf{1} = f(h(y))\mathbf{1}$ on U_ε, it follows from Theorem 3.11(b) that $H(k_{U_\varepsilon} f)(y) = Hf(y)$ and $H(k_{U_\varepsilon} f(h(y))\mathbf{1})(y) = H(f(h(y))\mathbf{1})(y)$.

(b) Suppose that H is pointwise strongly subadditive, $\mathbf{1}$-bounded and $Hf(y) = H(f(h(y))\mathbf{1})(y)$ for all $f \in C(X)$. Let $f_s \in C(X)$ be a net such that $f_s \to f$. Then if s is big enough so that $|f(h(y_s)) - f(h(y))| < \delta_{f(h(y))}$ (Def. 3.10), there exists

$P_y > 0$ and $D > 0$ such that

$$\begin{aligned}
|Hf(y) - Hf_s(y)| &= |H(f(h(y))\mathbf{1})(y) - H(f_s(h(y))\mathbf{1})(y)| \\
&\leq P_y |H((f(h(y)) - f_s(h(y)))\mathbf{1})(y)| \\
&\leq P_y \max_{z\in Y} |H((f(h(y)) - f_s(h(y))\mathbf{1}))(z)| \\
&= P_y \|H(f(h(y)) - f_s(h(y)))\mathbf{1}\| \\
&\leq P_y D |(f(h(y)) - f_s(h(y)))| \|H\mathbf{1}\| \\
&\leq P_y D \|f - f_s\| \|H\mathbf{1}\|
\end{aligned}$$

As $\|f_s - f\| \to 0$, the result follows. The converse follows from (a). □

Theorem 3.15. *Let H be pointwise strongly subadditive. Let $\{P_y : y \in Y\}$ be a bounded set of scalars satisfying inequality (1) of Def. 3.10. Let $\sup\{P_y : y \in Y\}$ be denoted by M.*

(a) *If H is **1**-bounded, then Y_c is closed.*

(b) *If H is continuous, then $Y_c = Y$. If H is **1**-bounded and $Y_c = Y$ then H is continuous.*

(c) *If H is pointwise expandable, then $h(\mathbf{C}Y_c)$ is a finite set.*

(d) *Suppose that H is bijective, biseparating, pointwise expandable,* **1***-bounded and detaching. Then $Y_c = Y$, and H is a continuous norm bounded pseudocomposition map.*

Proof. (a) Let (y_s) be a net from Y_c such that $y_s \to y$. Now
$|H(f(h(y_s))\mathbf{1})(y_s) - H(f(h(y))\mathbf{1})(y)| \leq$

$$\begin{gathered}
|H(f(h(y_s))\mathbf{1})(y_s) - H(f(h(y))\mathbf{1})(y_s)| + \\
|H(f(h(y))\mathbf{1})(y_s) - H(f(h(y))\mathbf{1})(y)|
\end{gathered}$$

Let $c = f(h(y))$. Since f is continuous, for sufficiently large s, $|f(h(y_s)) - f(h(y))| < \delta_c$ where δ_c is as in Def. 3.10. It follows that for such y_s
$|H(f(h(y_s))\mathbf{1})(y_s) - Hf((h(y))\mathbf{1})(y_s)|$
$\leq M |H((f(h(y_s)) - f(h(y)))\mathbf{1})(y_s)|$. Since

$$\begin{aligned}
&M\ |H((f(h(y_s)) - f(h(y)))\mathbf{1})(y_s)| \leq \\
&M\ \|H((f(h(y_s)) - f(h(y)))\mathbf{1})\|
\end{aligned}$$

and

$$\begin{aligned} & M \ \|H\left(\left(f\left(h(y_s)\right) - f\left(h(y)\right)\right)\mathbf{1}\right)\| \\ & < \ DM\left|f\left(h(y_s)\right) - f\left(h(y)\right)\right| \|H\mathbf{1}\| \end{aligned}$$

it follows that $\left|H\left(f\left(h(y_s)\right)\mathbf{1}\right)(y_s) - Hf\left(\left(h(y)\right)\mathbf{1}\right)(y_s)\right|$

$$\leq DM\left|f\left(h(y_s)\right) - f\left(h(y)\right)\right| \|H\mathbf{1}\| \longrightarrow 0$$

as $y_s \to y$. Since $H\left(f\left(h(y)\right)\mathbf{1}\right)$ is continuous,

$$\left|H\left(f\left(h(y)\right)\mathbf{1}\right)(y_s) - H\left(f\left(h(y)\right)\mathbf{1}\right)(y)\right| \longrightarrow 0.$$

Hence

$$\left|H\left(f\left(h(y_s)\right)\mathbf{1}\right)(y_s) - H\left(f\left(h(y)\mathbf{1}\right)(y)\right| \longrightarrow 0.$$

Thus $Hf\left(y_s\right) = H\left(f\left(h(y_s)\right)\mathbf{1}\right)(y_s) \to H\left(f\left(h(y)\mathbf{1}\right)(y)\right.$. By Theorem 3.14(a), since $y_s \in Y_c$, $Hf\left(y_s\right) = H\left(f\left(h(y_s)\right)\mathbf{1}\right)(y_s)$ for all y_s. Since Hf is continuous, $Hf\left(y_s\right) = H\left(f\left(h(y_s)\right)\mathbf{1}\right)(y_s)$ $\to Hf\left(y\right)$ therefore $Hf\left(y\right) = H\left(f\left(h(y)\right)\mathbf{1}\right)(y)$. By Theorem 3.14(b), it follows that $y \in Y_c$.

(b) If $Y_c \neq Y$, there exists $y \in Y$ such that $y\hat{} \circ H$ is discontinuous. This implies that H is discontinuous. Now suppose that $Y_c = Y$. Since H is pointwise strongly subadditive, for any $y \in Y = Y_c$ and any $f, g \in C\left(X\right)$,

$$\begin{aligned} \left|Hf\left(y\right) - Hg\left(y\right)\right| \ &\leq \ P_y\left|H\left(f-g\right)(y)\right| \\ &= \ P_y\left|H\left(\left(f\left(h\left(y\right)\right) - g\left(h\left(y\right)\right)\right)\mathbf{1}\right)(y)\right| \\ & \qquad \text{[Theorem 3.14]} \end{aligned}$$

Since H is $\mathbf{1}$-bounded there exists $D > 0$ such that

$$\begin{aligned} & \left|H\left(\left[f\left(h\left(y\right)\right) - g\left(h\left(y\right)\right)\right]\mathbf{1}\right)(y)\right| \\ & \qquad \leq \ P_y \max_{z\in Y}\left|H\left(\left[f\left(h\left(y\right)\right) - g\left(h\left(y\right)\right)\right]\mathbf{1}\right)(z)\right| \\ & \qquad = \ P_y\left\|H\left(f\left(h\left(y\right)\right) - g\left(h\left(y\right)\right)\right)\mathbf{1}\right\| \\ & \qquad \leq \ P_y D\left\|f-g\right\|\left\|H\mathbf{1}\right\| \end{aligned}$$

Thus $\max_{y\in Y}\left|Hf\left(y\right) - Hg\left(y\right)\right| = \left\|Hf - Hg\right\| \leq \sup\{P_y : y \in Y\}$ $D\left\|f-g\right\|\left\|H\mathbf{1}\right\|$ from which we deduce that H is continuous. (c) If $h\left(\mathbf{C}Y_c\right)$ is infinite, let $\left(y_n\right)$ be a sequence from $\mathbf{C}Y_c$ such that $\left\{h\left(y_n\right) : n \in \mathbf{N}\right\}$ is a denumerable subset of $h\left(\mathbf{C}Y_c\right)$.

By choosing a subsequence if necessary we may assume that for each $n \in \mathbf{N}$, U_n is a clopen neighborhood of $h(y_n)$ such that $U_n \cap \mathrm{cl} \bigcup_{m \neq n} U_m = \emptyset$. As $y_n\hat{} \circ H$ is discontinuous, $y_n\hat{} \circ H$ is discontinuous at 0 (Theorem 2.4). Thus for each n there exists $1 > \varepsilon_n > 0$ and a sequence $g_{nm} \in C(X)$ such that $\|g_{nm}\| \to 0$ as $m \to \infty$, while $|Hg_{nm}(y_n)| \geq \varepsilon_n$ for all m. Let $f_{nm} = k_{U_n} g_{nm}$. By Theorem 3.11(a), $Hf_{nm}(y_n) = Hg_{nm}(y_n)$. Thus $\|f_{nm}\| \to 0$ as $m \to \infty$, $|Hf_{nm}(y_n)| \geq \varepsilon_n$, and $\mathrm{coz}\, f_{nm} \subset U_n$ for all m. Suppose that for n_j, $j \in \mathbf{N}$, there exists an $m(n_j)$ such that $\left\|f_{n_j m(n_j)}\right\| \leq 1/n_j^2$ and $\left|Hf_{n_j m(n_j)}(y_{n_j})\right| > n_j$. Consider $f = \sum_{j \in \mathbf{N}} f_{n_j m(n_j)} \in C(X)$. Since $U_{n_j} \cap \mathrm{cl} \bigcup_{n_k \neq n_j} U_{n_k} = \emptyset$ and H is separating, it follows that $\left|Hf(y_{n_j})\right| = \left|Hf_{n_j m(n_j)}(y_{n_j})\right| > n_j$. Therefore Hf is unbounded, a contradiction, and the result is proved. It remains to consider the possibility that for all m such that $\|f_{nm}\| \leq 1/n^2$, $|Hf_{nm}(y_n)| \leq n$. Let $d_{\varepsilon_n, n}$ be as in Def. 3.1. For each $n \in \mathbf{N}$, choose $a_n \in K$ such that $|a_n| > d_{\varepsilon_n, n}$. Since $\|g_{nm}\| \to 0$ as $m \to \infty$, we may choose $m(n)$ such that $\left\|f_{nm(n)}\right\| \leq 1/\left(n^2 |a_n|\right)$ for each $n \in \mathbf{N}$. Since $\left\|a_n f_{nm(n)}\right\| \leq 1/n^2$, by Def. 3.1, $\left|H\left(a_n f_{nm(n)}\right)(y_n)\right| > n$. Denoting $a_n f_{nm(n)}$ by f_n for all $n \in \mathbf{N}$, $\sum_{n \in \mathbf{N}} f_n \in C(X)$ and $H\left(\sum_{n \in \mathbf{N}} f_n\right) \in C(Y)$. But since H is separating pointwise subadditive, and $U_n \cap \mathrm{cl} \bigcup_{m \neq n} U_m = \emptyset$,

$$\begin{aligned} \left|H\left(\textstyle\sum_{n \in \mathbf{N}} f_n\right)(y_k)\right| &\geq \left|\,|Hf_k(y_k)| - \left|-H\left(\textstyle\sum_{n \neq k} f_n\right)(y_k)\right|\,\right| \\ &= |Hf_k(y_k)| > k \end{aligned}$$

for all $k \in \mathbf{N}$. This implies the contradictory result that $H\left(\sum_{m \in \mathbf{N}} f_m\right)$ is unbounded. Thus, $h(\mathbf{C}Y_c)$ is finite.

(d) By 3.11(b,f) the support map h of H is a homeomorphism. By (a) and (c), if $\mathbf{C}Y_c \neq \emptyset$, both $\mathbf{C}Y_c$ and $h(\mathbf{C}Y_c)$ are finite sets of isolated points. Thus $\{h(y)\}$ is clopen for any $y) \in \mathbf{C}Y_c$. For any $f \in C(X)$, $f = f(h(y))\mathbf{1}$ on $\{h(y)\}$. By Theorem 3.11(a) it follows that $Hf(y) = H(f(h(y))\mathbf{1})(y)$. Since H is $\mathbf{1}$-bounded, there exists $D > 0$ such that

$$\begin{aligned} |Hf(y)| &= |H(f(h(y))\mathbf{1})(y)| \\ &\leq \|H(\mathbf{1}f(h(y)))\| \\ &\leq |f(h(y))|\, D\, \|H\mathbf{1}\| \end{aligned}$$

This implies that $y\hat{} \circ H$ is continuous which is contradictory. Thus $Y_c = Y$. Continuity of H follows from (b). □

Theorem 3.16. *Let H be a strongly pointwise subadditive, bijective, biseparating map and let H^{-1} be strongly pointwise subadditive. Let $\{P_y : y \in Y\}$ be a bounded set of scalars satisfying inequality (1) of Def. 3.10. If H is continuous at 0, then H is a continuous pseudocomposition map.*

Proof. By Theorem 3.13(b), H is strongly subadditive and

$$\|Hf - Hg\| \leq \sup\{P_y : y \in Y\}\, \|H(f-g)\|$$

for all $f, g \in C(X)$. By hypothesis, if (f_s) is a net from $C(X)$ such that $f_s \to 0$, then $Hf_s \to 0$. Therefore, if $f_s \to f \in C(X)$, then $H(f - f_s) \to 0$. As H is strongly subadditive, by Theorem 3.13(b),

$$\|Hf - Hf_s\| \leq \sup\{P_y : y \in Y\}\, \|H(f - f_s)\| \to 0$$

and H is continuous. Consequently $Y_c = Y$. As a consequence of Theorem 3.14(a), H is a pseudocomposition map. □

Separating linear continuous maps H must be **1**-bounded. Theorem 3.16 proves continuity of certain H that need not H be **1**- bounded**.** There exist continuous separating subadditive maps which are not **1**-bounded (see Example 4.1).

4. Examples

None of the examples of operators H given in this section are additive. In Example 4.1, H is not surjective. In Example 4.2, H is surjective when the map w is surjective.

Example 4.1. *An example of a pseudocomposition* $H : C(X) \to C(Y)$ *which is*

(1) *biseparating, injective;*
(2) *pointwise subadditive;*
(3) *neither surjective nor pointwise strongly subadditive;*
(4) *not norm bounded.*
(5) *The operators* $y\hat{} \circ H$ *for all* $y \in H$ *are continuous, and* H *is continuous.*

Proof. Let g map Y homeomorphically onto X. Let K be a discrete, non-Archimedean valued field. Let $a \in K$, $|a| > 1$, generate the value group $|K^*|$ of nonzero elements of K. Define $H : C(X) \to C(Y)$ as follows: If $f(g(y)) = 0$, let $Hf(y) = 0$; if $f(g(y)) \neq 0$, then $|f(g(y))| = |a|^n$ for some $n \in \mathbf{Z}$; in this case define $Hf(y) = a^n f(g(y))$. Since $Hf(y) = a^n f(g(y)) = H[f(g(y))\mathbf{1}](y)$, H is a pseudocomposition map. The continuity of H and $y\hat{} \circ H$ for all $y \in Y$ follows from the fact that $|Hf(y)| = |f(g(y))|^2$. The fact that H is ultrapointwise subadditive follows from the ultrametric inequality:

$$|H[f+\phi](y)| = |f(g(y)) + \phi(g(y))|^2$$

$$\begin{aligned} &\leq \max\left\{|f(g(y))|^2, |\phi(g(y))|^2, |2f(g(y))\phi(g(y))|\right\} \\ &= \max\left\{|f(g(y))|^2, |\phi(g(y))|^2\right\} \ (|2| \leq 1) \end{aligned}$$

for all $f, \phi \in C(X)$ and $y \in Y$.

To prove that H is not pointwise strongly subadditive, we need only observe that, for any $n \in \mathbf{Z}$ and $k < n$ there are scalars (and therefore function values) $f(g(y))$ and $\phi(g(y))$ where $|f(g(y))| = |\phi(g(y))| = |a|^n$ and $|f(g(y)) - \phi(g(y))| = |a|^k$. Therefore

$$\frac{|Hf(y) - H\phi(y)|}{|H(f-\phi)(y)|} = \frac{|a|^{n+k}}{|a|^{2k}} = |a|^{n-k}$$

can be arbitrarily large.

Since $\|Hf\| = \|f\|^2$, for any $c \in K$, $\|H(c\mathbf{1})\| = |c|^2$ which implies that H is not $\mathbf{1}$-bounded. H is not bijective because the nonzero values of $|Hf(y)|$ are even powers of $|a|$. Because of its form, H is not additive. Clearly, the support map of H is g. □

It is straightforward to verify the following result.

Example 4.2. *Let $g : Y \to X$ be a surjective homeomorphism. By Prop. 2.13, we may choose a continuous, strongly subadditive bijective isometry $w : K \to K$. Then $H : C(X) \to C(Y)$, $f \mapsto w \circ f \circ g$, is*

(1) *a bijective 0-isometry;*

(2) *biseparating;*

(3) *pointwise subadditive, and pointwise pointwise strongly subadditive;*

(4) *a pseudocomposition map;*

(5) ***1**-bounded, and norm-bounded.*

We use Example 4.3 in Example 4.4.

Example 4.3. *Let K be a complete, discretely valued field. There exist uncountably many bijective, valuation preserving (therefore subadditive by Prop. 2.9), discontinuous maps $w : K \to K$ which are not strongly subadditive.*

Proof. Let $a \in K$ be such that $|a| = r < 1$ generates the value group $|K^*|$ of K. Let $V = \{a \in K : |a| \leq 1\}$ denote the valuation ring of K, $P = \{a \in K : |a| < 1\} \subset V$ the maximal ideal of nonunits of V. Consider a subset $C = \{c_s : s \in S\}$ of V such that the residue class field $V/P = \{\overline{c_s} : s \in S\}$ and, for distinct $s, t \in S$, $\overline{c_s} \neq \overline{c_t}$. Every $b \in K$ can be written in the form $b = \sum_{i \geq N} c_i a^i$ for unique $c_i \in C$, for some integer N (see [B], p. 35). From this we deduce that the cardinality of K is at least equal to that of the continuum. Let c and d be distinct nonzero elements of C. For each $n \in \mathbf{N}$ let

$$x_n = \sum_{i=0}^{\infty} e_i a^i \quad e_i = \begin{cases} c, & i \leq n \\ 0, & i > n \end{cases}$$

$$y_n = \sum_{i=0}^{\infty} f_i a^i \quad f_i = \begin{cases} c, & i \leq n \\ d, & i > n \end{cases}$$

$$z_n = \sum_{i=0}^{\infty} g_i a^i \quad g_i = \begin{cases} d, & i \leq n \\ 0, & i > n \end{cases}$$

We define w as follows. Let $w(x_n) = z_n$, $w(z_n) = y_n$, $w(y_n) = x_n$, and $w(x_n - y_n) = x_n - y_n$. Since K is not denumerable, we may choose $a, b \in K$ distinct from $x_n, y_n, z_n, x_n - y_n$ for every n such that $0 < |a| = |b| < 1$. Let $w(0) = 0$, $w(a) = b$, $w(b) = a$. For all other $x \in K$, let $w(x) = x$. It is clear that w is bijective and preserves the valuation. Therefore w is subadditive. It is also clear that $w(x_n - y_n) = x_n - y_n \to 0$ but $|w(x_n) - w(y_n)| = 1$ for all $n \in \mathbf{N}$. Hence w is not strongly subadditive.

We observe that $x_n = \sum_{i=0}^{n} ca^i \to \sum_{i=0}^{\infty} ca^i$ and that $y_n = \sum_{i=0}^{n} ca^i + \sum_{i=n+1}^{\infty} da^i \to \sum_{i=0}^{\infty} ca^i$ but $w(x_n) = z_n = \sum_{i=0}^{n} da^i \to \sum_{i=0}^{\infty} da^i$ and $w(y_n) = x_n \to \sum_{i=0}^{\infty} ca^i$. Therefore w is discontinuous.

As there are uncountably many distinct pairs a, b which can be selected for the definition of w, there are uncountably many maps w of this type. □

In the final example, we show that H may satisfy most of the properties demanded in in Theorem 3.15, but be discontinuous when not pointwise strong subadditive.

Example 4.4. *Let x_0 be an isolated point of X. Let $H : C(X) \to C(X)$. For $f \in C(X)$ define $Hf(x) = f(x)$ if $x \neq x_0$, $Hf(x_0) = w(f(x_0))$ where w is any map of Example 4.3. H is clearly bijective and biseparating. Since the support map $h : X \to X$ is the identity map $x \mapsto x$, $Hf(x) = H(f(x)\mathbf{1})(x)$. Because of the properties of w, it follows that $|Hf(x)| = |f(x)|$ for all $f \in C(X)$ and all $x \in X$. Thus $\|Hf\| = \|f\|$ for all $f \in C(X)$. Since w is not strongly subadditive, it is clear that H is not pointwise strongly subadditive: There does not exist $k > 0$ such that when for all $f, g \in C(X)$ with $f(x_0)$ close to $f(x_0)$, $|Hf(x_0) - Hg(x_0)| \leq k|H(f-g)(x_0)|$. Since w is discontinuous, H is discontinuous.*

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