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COUNTABLY GENERATED INTERMEDIATE ALGEBRAS BETWEEN $C^*(X)$ AND $C(X)$

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Abstract

Let X be a completely regular space. We study those intermediate algebras between $C^*(X)$ and $C(X)$ that are obtained by adjoining to $C^*(X)$ a countable family of unbounded functions in $C(X)$. We shall call them countably generated intermediate algebras, and they are exactly the intermediate algebras that have a countable cofinal subset consisting of unbounded functions. They are far away from being isomorphic to any $C(T)$. In fact we show that no countably generated intermediate algebra is closed under composition with functions in $C(\mathbb{R})$. We also examine a classical intermediate algebra of real sequences, previously studied by R.M. Brooks and D. Plank, in order to show that it is not countably generated over $C^*(\mathbb{N})$.

Introduction

Let $C(X)$ be the algebra of all real-valued continuous functions on a nonempty completely regular space X, and $C^*(X)$ the subalgebra of bounded functions. This paper deals with subalgebras of $C(X)$ containing $C^*(X)$. We shall refer to them

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as *intermediate algebras* on X. They are sublattices of $C(X)$ and so they are Φ-algebras in the sense of Henriksen-Johnson. They have been studied from different points of view by several authors (see [DG]). The intermediate algebras that are isomorphic to some $C(T)$ will be called C-*type* intermediate algebras. It has been shown in [DGM] that any intermediate algebra on X is the ring of fractions of $C^*(X)$ with respect to a multiplicatively closed subset. As rings of fractions of $C^*(X)$ the intermediate algebras inherit some algebraic properties from $C^*(X)$ but, in general, they are neither C-type nor even closed under composition with functions in $C(\mathbb{R})$.

If one takes an intermediate space Y between X and βX, the restriction morphism from $C(Y)$ to $C(X)$, which sends $g \in C(Y)$ to $g|_X$, is clearly injective. We shall always see $C(Y)$ as an intermediate algebra on X. Let us now consider a family (Y_i) of intermediate spaces between X and βX, and assume that it is a directed family under inverse inclusion. Probably the intermediate algebra $\cup C(Y_i)$ is not C-type, but it should always be closed under composition with functions in $C(\mathbb{R})$. On the other hand, from an algebraic point of view, an easy way to get an intermediate algebra is to adjoin to $C^*(X)$ a family of functions $F \subseteq C(X)$, so that we get the smallest intermediate algebra containing F, which is just the set $C^*(X)[F]$ of all polynomial expressions in the members of F with coefficients in $C^*(X)$. Any intermediate algebra on X is $C^*(X)[F]$ for some convenient F. In this work we restrict our attention to the case of a countable family F of unbounded functions. We shall say that the intermediate algebras so obtained are *countably generated* over $C^*(X)$. We shall prove that they are never closed under composition with functions in $C(\mathbb{R})$, and so they are radically different from those previously obtained.

We conclude by examining the following classical intermediate algebra of real sequences previously studied by Brooks and Plank in [B] and [P] respectively.

Let H denote the intermediate algebra on $\mathbb{N}$ consisting of those functions $f \in C(\mathbb{N})$ such that

$$\limsup_{n\to\infty} \sqrt[n]{|f(n)|} \leq 1.$$

The algebra H is not closed under composition with functions in $C(\mathbb{R})$, since the function $f(n) = n$ is in H, but the composition $\exp \circ f = e^f$ does not belong to H. We show that H is not countably generated over $C^*(\mathbb{N})$.

1. Preliminaries

Concerning rings of continuous functions we shall basically adhere to the notation and terminology in [GJ].

A subalgebra A of $C(X)$ is said to be *absolutely convex* if, whenever $|f| \leq |g|$, with $f \in C(X)$ and $g \in A$, then $f \in A$.

Proposition 1.1. *If A is an intermediate algebra on X, then A is an absolutely convex subalgebra of $C(X)$ and so a sublattice of $C(X)$.*

Proof. Let us repeat the short argument in [DGM, 3.3]. If $|f| \leq |g|$, with $f \in C(X)$ and $g \in A$, then $f(1+g^2)^{-1}$ is in $C^*(X) \subseteq A$, and so $f = (1+g^2)f(1+g^2)^{-1}$ is in A. □

Next we shall examine the more simple intermediate algebras containing unbounded functions.

Definition 1.2. Singly Generated Intermediate Algebras Let $f \in C(X)$. We shall denote by $C^*(X)[f]$ the smallest intermediate algebra containing f, that is,

$$C^*(X)[f] = \{\sum_{i=0}^{n} g_i f^i : g_i \in C^*(X), n = 0, 1, 2, \dots\}.$$

If $f \notin C^*(X)$ we shall say that $C^*(X)[f]$ is *singly generated* over $C^*(X)$.

In [DGM, 3.4] it has been shown:

(a) Let c be a real number, $c > 1$. Every singly generated intermediate algebra on X is $C^*(X)[f]$ for some $f \geq c$.

(b) If $f \geq c > 1$ for some $c \in \mathbb{R}$, then
$$C^*(X)[f] = \{g \in C(X) : |g| \leq f^k \text{ for some } k \in \mathbb{N}\}.$$

(c) Every finitely generated intermediate algebra is singly generated. Explicitly,
$$C^*(X)[f_1, \ldots, f_n] = C^*(X)[|f_1| + \ldots + |f_n|], \text{ for any } n \in \mathbb{N}.$$

Now, taking into account that $\lim_{r \to \infty} e^r/r^n = \infty$, one may easily conclude that, for any nonnegative unbounded function $f \in C(X)$, the composition $\exp \circ f = e^f$ does not belong to $C^*(X)[f]$. Hence, any singly generated intermediate algebra is a proper intermediate algebra that is not closed under composition with functions in $C(\mathbb{R})$, and so it is not isomorphic to any $C(T)$.

2. Countably Generated Intermediate Algebras

We shall say that an intermediate algebra A is *countably generated* over $C^*(X)$ if A may be obtained by adjoining to $C^*(X)$ a countable family $\{f_1, f_2, \ldots\}$ of unbounded functions in $C(X)$. The functions f_n will be called the *generators* for A over $C^*(X)$.

We shall denote by $C^*(X)[f_1, f_2, \ldots]$ the smallest intermediate algebra containing $\{f_1, f_2, \ldots\}$, and we shall say that $\{f_1, f_2, \ldots\}$ is a *well-chosen* set of generators for $C^*(X)[f_1, f_2, \ldots]$ if $1 < c \leq f_1 \leq f_2 \leq \ldots$, for some real number $c \geq 1$. From any countable set of generators $\{f_1, f_2, \ldots\}$ we may always obtain a well-chosen one: Take any real number $c \geq 1$, and replace f_n by $g_n = c + |f_1| + \ldots + |f_n|$. As any intermediate algebra is an absolutely convex subalgebra of $C(X)$, $C^*(X)[f_1, f_2, \ldots] = C^*(X)[g_1, g_2, \ldots]$.

Any singly generated intermediate algebra is indeed a countably generated one, and we have seen in 1.2 (c) that every finitely generated intermediate algebra is singly generated. We shall see next that, for any nonpseudocompact X, there exist countably generated intermediate algebras on X that are not singly generated.

Example 2.1. Let X be a nonpseudocompact space. Take an unbounded function $f_1 \in C(X)$ such that $f_1 \geq c > 1$, for some $c \in \mathbb{R}$. Set inductively $f_{n+1} = e^{f_n}$.

Then $\{f_1, f_2, \ldots\}$ is a well-chosen set of generators for $C^*(X)[f_1, f_2, \ldots] = \cup_{n=1}^{\infty} C^*(X)[f_n]$. If $C^*(X)[f_1, f_2, \ldots]$ were singly generated over $C^*(X)$, then its generator would belong to some $C^*(X)[f_n]$, and therefore $C^*(X)[f_1, f_2, \ldots] \subseteq C^*(X)[f_n]$. That is impossible, since $f_{n+1} = e^{f_n} \notin C^*(X)[f_n]$.

Lemma 2.2. *Assume that* $\{f_1, f_2, \ldots\}$ *is a well-chosen set of generators for* $C^*(X)[f_1, f_2, \ldots]$*. Then*

$$C^*(X)[f_1, f_2, \ldots] = \{g \in C(X) : |g| \leq f_n^k \text{ for some } n, k \in \mathbb{N}\}.$$

Proof. It follows from 1.2 (b), taking into account that $C^*(X)[f_1, f_2, \ldots] = \cup_{n=1}^{\infty} C^*(X)[f_n]$. □

Now it is clear that each countably generated intermediate algebra has a countable cofinal subset, namely, the set $\{f_n^k : n, k \in \mathbb{N}\}$, for any well-chosen set of generators $\{f_1, f_2, \ldots\}$. We shall see next that the converse also holds.

Proposition 2.3. *An intermediate algebra on* X *is countably generated over* $C^*(X)$ *if and only if it has a countable cofinal subset consisting of unbounded functions.*

Proof. Let A be an intermediate algebra with a countable cofinal subset consisting of unbounded functions, say $\{f_1, f_2, \ldots\}$. Clearly $A \supseteq C^*(X)[f_1, f_2, \ldots]$. Let us prove the reverse inclusion. By cofinality of the set $\{f_1, f_2, \ldots\}$, for any $f \in A$, there exist $i, j \in \mathbb{N}$ such that $f \leq f_i$, $-f \leq f_j$, and there exists $n \in \mathbb{N}$ such that $f_i \vee f_j \leq f_n$, whence $|f| \leq f_n$. Thus, $f \in C^*(X)[f_1, f_2, \ldots]$. □

Notice that if we start with a countable cofinal subset $\{f_1, f_2, \ldots\}$ of an intermediate algebra A, and we replace f_n by $g_n = c + |f_1| + \ldots + |f_n|$, for some real number $c > 1$, then we get another cofinal subset $\{g_1, g_2, \ldots\}$, which is also a well-chosen set of generators for A.

Definition 2.4. Let A be an intermediate algebra. It is said that A is *closed under (finite) composition* if whenever $g \in C(\mathbb{R}^n)$ and $f_1, \ldots, f_n \in A$, the composition $g \circ (f_1, \ldots, f_n)$ is in A, for any $n \in \mathbb{N}$. Similarly, A is said to be *closed under countable composition* if whenever $g \in C(\mathbb{R}^{\mathbb{N}})$ and (f_n) is a sequence from A, the composition $g \circ (f_1, f_2, \ldots)$ is in A. We say that A is a *C-type* intermediate algebra if it is isomorphic as a ring (or, equivalently, as an algebra) to some $C(T)$.

Each one of the above properties is weaker than the next one, and they are all algebraic properties (see [I] and [HIJ]).

One can see in [DGM, 4.9] that an intermediate algebra A is closed under finite composition if and only if it is *closed under composition with functions in* $C(\mathbb{R})$, i.e., $g \circ f$ is in A whenever $f \in A$ and $g \in C(\mathbb{R})$.

Let A be a countably generated intermediate algebra, $\{f_n : n \in \mathbb{N}\}$ a cofinal subset of A, and M an hyper-real maximal ideal of A. Professor M. Henriksen pointed out to us that in this situation the set of residual classes $\{M(f_n) : n \in \mathbb{N}\}$ is a countable cofinal subset in the quotient field A/M, and so, according to [GJ, 14.15], A can not be a C-type intermediate algebra. It follows from [HIJ, 2.7] that A is not closed under countable composition either. In short:

Proposition 2.5 [HIJ, 2.7] *Any countably generated intermediate algebra is neither C-type nor closed under countable composition.*

An elementary proof of 2.5, similar to the proof of 2.6 below, can be seen in [G].

The existence of a countable cofinal subset in the quotient of an intermediate algebra A by an hyper-real maximal ideal does not allow us to assure that A is not closed under composition (see [HIJ, 1.9]).

Theorem 2.6. *No countably generated intermediate algebra is closed under composition.*

Proof. Let $C^*(X)[f_1, f_2, \ldots]$ be a countably generated intermediate algebra. We may assume that $\{f_1, f_2, \ldots\}$ is a well-chosen set of generators that is also a cofinal subset consisting of unbounded functions. Let x_1 be any point in X. Then $f_1(x_1) \geq c > 1$, for some $c \in \mathbb{R}$. Inductively, for $n \geq 1$, take $x_{n+1} \in X$ such that $f_1(x_{n+1}) \geq f_1(x_n) + 1$. The set $D = \{f_1(x_1), f_1(x_2), \ldots\}$ is a C-embedded countable discrete subset of $\mathbb{R}$. Let g be the continuous function on D defined by $g(f_1(x_n)) = e^{f_n(x_n)}$, and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous extension of g. Then $(h \circ f_1)(x_n) = e^{f_n(x_n)} > f_n(x_n)$, for any $n \in \mathbb{N}$. So that, no f_n is an upper bound for $h \circ f_1$. By the cofinality of the set $\{f_1, f_2, \ldots\}$, $h \circ f_1 \notin C^*(X)[f_1, f_2, \ldots]$. □

We shall conclude by examining an intermediate algebra of real sequences previously studied by Brooks and Plank (see [B] and [P]).

Let H denote the intermediate algebra on $\mathbb{N}$ consisting of those sequences $(x_n) \in C(\mathbb{N})$ such that

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} \leq 1.$$

We have already seen in the introduction that H is not closed under composition. We are just going to show that H is not singly generated over $C^*(\mathbb{N})$, later we shall prove that it is not countably generated either.

We shall make use of the following elementary result: Let (x_n) be a sequence of real numbers such that $x_n \geq 1$, for any $n \in \mathbb{N}$. The sequence (x_n) is in H if and only if $\lim_{n\to\infty} \ln x_n/n = 0$.

Suppose that H is generated over $C^*(\mathbb{N})$ by a single sequence (x_n), and assume that $(x_n) \geq c > 1$, for some $c \in \mathbb{R}$. We define

$$y_n = x_n^{\frac{1}{\sqrt{\ln x_n/n}}}.$$

Firstly, observe that $y_n \geq 1$, since $\ln y_n \geq 0$. Secondly, notice that

$$\frac{1}{n}\ln y_n = \frac{1}{n}\frac{1}{\sqrt{\frac{1}{n}\ln x_n}}\ln x_n = \frac{\frac{1}{n}\ln x_n}{\sqrt{\frac{1}{n}\ln x_n}} = \sqrt{\frac{1}{n}\ln x_n}.$$

Hence, $\lim_{n\to\infty} \ln y_n/n = 0$, and so (y_n) is in $H = C^*(\mathbb{N})[(x_n)]$. Finally, by 1.2 (b), there exists $k \in \mathbb{N}$ such that

$$(y_n) = (x_n^{\frac{1}{\sqrt{\ln x_n/n}}}) \leq (x_n^k).$$

Thus,

$$\frac{1}{\sqrt{\ln x_n/n}} \leq k, \text{ for every } n \in \mathbb{N}.$$

This is, of course, a contradiction, since

$$\lim_{n\to\infty} \frac{1}{\sqrt{\ln x_n/n}} = \infty.$$

Proposition 2.7. *The Brooks-Plank subalgebra H is not countably generated over $C^*(\mathbb{N})$.*

Proof. We are going to see that any subalgebra of H that is countably generated over $C^*(\mathbb{N})$ is properly contained in H. Let A be such a subalgebra of H. We may assume that $\{(x_{n,1}), (x_{n,2}), \ldots\}$ is a well-chosen set of generators that is also a cofinal set of A. Set $y_{n,k} = \ln x_{n,k}/n$, and observe that $y_{n,k} \leq y_{n,k+1}$, for every $n \in \mathbb{N}$. Moreover, for every $k \in \mathbb{N}$, $\lim_{n\to\infty} y_{n,k} = 0$. Hence, for every $k \in \mathbb{N}$ and every $\varepsilon > 0$, there exists $n_k \in \mathbb{N}$ such that $y_{n,k} < \varepsilon$ whenever $n_k \leq n$. Accordingly, for $\varepsilon = 1$, there exists $n_1 \in \mathbb{N}$ such that $y_{n,1} < 1$ whenever $n_1 \leq n$. For $\varepsilon = 1/2$, there exists $n_2 \in \mathbb{N}$, $n_2 > n_1$ such that $y_{n,2} < 1/2$ whenever $n_2 \leq n$. Inductively, for every $p \in \mathbb{N}$, $p \geq 2$, there exists $n_p \in \mathbb{N}$, $n_p > n_{p-1}$, such that $y_{n,p} < 1/p$ whenever $n_p \leq n$. Observe that $n_p \geq p$, for all $p \in \mathbb{N}$. Now we define the sequence

$$z_n = y_{n,p} + \frac{1}{n_p}, \text{ for } n_p \leq n < n_{p+1}.$$

We have

$$0 \le \lim_{n\to\infty} z_n = \lim_{p\to\infty}(y_{n,p} + \frac{1}{n_p}) \le \lim_{p\to\infty}(\frac{1}{p} + \frac{1}{n_p}) = 0.$$

Set $a_n = e^{nz_n}$, and observe that $(a_n) \in H$, since $\lim_{n\to\infty} \ln a_n/n = \lim_{n\to\infty} z_n = 0$. Finally, let us consider the subsequence (a_{n_p}), and observe that

$$a_{n_p} = e^{n_p z_{n_p}} = e^{n_p(y_{n_p,p}+1/n_p)} = ee^{n_p y_{n_p,p}} = ex_{n_p,p} > x_{n_p,p}.$$

Therefore, the sequence (a_n) is not in

$$C^*(\mathbb{N})[(x_{n,1}), (x_{n,2}), \ldots]. \qquad \square$$

Remark 2.8. The smallest closed under composition intermediate algebra containing a countable family $\{f_1, f_2, \ldots\} \subseteq C(X)$ is the union

$$\bigcup_{n=1}^{\infty} C(\bigcap_{i=1}^{n} \upsilon_{f_i} X).$$

One can see, as a consequence of [HIJ, 2.4], that the smallest closed under countable composition intermediate algebra containing $\{f_1, f_2, \ldots\}$ is precisely

$$C(\bigcap_{n=1}^{\infty} \upsilon_{f_n} X).$$

In general, these two algebras do not agree (see [HIJ, 1.9]).

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