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ON TWO GENERALIZATIONS OF PSEUDOCOMPACTNESS

Salvador Garcia-Ferreira

Abstract

The two generalizations of pseudocompactness considered in this paper are α-pseudocompactness, where α is an infinite cardinal, and p-pseudocompactness, where p is an uniform ultrafilter on an infinite cardinal. Let $\leq_{RK}$ be the Rudin-Keisler order on $\beta_(\alpha) - \alpha = \alpha^*$ and let $P_{RK}(p) = \{q \in \alpha^* : q \leq_{RK} p\}$. We prove that if $p \in U(\alpha)$, $\alpha^{<\alpha} = \alpha$, and $P_{RK}(p)$ is α-pseudocompact, then p is decomposable. Under **GCH**, if $p \in U(\alpha)$ and $P_{RK}(p)$ is α-pseudocompact, then p is decomposable. It is also shown that, for every cardinal α, there is $p \in U(\alpha)$ such that $P_{RK}(p)$ is α-pseudocompact. Assuming a set-theoretic axiom, we may find $p \in U(\omega_1)$ such that $P_{RK}(p)$ is not ω_1-pseudocompact. For $p \in U(\omega_1)$, we let $\beta_p(\omega)$ be the p-compact reflexion of the discrete space ω. We proved that $2^{\omega_1} < 2^{2^\omega}$ if and only if $\beta_p(\omega) \neq \beta(\omega)$, for every $p \in U(\omega_1)$; and **CH** implies that $\beta_p(\omega) = \beta(\omega)$ for every $p \in U(\omega_1)$. Now, for $p \in U(\omega_1)$ and $q \in \omega^*$, we put $\Gamma_{p,q} = \beta_p(\omega) - (\{q\} \cup \omega)$. These spaces satisfy the following properties: 1. if $2^{\omega_1} < 2^{2^\omega}$, then for every $p \in U(\omega_1)$ there is $q \in \omega^*$ such that $\Gamma_{p,q}$ is p-pseudocompact; 2. **CH** implies that for every $q \in \omega^*$, $\Gamma_{p,q}$ is not p-pseudocompact for

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all $p \in U(\omega_1)$; 3. if $q \in \omega^*$ is a P_{ω_2}-point, then $\Gamma_{p,q}$ is p-pseudocompact for every $p \in U(\omega_1)$; and 4. if $p \in U(\omega_1)$, then $\Gamma_{p,q}$ is p-pseudocompact for every $q \in \omega^*$ with $\pi\chi(q, \omega^*) > \omega_1$. In the last Section, we give several conditions that are equivalent to the statement "every p-pseudocompact space is pseudocompact", and some necessary and sufficient conditions are listed to guarantee that every p-compact space is α-pseudocompact, for $p \in U(\alpha)$.

0. Introduction and Preliminaries

All spaces are assumed to be Tychonoff. For a space X, $\mathcal{Z}(X)$ will denote the set of zero-sets of X. If $f : X \to Y$ is a continuous function, then $\overline{f} : \beta(X) \to \beta(Y)$ will denote the Stone-Čech extension of f. The Greek letters α, γ and κ will stand for infinite cardinal numbers. If X is a set, then $[X]^\alpha = \{A \subseteq X : |A| = \alpha\}$ and the definition of $[X]^{<\alpha}$, $[X]^{\leq\alpha}$ and $[X]^{\geq\alpha}$ should be clear. For a cardinal α, a subset G of a space X is called a G_α-*set* if G is the intersection of α-many open subsets of X (the G_ω-sets are usually called G_δ-sets). We say that D is G_α-*dense* in X if every non-empty G_α-set of X intersects D. The Stone-Čech compactification of the discrete space of size α will be denoted by $\beta(\alpha)$ and its remainder by $\alpha^* = \beta(\alpha) - \alpha$. For $A \subseteq \alpha$, $\hat{A} = \{p \in \beta(\alpha) : A \in p\} = \mathrm{cl}_{\beta(\alpha)} A$ and $A^* = \hat{A} \cap \alpha^*$. The set of uniform ultrafilters on α is $U(\alpha) = \{p \in \alpha^* : \forall A \in p(|A| = \alpha)\}$.

A natural generalization of pseudocompactness is the following.

Definition 0.1. [**Ke**] *A space X is said to be α-pseudocompact if $f[X]$ is a compact subset of $\mathbf{R}^\alpha$ for every $f \in C(X, \mathbf{R}^\alpha)$.*

If α is a cardinal with $cf(\alpha) > \omega$, then $[0, \alpha)$ is γ-pseudocompact for every $\omega \leq \gamma < cf(\alpha)$ and cannot be $cf(\alpha)$-pseudocompact (for details see [GST]). In the next Theorem, we state several useful conditions that are equivalent to α-pseudocompactness (a proof is available in [GST]). The equivalence (1) $\Leftrightarrow$ (2) was established in [Ke] and the equivalences (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) are taken from [Re].

Theorem 0.2. *For a space X, the following are equivalent.*

1: *X is α-pseudocompact;*

2: *X is G_α-dense in $\beta(X)$;*

3: *X is G_α-dense in $K(X)$ for every compactification $K(X)$ of X;*

4: *every cover of X of cardinality $\leq \alpha$ consisting of cozero sets of X has a finite subcover;*

5: *if $\{Z_\xi : \xi < \alpha\} \subseteq \mathcal{Z}(X)$ and $\bigcap_{\xi \in I} Z_\xi \neq \emptyset$ for every finite subset I of α, then $\bigcap_{\xi<\alpha} Z_\xi \neq \emptyset$;*

To state our second generalization of pseudocompactness we need the following notion.

Definition 0.3. **[GS]** *Let $p \in \alpha^*$ and $(S_\xi)_{\xi<\alpha}$ an α-sequence of non-empty subsets of a space X. A point $x \in X$ is a p-limit point of the α-sequence $(S_\xi)_{\xi<\alpha}$ if for every neighborhood V of x, $\{\xi < \alpha : V \cap S_\xi \neq \emptyset\} \in p$.*

If $x_n \in X$ and $S_n = \{x_n\}$ for each $n < \omega$, then a p-limit point, for $p \in \omega^*$, of $(S_n)_{n<\omega}$ is the original Bernstein's concept of p-limit introduced in [Be]. If $(S_\xi)_{\xi<\alpha}$ is an α-sequence of nonempty subsets of a space X and $p \in \alpha^*$, then the set of all p-limit points of $(S_\xi)_{\xi<\alpha}$ will be denoted by $L(p, (S_\xi)_{\xi<\alpha})$. If $(x_\xi)_{\xi<\alpha}$ is an α-sequence of points of a space X and $x \in L(p, (x_\xi)_{\xi<\alpha})$, then we simply write $x = p\text{-}\lim\ x_\xi$, for $p \in \alpha^*$, since in this case the p-limit point is unique.

Example 0.4. *Let $p \in U(\alpha)$ and let $\{A_\xi : \xi < \alpha\}$ be a partition of α into sets of cardinality α such that $A_\xi \notin p$ for every $\xi < \alpha$. Then, $|L(p, (\hat{A}_\xi)_{\xi<\alpha})| \geq \alpha$, where the p-limit is taken inside of $\beta(\alpha)$. Indeed, enumerate each A_ξ as $\{\theta(\xi,\zeta) : \zeta < \alpha\}$. For every $\zeta < \alpha$, we define $f_\zeta : \alpha \to \alpha$ by $f_\zeta(\xi) = \theta(\xi,\zeta)$. If $\overline{f_\zeta}(p) = q_\zeta$, for $\zeta < \alpha$, then $q_\zeta \in L(p, (\hat{A}_\xi)_{\xi<\alpha})$ and $q_\zeta \neq q_\eta$ whenever $\zeta < \eta < \alpha$.*

Lemma 0.5. *Let X be a space. For every $x \in X$ there are a family $\{V_\xi : \xi < \alpha\}$ of open subsets of X and $p \in U(\alpha)$ such that $\{x\} = L(p, (V_\xi)_{\xi<\alpha})$, where $\alpha = \pi\chi(x, X)$.*

Proof. Fix $x \in X$. Let $\{V_\xi : \xi < \alpha\}$ be a π-base at x. For every open neighborhood V of x, we put $S(V) = \{\xi < \alpha : V_\xi \subseteq V\}$. It is clear that $|S(V)| = \alpha \geq \omega$ and $S(V) \cap S(U) = S(V \cap U)$ for every $V, U \in \mathcal{N}(x)$. Hence, we may find $p \in U(\alpha)$ such that $S(V) \in p$ for every $V \in \mathcal{N}(x)$. It is clear that $x \in L(p, (V_\xi)_{\xi<\alpha})$. We claim that $\{x\} = L(p, (V_\xi)_{\xi<\alpha})$. Indeed, suppose that $y \in L(p, (V_\xi)_{\xi<\alpha})$ and $x \neq y$. Let V and U be two disjoint open sets such that $x \in V$ and $y \in U$. Then, $\{\xi < \alpha : U \cap V_\xi \neq \emptyset\} \cap S(V) \in p$. If $\zeta \in \{\xi < \alpha : U \cap V_\xi \neq \emptyset\} \cap S(V)$, then $\emptyset \neq U \cap V_\zeta \subseteq U \cap V$, which is a contradiction. □

It should be remarked that if $p \in \alpha^*$ and $(V_\xi)_{\xi<\alpha}$ is an α-sequence of nonempty subsets of a space X, then

$$L(p, (V_\xi)_{\xi<\alpha}) = \bigcap_{A \in p} cl_X(\bigcup_{\xi \in A} V_\xi).$$

Hence, it is evident that any α-sequence of nonempty subsets of a compact space has a p-limit point for every $p \in \alpha^*$.

The following definition is a generalization of pseudocompactness in the context of p-limit points (The particular case $\alpha = \omega$ was considered in [GS]):

Definition 0.6. *Let $p \in U(\alpha)$. A space X is called p-pseudocompact if for every α-sequence $(V_\xi)_{\xi<\alpha}$ of nonempty open subsets of X,*

$$L(p, (V_\xi)_{\xi<\alpha}) \neq \emptyset.$$

We know that a space is pseudocompact iff for every sequence $(V_n)_{n<\omega}$ of nonempty open subsets of X there is $p \in \omega^*$ such that $L(p,(V_n)_{n<\omega}) \neq \emptyset$. Thus, every p-pseudocompact space, for $p \in \omega^*$, is pseudocompact. It is pointed out in [GS] that p-pseudocompactness is closed under arbitrary products, for each $p \in U(\alpha)$; hence, pseudocompactness does not necessarily imply p-pseudocompactness for some $p \in \omega^*$. We shall show in Theorem 2.4 that if α is not an Ulam-measurable cardinal, then p-pseudocompactness implies pseudocompactness for every $p \in U(\alpha)$.

The following property is very useful in the characterization of those spaces that all of whose powers are initially α-compact (see [Ca], [G1], [Li] and [Sa]). We recall that a space is called *initially α-compact* if every open cover of size $\leq \alpha$ has a finite subcover ([St] offers a good survey on initially α-compact spaces).

Definition 0.7. *Let $p \in U(\alpha)$. A space X is said to be p-compact if for every α-sequence of points $(x_\xi)_{\xi<\alpha}$ of X there is $x \in X$ such that $x = p - \lim_{\xi\to\alpha} x_\xi$.*

Alternatively, V. Saks [Sa] gave the following definition of p-compactness: For a $p \in U(\alpha)$, a space X is p-compact if $\overline{f}(p) \in X$ for every function $f : \alpha \to X$. It is evident that every p-compact space is p-pseudocompact, for each $p \in U(\alpha)$. For every $p \in \omega^*$, there is a p-pseudocompact space that is not p-compact (an example is available in [G4]). For every $p \in U(\alpha)$, p-compactness is preserved under arbitrary products, it is closed-hereditary, and every compact space is p-compact. It follows that every space X has a p-compact reflexion, for every $p \in U(\alpha)$, which is denoted by $\beta_p(X)$ and it satisfies the following:

1: X is dense in $\beta_p(X)$;

2: $\beta_p(X)$ is p-compact;

3: if $f : X \to Z$ is a continuous function and Z is p-compact, then $\overline{f}[\beta_p(X)] \subseteq Z$; and

4: the space $\beta_p(X)$ is the only one satisfying (1), (2) and (3).

We pointed out, in the paragraph after Definition 0.6, that the condition "every countable sequence of nonempty open subsets has a p-limit point in the space, for some $p \in \omega^*$" characterizes pseudocompactness. But, in any separable space every ω_1-sequence of nonempty open subsets has a p-limit point for some $p \in U(\omega_1)$. This means that we cannot get a characterization of ω_1-pseudocompactness, just by replacing "countable sequence" by "ω_1-sequence", similar to the one for pseudocompactness given above. A related result is the following which was suggested by A. Dow, in conversation with the author.

Theorem 0.8. [**Dow**] **CH** *is equivalent to the statement "there is not a locally countable ω_1-pseudocompact space".*

Proof. Necessity. Assume **CH**. Let X be a locally countable, ω_1-pseudocompact spaces. Since X is locally countable and Tychonoff, we must have that X is zero-dimensional. Hence, every point $x \in X$ has a countable clopen neighborhood V_x. Since V_x is pseudocompact and countable for every $x \in X$, V_x is countable, it is a compact space. Therefore, X is first-countable. Hence, if $B \subseteq X$ and $|B| \leq \omega_1$, then $|cl_X(B)| \leq \omega_1^\omega = \omega_1$. Now, suppose that for every $\mu < \theta < \omega_1$ we have defined an open subset U_ν of X such that

1: $|U_\mu| \leq \omega_1$ for every $\mu < \theta$; and

2: $cl_X(U_\mu) \subseteq U_\nu$ whenever $\mu < \nu < \theta$.

First, we suppose that $\theta = \mu + 1$. Since $|U_\mu| \leq \omega_1$, $|cl_X(U_\mu)| \leq \omega_1$. By the local countability of X, we may find an open subset U_θ of size ω_1 such that $cl_X(U_\mu) \subseteq U_\theta$. If θ is a limit ordinal, then we choose an open subset U_θ of size ω_1 such that $cl_X(\bigcup_{\mu<\theta} U_\mu) \subseteq U_\theta$. Let us consider the open set $U = \bigcup_{\theta<\omega_1} U_\theta$. If $x \in cl_X(U)$, then there is $\theta < \omega_1$ such that $x \in cl_X(U_\theta) \subseteq U_{\theta+1}$, this is possible since X is locally countable. Thus, U is also a closed subset of X. Then, U is an ω_1-pseudocompact space of size ω_1; hence, U has to be compact which is impossible since U is locally countable and of cardinality ω_1. Therefore, there is not a locally countable, ω_1-pseudocompact spaces.

Sufficiency. Suppose that $\omega_1 < 2^\omega$. S. Mrówka [Mr] showed that there is a Ψ-space X such that $|X| = 2^\omega$ and $\beta(X)$ is the one point compactification of X. By Corollary 2.12 of [GST], X is ω_1-pseudocompact, and X is locally countable, which contradicts our hypothesis. Therefore, **CH** holds. □

Question 0.9. *Is there an ω_1-pseudocompact space which has an ω_1-sequences of nonempty open subsets having no p-limit point for any $p \in U(\omega_1)$?.*

1. Subspaces of $\beta(\alpha)$

We start this section with some preliminary results.

Lemma 1.1. *Let $p, q \in \alpha^*$ and let $f : \alpha \to \alpha$ be a function such that $\overline{f}(q) = p$. If $(W_\xi)_{\xi<\alpha}$ is an α-sequence of nonempty subsets of a space X, then*

$$L(p, (W_\xi)_{\xi<\alpha}) = L(q, (U_\xi)_{\xi<\alpha})$$

where $U_\xi = W_{f(\xi)}$ for every $\xi < \alpha$.

Proof. Fix $x \in L(p, (W_\xi)_{\xi<\alpha})$ and let V be an open neighborhood of x. By definition, $A = \{\xi < \alpha : V \cap W_\xi \neq \emptyset\} \in p$ and $B = f^{-1}(A) \in q$. Hence, $V \cap W_{f(\xi)} \neq \emptyset$, for each $\xi \in B$, which implies that $B \subseteq \{\zeta < \alpha : V \cap U_\zeta \neq \emptyset\} \in q$. So, $x \in L(q, (U_\xi)_{\xi<\alpha})$. Therefore $L(p, (W_\xi)_{\xi<\alpha}) \subseteq L(q, (U_\xi)_{\xi<\alpha})$. Now, fix $x \in L(q, (U_\xi)_{\xi<\alpha})$. If V is an open neighborhood of x, then $A = \{\xi < \alpha : V \cap U_\xi \neq \emptyset\} \in q$, and $B = f(A) \in p$. If $\xi \in A$, then $V \cap U_\xi \neq \emptyset$; that is, $V \cap W_{f(\xi)} \neq \emptyset$. Thus, $B \subseteq \{\zeta < \alpha : V \cap W_\zeta \neq \emptyset\} \in p$. So, $x \in L(p, (W_\xi)_{\xi<\alpha})$. This shows that $L(q, (U_\xi)_{\xi<\alpha}) \subseteq L(p, (W_\xi)_{\xi<\alpha})$. □

The follwing pre-orderings of ultrafilters and notions will be useful in the study of p-pseudocompactness like properties.

a: *Rudin-Keisler order*: If $p, q \in \alpha^*$, then we say $q \leq_{RK} p$ if there is a function $f : \alpha \to \alpha$ such that $\overline{f}(p) = q$. For $p, q \in \alpha^*$, we write $p \approx_{RK} q$ if $p \leq_{RK} q$ and $q \leq_{RK} p$ (equivalently, there is a bijection $\sigma : \alpha \to \alpha$ such that $\overline{f}(p) = q$).

b: *Comfort order*: If $p, q \in \alpha^*$, the we say $q \leq_C p$ if every p-compact space is q-compact (this order was introduced and studied in [G2]).

c: An ultrafilter $p \in U(\alpha)$, for $\alpha > \omega$, is called *decomposable* if for every $\omega \leq \gamma < \alpha$ there is $q \in U(\gamma)$ such that $q \leq_{RK} p$.

d: The *norm* of a free ultrafilter p on a cardinal α is $\|p\| = \min\{|A| : A \in p\}$.

We have that $\leq_{RK} \subseteq \leq_C$ and this containment is proper (see [G2]). The *type* of $p \in U(\alpha)$ is the set $T(p) = \{q \in \alpha^* : p \approx_{RK} q\}$. It is not difficult to see that $T(p)$ is a dense subset of $U(\alpha)$ for every $p \in U(\alpha)$. If $p \in U(\alpha)$, then $P_{RK}(p) = \{\alpha\} \cup \{q \in \alpha^* : q \leq_{RK} p\}$. We have that $\beta(P_{RK}(p)) = \beta(\alpha)$, for each $p \in U(\alpha)$. Next, we list some facts about these orderings (proofs are available in [G1], [G2] and [G3]):

Lemma 1.2. *Let $p, q \in \alpha^*$.*

(1): $p \leq_C q \Leftrightarrow \beta_p(\alpha) \subseteq \beta_q(\alpha) \Leftrightarrow p \in \beta_q(\alpha)$.

(2): $P_{RK}(p) \subseteq \beta_p(\alpha)$.

The next Lemma provides nice examples of p-pseudocompact spaces, for each $p \in U(\alpha)$ (the particular case $\alpha = \omega$ is Lemma 1.4 from [G4]).

Lemma 1.3. *For $p \in U(\alpha)$, the space $P_{RK}(p)$ is r-pseudocompact for every $r \in P_{RK}(p) \cap \alpha^*$.*

Proof. Let $p \in U(\alpha)$ and $r \in P_{RK}(p) \cap \alpha^*$. Put $\|r\| = \gamma$ and, without loss of generality, assume that $r \in U(\gamma)$. Notice that $\gamma \leq \alpha$. To verify that $P_{RK}(p)$ is r-pseudocompact, fix a γ-sequence $(\hat{A}_\xi)_{\xi<\gamma}$ of nonempty open subsets of $\beta(\alpha)$, where $\emptyset \neq A_\xi \subseteq \alpha$ for each $\xi < \gamma$. We choose a function $f : \gamma \to \alpha$ so that $f(\xi) \in A_\xi$ for every $\xi < \gamma$. Then, $\overline{f}(r) = q \leq_{RK} r \leq_{RK} p$; hence, $q \in P_{RK}(p)$. Let $A \in q$. Since $q = \overline{f}(r) = r - \lim_{\xi \to \gamma} f(\xi)$, $\{\xi < \gamma : f(\xi) \in A\} \in r$ and so $\{\xi < \gamma : \emptyset \neq \hat{A} \cap \hat{A}_\xi \cap P_{RK}(p)\} \in r$. That is, $q \in L(r, (\hat{A}_\xi \cap P_{RK}(p))_{\xi<\gamma})$. Therefore, $P_{RK}(p)$ is r-pseudocompact. □

The next Lemma slightly generalizes Theorem 1.5 from [G4]. For the sake of completeness, we give the proof of the general case.

Lemma 1.4. *For $p \in \alpha^*$ and $q \in U(\alpha)$, the following are equivalent.*

(1): $p \leq_{RK} q$;

(2): *every q-pseudocompact space is p-pseudocompact;*

(3): $P_{RK}(q)$ *is p-pseudocompact.*

Proof. (1) $\Rightarrow$ (2). Put $\|q\| = \gamma$ and choose a function $f : \alpha \to \gamma$ so that $\overline{f}(q) = p$. Let X be a q-pseudocompact space and let $(V_\xi)_{\xi<\gamma}$ be a γ-sequence of nonempty open subsets of X. By Lemma 1.1, $L(p, (V_\xi)_{\xi<\gamma}) = L(q, (U_\xi)_{\xi<\gamma})$, where $U_\xi = V_{f(\xi)}$ for every $\xi < \gamma$. By assumption, $L(q, (U_\xi)_{\xi<\gamma}) \neq \emptyset$ and so $L(p, (V_\xi)_{\xi<\gamma}) \neq \emptyset$. Thus, X is p-pseudocompact.

(2) $\Rightarrow$ (3). This follows from Lemma 1.3.

(3) $\Rightarrow$ (1). Consider the α-sequence $(\{\xi\})_{\xi<\alpha}$ of nonempty open subsets of $P_{RK}(q)$. Then, there is $r \in L(p, (\{\xi\})_{\xi<\alpha}) \cap P_{RK}(q)$. Hence, $r = p = p - \lim_{\xi \to \alpha}\{\xi\} \in P_{RK}(q)$. □

We turn next to the study of the γ-psedocompactness of the space $P_{RK}(p)$, for $p \in U(\alpha)$. We first give two Lemmas.

Lemma 1.5. *Let $p \in U(\alpha)$, $\kappa \leq \gamma$ and suppose that $P_{RK}(p)$ is γ-pseudocompact for $\gamma \leq \alpha$. Then, if either*

(1): $\kappa^{<\kappa} \leq \gamma$,

(2): *or κ is regular,*

(3): *or γ is a strong limit cardinal,*

then $P_{RK}(p) \cap U(\kappa) \neq \emptyset$.

Proof. We know that $U(\kappa) \subseteq \beta(\kappa) \subseteq \beta(\alpha)$, $\beta(\kappa)$ is a clopen subset of $\beta(\alpha)$, and $\kappa \leq \psi(U(\kappa), \beta(\kappa)) \leq \kappa^{<\kappa}$ (notice that $\psi(U(\kappa), \beta(\kappa)) = \kappa$ iff κ is regular). If either $\kappa^{<\kappa} \leq \gamma$, or κ is regular, or γ is a strong limit cardinal, then, in any case, $U(\kappa)$ is G_γ-set in $\beta(\alpha)$ and so, by Theorem 0.2, $P_{RK}(p) \cap U(\kappa) \neq \emptyset$, since $P_{RK}(p)$ is γ-pseudocompact. □

Applying Lemma 1.5, we have the following two Theorems:

Theorem 1.6. *If $p \in U(\alpha)$, $\alpha^{<\alpha} = \alpha$ and $P_{RK}(p)$ is α-pseudocompact, then p is decomposable.*

Theorem 1.7. **[GCH]**. *For every cardinal α and for every $p \in U(\alpha)$, if $P_{RK}(p)$ is α-pseudocompact, then p is decomposable.*

Theorem 1.8. *For every cardinal α there is $p \in U(\alpha)$ such that $P_{RK}(p)$ is α-pseudocompact.*

Proof. First observe that the family $\mathcal{A} = \{\bigcap_{\nu<\gamma} \hat{A}_\nu : \alpha^* \cap (\bigcap_{\nu<\gamma} \hat{A}_\nu) \neq \emptyset, A_\nu \in [\alpha]^{\geq\omega}$ for every$\nu < \gamma$, and $\gamma \leq \alpha\}$ has cardinality $\leq 2^\alpha$. Enumerate $\mathcal{A}$ as $\{Z_\nu : \nu < 2^\alpha\}$. For each $\nu < 2^\alpha$, choose $p_\nu \in Z_\nu$ and consider the set $\{p_\nu : \nu < 2^\alpha\}$. W. W. Comfort and S. Negrepontis have shown that the Rudin-Keisler order on α^* is 2^α upward directed (a proof of this fact is available in [Co, Th. 6.4], [CN, Th. 10.9-10.13] and [G1]). By applying this result, we may find, $p \in U(\alpha)$ such that $p_\nu \leq_{RK} p$ for every $\nu < 2^\alpha$. That is, $\{p_\nu : \nu < 2^\alpha\} \subseteq P_{RK}(p)$. It is then evident that $P_{RK}(p)$ is G_α-dense in $\beta(P_{RK}(p)) = \beta(\alpha)$ and hence, by Theorem 0.2, $P_{RK}(p)$ is α-pseudocompact. □

Assuming a set-theoretic axiom, we shall next show that there is $p \in U(\omega_1)$ such that $P_{RK}(p)$ is not ω_1-pseudocompact. First, we state the axiom:

$$(*) = \diamondsuit + \exists \text{ a normal, } \omega_1 - \text{dense ideal } I \text{ over } \omega_1.$$

Example 1.9. *$(*)$ implies that there is $p \in U(\omega_1)$ such that $P_{RK}(p)$ is not ω_1-pseudocompact.*

Proof. A. Kanamori [Ka] used $(*)$ to construct two ultrafilters $p \in U(\omega_1)$ and $q \in \omega^*$ so that $r <_{RK} p$ iff $r \approx_{RK} q$. Let us consider these two ultrafilters p and q. Since ω^* has 2^{2^ω} distinct types, we may fix $s \in \omega^* - T(q)$. Since $(*) \Rightarrow \diamondsuit$ and $\diamondsuit \Rightarrow CH$, we must have that $\omega < \psi(s, \beta(\omega)) \leq 2^\omega = \omega_1$. Hence, $\psi(s, \beta(\omega_1)) = \omega_1$ and then $\{s\}$ is a G_{ω_1}-set in $\beta(\omega_1)$ which does not intercept $P_{RK}(p)$. Thus, $P_{RK}(p)$ cannot be ω_1-pseudocompact. □

The ultrafilter p defined in Example 1.9 is decomposable and $P_{RK}(p)$ is not ω_1-pseudocompact. This shows that the converse of Theorem 1.7 does not hold, even under the assumption of **CH**.

Example 1.10. *Let $\omega < \alpha$. If $p \in U(\alpha)$ is RK-minimal (selective), then $P_{RK}(p) = \alpha \cup T(p)$ and hence $P_{RK}(p)$ is a p-pseudocompact, non-pseudocompact space. We remark that if $p \in U(\alpha)$ is RK-minimal, then α is a measurable cardinal (see [Co, 4.5]), and, conversely, if α is a measurable cardinal and $\alpha > \omega$, then there is $p \in U(\alpha)$ which is RK-minimal in α^*.*

Now, we turn our attention to the study of the p-compact reflexion $\beta_p(\omega)$ of ω, for $p \in U(\omega_1)$ ([G2] contains basic information about the p-compact reflexion of ω when $p \in \omega^*$). First, we prove a Lemma.

Lemma 1.11. *If $q \in \omega^*$ satisfies $\chi(q, \omega^*) = \omega_1$, then $q \in \beta_p(\omega)$ for every $p \in U(\omega_1)$ (i. e., $q \leq_C p$).*

Proof. Let $\{A_\nu : \nu < \omega_1\} \subseteq q$ be such that $\{A_\nu^* : \nu < \omega_1\}$ is a local base at q in ω^*. Since ω^* is an almost P-space (see [Le]), we may find a π-base $\{B_\nu^* : \nu < \omega_1\}$ of q so that $B_\nu^* \subseteq \bigcap_{\mu \leq \nu} A_\mu^*$ for each $\nu < \omega_1$. By the proof of Lemma 0.5, $\{S(A) : A \in q\}$ generates a filter on ω_1, where $S(A) = \{\nu < \omega_1 : B_\nu^* \subseteq A^*\}$. Let $A \in q$. Then, there is $\theta < \omega_1$ such that $B_\theta^* \subseteq A_\theta^* \subseteq A^*$. If $\theta < \mu < \omega_1$, then $B_\mu^* \subseteq A_\theta^* \subseteq A^*$; that is, $\mu \in S(A)$ for all $\theta < \mu < \omega_1$. Thus, $\omega_1 - S(A)$ is countable for every $A \in q$. This implies that $\{S(A); A \in q\} \in p$ for every $p \in U(\omega_1)$. By the proof of Lemma 0.5, $\{q\} = L(p, (B_\nu^*)_{\nu<\omega_1})$ for every $p \in U(\omega_1)$. Fix $p \in U(\omega_1)$. It is a result of G. V. Čudnovskij and D. V. Čudnovskij [CC] and K. Kunen and K. Prikry [KP] that there is $r \in \omega^*$ such that $r \leq_{RK} p$. By Lemma 1.2, $T(r) \subseteq \beta_p(\omega)$. Define $f : \omega_1 \to T(r)$ so that $f(\nu) \in B_\nu^* \cap T(r)$ for every $\nu < \omega_1$. Since $\beta_p(\omega)$ is p-compact and $f[\omega] \subseteq \beta_p(\omega)$, $\overline{f}(p) \in \beta_p(\omega)$. But, we have that $\overline{f}(p) = p\text{-}\lim\ f(\nu) \in L(p, (B_\nu^*)_{\nu<\omega_1}) = \{q\}$, since $f(\nu) \in B_\nu^*$ for every $\nu < \omega_1$. So, by p-compactness, $q = \overline{f}(p) \in \beta_p(\omega)$ as desired. □

In the next Lemma, we list some basic properties of $\beta_p(\omega)$, for $p \in U(\omega_1)$.

Lemma 1.12. *The following holds.*

1: $|\beta_p(\omega)| \leq 2^{\omega_1}$ *for every* $p \in U(\omega_1)$.

2: *If* $\omega^* \subseteq P_{RK}(p)$, *for* $p \in U(\omega_1)$, *then* $\beta_p(\omega) = \beta(\omega)$.

3: $2^{\omega_1} < 2^{2^\omega}$ *if and only if* $\beta_p(\omega) \neq \beta(\omega)$, *for every* $p \in U(\omega_1)$.

4: **[CH]** $\beta_p(\omega) = \beta(\omega)$ *for every* $p \in U(\omega_1)$.

Proof. 1. We know that the space $\beta_p(\omega)$ can be defined by an inductive process as follows:
$(X)_0 = \omega$ and $(X)_\nu = \{\overline{f}(p) : f : \omega_1 \to \bigcup_{\mu<\nu}(X)_\mu\}$ for each $\nu < \omega_2$. It is not hard to prove that $|(X)_\nu| \leq 2^{\omega_1}$ for every $\nu < \omega_2$ and hence $|\beta_p(\omega)| \leq 2^{\omega_1}$.

2. If $s \in \omega^*$, then there is a function $f : \omega_1 \to \omega$ such that $\overline{f}(p) = s$. Using the notation introduced in the proof of the the first clause, $s = \overline{f}(p) \in (\omega)_1 \subseteq \beta_p(\omega)$. So, $\beta_p(\omega) = \beta(\omega)$.

3. Necessity. If there is $p \in U(\omega_1)$ such that $\beta_p(\omega) = \beta(\omega)$, by clause 1, $2^{2^\omega} = |\beta(\omega)| = |\beta_p(\omega)| \leq 2^{\omega_1}$. So, $2^{\omega_1} = 2^{2^\omega}$ which contradicts our assumption.

Sufficiency. If $2^{\omega_1} = 2^{2^\omega}$ by the upward directed property of the RK-order (see [CN, Th. 10.9] and [G1, Lemma 2.9]) there is $p \in U(\omega_1)$ such that $\omega^* \subseteq P_{RK}(p)$ and, by clause 2, $\beta_p(\omega) = \beta(\omega)$, which is a contradiction.

4. Assume **CH** and let $p \in U(\omega_1)$. By Lemma 1.11, $\omega^* \subseteq \beta_p(\omega)$ and so $\beta_p(\omega) = \beta(\omega)$. □

M. Bell and K. Kunen [BK] showed the existence of a model of ZFC in which $2^\omega = \aleph_{\omega_1}$ and $\pi\chi(q, \omega^*) = \omega_1$ for all $q \in \omega^*$. But, it is a known result that there is, in ZFC, $p \in \omega^*$ such that $\chi(q, \omega^*) = 2^\omega$ (see [Co]).

Now, for $p \in U(\omega_1)$ and $q \in \omega^*$, we will study the space $\Gamma_{p,q} = \beta_p(\omega) - (\{q\} \cup \omega)$. We should remark that $\beta_p(\omega) - \omega$ is a p-compact space for every $p \in U(\omega_1)$. Hence, if $p \in U(\omega_1)$ and $q \in \omega^* - \beta_p(\omega)$, then $\Gamma_{p,q} = \beta_p(\omega) - \omega$ is p-compact. We will see that under certain conditions on the ultrafilters p and q the space $\Gamma_{p,q}$ is not p-pseudocompact.

Theorem 1.13. *If $2^{\omega_1} < 2^{2^\omega}$, then for every $p \in U(\omega_1)$ there is $q \in \omega^*$ such that $\Gamma_{p,q}$ is p-pseudocompact.*

Proof. Given $p \in U(\omega_1)$. By clause 3 of Lemma 1.13, there is $q \in \omega^* - \beta_p(\omega)$. So, $\Gamma_{p,q} = \beta_p(\omega) - \omega$ which is a p-compact space and so it is p-pseudocompact. □

Next, we will prove that **CH** implies that for every $p \in U(\omega_1)$ there is $q \in \omega^*$ such that $\Gamma_{p,q}$ is not p-pseudocompact. We need several preliminary results.

Lemma 1.14. *Let $p \in U(\omega_1)$, $q \in \omega^*$ and $(A_\nu)_{\nu<\omega_1}$ an ω_1-sequence of elements in $[\omega]^\omega$. If $\{q\} = L(p, (A^*_\nu)_{\nu<\omega_1}) \cap \beta_p(\omega)$, then $\{q\} = L(p, (A^*_\nu)_{\nu<\omega_1})$.*

Proof. Suppose that $s \in L(p,(A^*_\nu)_{\nu<\omega_1}) - \{q\}$. Let $B \in s$ be such that $\omega - B \in q$. Then, $D = \{\nu < \omega_1 : A^*_\nu \cap B^* \neq \emptyset\} \in p$. Now, for each $\nu \in D$ we choose $q_\nu \in A^*_\nu \cap B^* \cap \beta_p(\omega)$, and if $\nu \in \omega_1 - D$, we choose any $q_\nu \in A^*_\nu \cap \beta_p(\omega)$. By the p-compactness of $\beta_p(\omega) - \omega$, there is $r \in \beta_p(\omega) - \omega$ such that $r = p\text{-}\lim\ q_\nu$ and so $r \in L(p,(A^*_\nu)_{\nu<\omega_1}) \cap \beta_p(\omega_1)$. By assumption, we must have that $r = q$. Thus, $E = \{\nu < \omega_1 : q_\nu \in (\omega - B)^*\} \in p$ and hence $D \cap E \in p$, but this is impossible since $D \cap E = \emptyset$. □

Lemma 1.15. *Let $p \in U(\omega_1)$ and $q \in \omega^*$. Then, $\Gamma_{p,q}$ is not p-pseudocompact if and only if there is an ω_1-sequence $(A_\nu)_{\nu<\omega_1}$ of elements in $[\omega]^\omega$ such that $\{q\} = L(p,(A^*_\nu)_{\nu<\omega_1})$.*

Proof. Necessity. Suppose that $\Gamma_{p,q}$ is not p-pseudocompact. It is clear that $q \in \beta_p(\omega)$. Then, there is an ω_1-sequence $(V_\nu)_{\nu<\omega_1}$ of nonempty open subsets of $\Gamma_{p,q}$ without any p-limit point in $\Gamma_{p,q}$. For every $\nu < \omega_1$, we may assume that $V_\nu = \Gamma_{p,q} \cap A^*_\nu$, for some $A_\nu \in [\omega]^\omega$, and $A_\nu \notin q$ for every $\nu < \omega_1$. Since $\beta_p(\omega) - \omega$ is p-compact, hence it is p-pseudocompact, the ω_1-sequence $(\beta_p(\omega) \cap A^*_\nu)_{\nu<\omega_1}$ has a p-limit point in $\beta_p(\omega) - \omega$, and since the ω_1-sequence $(\Gamma_{p,q} \cap A^*_\nu)_{\nu<\omega}$ has no p-limit point in $\Gamma_{p,q} = \beta_p(\omega) - (\{q\} \cup \omega)$, we must have that $\{q\} = L(p,(A^*_\nu)_{\nu<\omega_1}) \cap \beta_p(\omega)$. By Lemma 1.14, $\{q\} = L(p,(A^*_\nu)_{\nu<\omega_1})$.

Sufficiency. Suppose that $\{q\} = L(p,(A^*_\nu)_{\nu<\omega_1})$. Then the ω_1-sequence $(A^*_\nu \cap \Gamma_{p,q})_{\nu<\omega_1}$ has no p-limit point in $\Gamma_{p,q}$. Thus, $\Gamma_{p,q}$ cannot be p-pseudocompact. □

Lemma 1.16. *If $q \in \omega^*$ and $\chi(q,\omega^*) = \omega_1$, then $\Gamma_{p,q}$ is not p-pseudocompact for all $p \in U(\omega_1)$.*

Proof. Fix $q \in \omega^*$ with $\chi(q,\omega^*) = \omega_1$. By the proof of Lemma 1.11, there are is a π-base $\{B^*_\nu : \nu < \omega_1\}$ of q such that $L(p,(B^*_\nu)_{\nu<\omega_1}) = \{q\}$, for all $p \in U(\omega_1)$. By Lemma 1.15, $\Gamma_{p,q}$ is not p-pseudocompact. □

As a corollary we have:

Theorem 1.17. **[CH]** *For every $q \in \omega^*$, $\Gamma_{p,q}$ is not p-pseudocompact for all $p \in U(\omega_1)$.*

We say that $p \in \omega^*$ is a *P_κ-point* if p lies in the interior of the intersecction of less than κ neighborhoods of it. A P_{ω_1}-point is simply called *P-point*. A. Szymański [Sz] showed that MA implies that for every regular cardinal $\omega < \kappa < 2^\omega$ there is a P_κ-point of ω^* which is not a *P_{κ^+}-point*, and, on the other hand, S. Shelah ([Mi], [Wi]) found a model of ZFC in which ω^* does not have any P-point.

Theorem 1.18. *If $q \in \omega^*$ is a P_{ω_2}-point, then $\Gamma_{p,q}$ is p-pseudocompact for every $p \in U(\omega_1)$.*

Proof. Suppose that $\Gamma_{p,q}$ is not p-pseudocompact. By Lemma 1.17, there exists an ω_1-sequence $(A_\nu)_{\nu<\omega_1}$ of elements in $[\omega]^\omega$ such that $\{q\} = L(p, (A^*_\nu)_{\nu<\omega_1})$. Without loss of generality, we may assume that $A_\nu \notin q$ for every $\nu < \omega_1$. Since q is a P_{ω_2}-point of ω^* there is $B \in q$ such that $B^* \cap A^*_\nu = \emptyset$ for every $\nu < \omega_1$ which is impossible. □

Lemma 1.19. *Let $p \in U(\alpha)$ and $q \in \omega^*$. If $\{q\} = L(p, (A^*_\nu)_{\nu<\alpha})$ for some α-sequence $(A_\nu)_{\nu<\alpha}$ of elements in $[\omega]^\omega$, then $\alpha \geq \omega_1$ and $\pi\chi(q, \omega^*) \leq \alpha$.*

Proof. Without loss of generality, we may assume that $A_\nu \notin q$ for every $\nu < \alpha$. We claim that $\{\nu < \alpha : A^*_\nu \subseteq B^*\} \in p$ for every $B \in q$. Suppose the contrary, $D = \{\nu < \alpha : A^*_\nu - B^* \neq \emptyset\} \in p$ for some $B \in q$. For every $\nu \in D$, fix $q_\nu \in A^*_\nu - B^*$, and for $\nu \in \alpha - D$, fix any $q_\nu \in \omega^*$. Since ω^* is compact (hence, it is p-compact), there is $s \in \omega^*$ such that $s = p\text{-}\lim\ q_\nu$, and so $s \in L(p, (A^*_\nu)_{\nu<\alpha})$. By assumption, we must have that $s = q$. But, this implies that $\{\nu \in D : q_\nu \in B^*\} \in p$ which is a contradiction. This shows our claim and it implies that $\pi\chi(q, \omega^*) \leq \alpha$.

Now, let us assume that $\alpha = \omega$. By using the Disjoint Refinement Lemma [CN, Lemma 7.5], we may find a pairwise disjoint family $\{B_\nu : \nu < \omega\}$ of infinite subsets of ω such that $B_\nu \subseteq A_\nu$ for every $\nu < \omega$. If $B = \bigcup_{\nu<\omega} B_\nu$, then $B \in q$: Otherwise there would be $\mu < \omega$ such that $B_\mu^* \subseteq A_\mu^* \subseteq (\omega - B)^*$ which would be a contradiction. We choose two infinite disjoint subsets N and M of B such that $B = N \cup M$ and $|N \cap B_\nu| = |M \cap B_\nu| = \omega$ for every $\nu < \omega$. If $N \in q$, then there is $\nu < \omega$ such that $B_\nu^* \subseteq A_\nu^* \subseteq N^*$, but this is impossible since $|M \cap A_\nu| = \omega$. If we assume that $M \in q$, then we obtain an analogous contradiction. This shows that $\alpha \geq \omega_1$. □

One consequence of Lemma 1.15 and Lemma 1.19 is that if $\Gamma_{p,q}$ is not p-pseudocompact for some $q \in \omega^*$ and some $p \in U(\omega_1)$, then $\pi\chi(q, \omega^*) = \omega_1$. That is:

Theorem 1.20. *If $p \in U(\omega_1)$, then $\Gamma_{p,q}$ is p-pseudocompact for every $q \in \omega^*$ with $\pi\chi(q, \omega^*) > \omega_1$.*

It is known that under MA the π-character of any free ultrafilter on ω is 2^ω. The authors of [BK] proved, in ZFC, that there is $q \in \omega^*$ such that $\pi\chi(q, \omega^*) \geq cf(2^\omega)$. Hence, in a model of ZFC in which $cf(2^\omega) > \omega_1$ we will have that for each $p \in U(\omega_1)$, $\Gamma_{p,q}$ is p-pseudocompact for some $q \in \omega^*$.

We end this Section with a characterization of some P-points of ω^* in terms of p-limits for, $p \in U(\omega_1)$.

Theorem 1.21. *For $q \in \omega^*$, the follwing are equivalent.*

1: *There is an ω_1-sequence $(A_\nu)_{\nu<\omega_1}$ of elements of q such that $\{q\} = L(p, (A_\nu^*)_{\nu<\omega_1})$ for every $p \in U(\omega_1)$.*

2: *q is a P-point of ω^* and $\chi(q, \omega^*) = \omega_1$.*

Proof. (1) $\Rightarrow$ (2). Let $\{B_n : n < \omega\} \subseteq q$. Following the proof of Lemma 1.19, for every $n < \omega$, $S_n = \{\nu < \omega_1 : A^*_\nu \subseteq B^*_n\} \in p$ for every $p \in U(\omega_1)$. Hence, for each $n < \omega$, $|\omega_1 - S_n| \leq \omega$. Then, there is $\nu \in \bigcap_{n<\omega} S_n$ which satisfies that $A^*_\nu \subseteq B^*_n$ for all $n < \omega$. This shows, at the same time, that q is a P-point of ω^* and that $\{A^*_\nu : \nu < \omega_1\}$ is a local base at q.

(1) $\Rightarrow$ (2). Let $\{A^*_\nu : \nu < \omega_1\}$ be a local base at q. Since q is a P-point of ω^*, we may assume that $A^*_\nu \subseteq A^*_\mu$ whenever $\mu < \nu < \omega_1$. Fix $p \in U(\omega_1)$. It is clear that $q \in L(p, (A^*_\nu)_{\nu<\omega_1})$. Let $s \in L(p, (A^*_\nu)_{\nu<\omega_1}) - \{q\}$ and let $S \in s$ be such that $\omega - S \in q$. Then there is $\theta < \omega_1$ such that $A^*_\mu \subseteq (\omega - S)^*$ for all $\theta < \mu < \omega_1$, and since $\{\mu < \omega_1 : A^*_\mu \subseteq S^*\} \in p \in U(\omega_1)$, there is $\lambda < \omega_1$ for which $\theta < \lambda$ and $A^*_\lambda \subseteq S^* \cap (\omega - S)^*$, but this is a contradiction. $\square$

Question 1.22. *Is it consistent with ZFC the existence of $p \in U(\omega_1)$ such that $\Gamma_{p,q}$ is p-pseudocompact for all $q \in \omega^*$?.*

Question 1.23. *Is it consistent with ZFC that there are $p, q \in U(\omega_1)$ such that $\beta_p(\omega) = \beta(\omega)$ and $\beta_q(\omega) \neq \beta(\omega)$?.*

For $q \in \omega^*$, we have shown that $\chi(q, \omega^*) = \omega_1 \Rightarrow \Gamma_{p,q}$ is not p-pseudocompact for any $p \in U(\omega_1) \Rightarrow \Gamma_{p,q}$ is not p-pseudocompact for some $p \in U(\omega_1) \Rightarrow \pi\chi(q, \omega^*) = \omega_1$. I belive that none of these arrows can be reversed.

2. p-pseudocompactness and α-pseudocompactness

We start with some terminology: For cardinals α and γ with $\gamma \leq \alpha$, an ultrafilter $p \in U(\alpha)$ is said to be γ-*complete* if $\bigcap_{\xi<\kappa} A_\xi \in p$ whenever $\{A_\xi : \xi < \kappa\} \subseteq p$ and $\kappa < \gamma$. Observe that every free ultrafilter is ω-complete. A cardinal number α is called *Ulam-measurable* if there is an ω_1-complete ultrafilter $p \in U(\alpha)$. A characterization of γ-completeness in terms of functions and p-pseudocompactness is given in the next Lemma.

Lemma 2.1. [**Folklore**] *Let $\gamma \leq \alpha$. For $p \in U(\alpha)$ the following are equivalent.*

1: *p is γ-complete;*

2: *for every $\kappa < \gamma$ and for every surjection $f : \alpha \to \kappa$, there is $\xi < \kappa$ such that $f^{-1}(\xi) \in p$;*

3: *the discrete space κ is p-compact for every $\kappa < \gamma$;*

4: *every space X with $|X| < \gamma$ is p-compact;*

5: *the discrete space κ is p-pseudocompact for every $\kappa < \gamma$;*

6: *every space X with $|X| < \gamma$ is p-pseudocompact;*

7: *$\beta_p(X) = X$ for every space X with $|X| < \gamma$.*

Proof. (1) $\Rightarrow$ (2). Let $\kappa < \gamma$ and let $f : \alpha \to \kappa$ be a surjection. Suppose that $f^{-1}(\xi) \notin p$ for every $\xi < \kappa$. Then, $\emptyset = \bigcap_{\xi<\kappa}(\alpha - f^{-1}(\xi)) \in p$, which is a contradiction.

(2) $\Rightarrow$ (4). Let X be a space such that $\kappa = |X| < \gamma$. Suppose that there is a surjection $f : \alpha \to X$ such that $\overline{f}(p) \in \beta(X) - X$. Fix a bijection $i : X \to \kappa$. By hypothesis, there is $\xi < \kappa$ such that $(i \circ f)^{-1}(\xi) \in p$. Put $\xi = i(x)$. Now, we choose two disjoint open subsets U and V of $\beta(X)$ so that $x \in U$ and $y = \overline{f}(p) \in V$. Then, $p \in \overline{f}^{-1}(V)$ and hence $\alpha \cap \overline{f}^{-1}(V) \in p$. If $\zeta \in \alpha \cap \overline{f}^{-1}(V) \cap (i \circ f)^{-1}(\xi)$, then $f(\zeta) = x \in V$, but this is impossible. Therefore, X is p-compact.

The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (5) and the equivalence (4) $\Leftrightarrow$ (3) are evident.

(5) $\Rightarrow$ (6). This follows from the fact that p-pseudocompactness is preserved under continuous surjections.

(6) $\Rightarrow$ (1). Assume that p is not γ-complete. Let $\lambda = \min\{\gamma : p \text{ is not } \gamma\text{-complete}\}$. Then, it is easily checked that λ is a successor cardinal, i.e., $\lambda = \kappa^+$. So, there is $\{A_\xi : \xi < \kappa\} \subseteq p$ such that $\bigcap_{\xi<\kappa} A_\xi \notin p$. Notice that p is κ-complete. Without loss of generality, suppose that $A_0 = \alpha$ and $\bigcap_{\xi<\kappa} A_\xi = \emptyset$.

Define $B_0 = A_0$ and $B_\xi = \bigcap_{\eta<\xi} A_\eta$ for each $\xi < \kappa$. By assumption, we have that $B_\xi \in p$ for every $0 < \xi < \kappa$. We may assume that $B_\xi - B_{\xi+1} \neq \emptyset$ for every $\xi < \kappa$. Define $f : \alpha \to \kappa$ by $f^{-1}(\xi) = B_\xi - B_{\xi+1}$ for every $\xi < \kappa$. Now, consider the sequence $(\{f(\zeta)\})_{\zeta<\alpha}$ of open subsets of κ. Since κ is p-pseudocompact, there is $\theta < \kappa$ such that $\{\zeta < \alpha : f(\zeta) = \theta\} \in p$. It then follows that $f^{-1}(\theta) = B_\theta - B_{\theta+1} \in p$, but this is a contradiction. □

It is well-known that the ultrafilters that are not γ-complete can be characterized in terms of the Rudin-Keisler order.

Lemma 2.2. **[Folklore]** *Let $\gamma \leq \alpha$. For $p \in U(\alpha)$, the following are equivalent.*

1: *p is not γ-complete*;

2: *there are $\kappa < \gamma$ and $q \in U(\kappa)$ such that $q \leq_{RK} p$.*

Proof. (1) ⇒ (2). By Lemma 2.1 there is a cardinal $\kappa < \gamma$ for which κ is not p-compact. Let κ be the smallest cardinal with this property. Then, there is a function $f : \alpha \to \kappa$ such that $\overline{f}(p) = q \in \beta(\kappa) - \kappa$. By the minimality of κ, $q \in U(\kappa)$. Thus, $q \in U(\kappa)$ and $q \leq_{RK} p$.

(2) ⇒ (1). Let $\kappa < \gamma$ and $q \in U(\kappa)$ satisfy $q \leq_{RK} p$. Then there is $f : \alpha \to \kappa$ such that $\overline{f}(p) = q$ and so, κ is not p-compact. According to Lemma 2.1, p cannot be γ-complete. □

Theorem 2.3. *Let α and γ be cardinals with $\gamma \leq \alpha$. For $p \in U(\alpha)$, the following are equivalent.*

1: *every p-compact space is initially γ-compact;*

2: *every p-compact space is γ-pseudocompact;*

3: *$\beta_p(X)$ is γ-pseudocompact for every space X;*

4: *$\beta_p(\alpha)$ is γ-pseudocompact;*

5: *$\beta_p(\alpha) \cap U(\kappa) \neq \emptyset$ for every regular cardinal κ with $\kappa \leq \gamma$;*

6: *$\beta_p(\alpha) \cap U(\kappa) \neq \emptyset$ for every cardinal κ with $\kappa \leq \gamma$;*

7: *there is $q \in U(\gamma)$ such that q is decomposable and $q \leq_C p$;*

8: *$\beta_p(\alpha)$ is initially γ-compact.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (5). Suppose that $\beta_p(\alpha)$ is γ-pseudocompact. By Theorem 0.2, $\beta_p(\alpha)$ is G_γ-dense in $\beta(\alpha)$. Let κ be a regular cardinal with $\omega \leq \kappa \leq \gamma$. For every $\xi < \kappa$, we define $A_\xi = \{\eta : \xi \leq \eta < \kappa\}$. By Theorem 0.2 (6), we have that $\emptyset \neq (\bigcap_{\xi<\kappa} \hat{A}_\xi) \cap \beta_p(\alpha)$. If $q \in (\bigcap_{\xi<\kappa} \hat{A}_\xi) \cap \beta_p(\alpha)$, then $q \in U(\kappa) \cap \beta_p(\alpha)$, because κ is regular. Therefore, $\beta_p(\alpha) \cap U(\kappa) \neq \emptyset$

(5) $\Rightarrow$ (6). Let $\kappa \leq \gamma$ be a singular cardinal, and let $\{\kappa_\xi : \xi < cf(\kappa)\}$ be an increasing unbounded subset of regular cardinal numbers in κ. Fix $q \in U(cf(\kappa)) \cap \beta_p(\alpha)$ and, for each $\xi < cf(\kappa)$, we fix $q_\xi \in U(\kappa_\xi) \cap \beta_p(\alpha) \cap A_\xi^*$, where $\{A_\xi : \xi < cf(\kappa)\}$ forms a partition of α. Now, we define $f : cf(\kappa) \to \beta_p(\alpha)$ by $f(\xi) = q_\xi$ for each $\xi < cf(\kappa)$. Since $\beta_p(\alpha)$ is q-compact (by Lemma 1.2), $\overline{f}(q) \in \beta_p(\alpha)$. We leave the reader the proof that $\overline{f}(q) \in U(\kappa)$.

(6) $\Rightarrow$ (7). We apply inductively Theorem 2.12 from [G1].

(7) $\Rightarrow$ (8). Let $q \in U(\gamma)$ be such that q is decomposable and $q \leq_C p$. By the Definition of Comfort order, $\beta_p(\alpha)$ is q-compact and, by Corollary 2.15 from [G1], $\beta_p(\alpha)$ is initially γ-compact.

(8) $\Rightarrow$ (1). Assume that $\beta_p(\alpha)$ is initially γ-compact. Let X be a p-compact space. It suffices to show that every infinite set $E \in [X]^{\leq\gamma}$ has a complete accumulation point in X (see [St, Th. 2.2]). Let $E = \{x_\xi : \xi < \gamma\}$ be a subset of X. Choose a partition $\{A_\xi : \xi < \gamma\}$ of α in infinite subsets. Now, define $f : \alpha \to X$ by $A_\xi = f^{-1}(x_\xi)$ for every $\xi < \gamma$. Since X is p-compact, $\overline{f}[\beta_p(\alpha)] \subseteq X$ and since $\beta_p(\alpha)$ is initially γ-compact, $\overline{f}[\beta_p(\alpha)]$ is also initially γ-compact. Since $E \subseteq \overline{f}[\beta_p(\alpha)]$, E has a complete accumulation point in X. Therefore, X is initially γ-compact. □

In virtue of Theorem 2.3,if $p \in U(\alpha)$ is decomposable, then every p-compact space is α-pseudocompact (this is not a new fact since it was shown in [G1, Corollary 2.15] that a p-compact space is initially α-compact provided that $p \in U(\alpha)$ is decomposable). The ultrafilter $p \in U(\omega_1)$ given in Example 1.9 is decomposable and satisfies that $P_{RK}(p)$ is p-pseudocompact and is not ω_1-pseudocompact.

If $p \in U(\alpha)$ and every p-pseudocompact space is γ-pseudocompact, for $\gamma \leq \alpha$, then p is not γ-complete: In fact, if p is γ-complete, by Lemma 2.1, then the discrete space κ would be p-pseudocompact for every $\kappa < \gamma$, which would be impossible. In Example 1.9, under the assumption of the axiom $(*)$, we found $p \in U(\omega_1)$ such that $P_{RK}(p)$ is p-pseudocompact, is not ω_1-pseudocompact and p is not ω_1-complete. In connection with these remarks we have the following.

Theorem 2.4. *For $p \in U(\alpha)$, the following are equivalent.*

1: *every p-compact space is countably compact;*

2: *every p-compact space is pseudocompact;*

3: *$\beta_p(X)$ is pseudocompact for every space X;*

4: *$\beta_p(\alpha)$ is pseudocompact;*

5: $\beta_p(\alpha) \cap \omega^* \neq \emptyset$*;*

6: *there is $q \in \omega^*$ such that $q \leq_C p$;*

7: *$\beta_p(\alpha)$ is countably compact;*

8: *p is not ω_1-complete;*

9: *every p-pseudocompact space is pseudocompact;*

10: *$P_{RK}(p)$ is pseudocompact.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follow from Theorem 2.3 (when $\gamma = \omega$).

(8) $\Rightarrow$ (9). Suppose that $p \in U(\alpha)$ is not ω_1-complete and let X be a p-pseudocompact space. Then, by Lemma 2.2, there is $q \in \omega^*$ such that $q \leq_{RK} p$. By Lemma 1.4, X is also q-pseudocompact. We shall verify that X is G_δ-dense in $\beta(X)$. In fact, let $\emptyset \neq Z \in \mathcal{Z}(\beta(X))$. Then, there is a set $\{V_n : n < \omega\}$ of open subsets of $\beta(X)$ such that $\mathrm{cl}_{\beta(X)} V_{n+1} \subseteq V_n$ for every $n < \omega$ and $Z = \bigcap_{n<\omega} V_n = \bigcap_{n<\omega} \mathrm{cl}_{\beta(X)} V_n$. For each $n < \omega$ we put $U_n = X \cap V_n$. Since X is q-pseudocompact there is $x \in X$ such that for every neighborhood V of x, $\{n < \omega : V \cap U_n \neq \emptyset\} \in q$. Suppose that $x \notin Z$. Since $Z = \bigcap_{n<\omega} \mathrm{cl}_{\beta(X)} V_n$, there is $m < \omega$ such that $x \notin \mathrm{cl}_X U_m$. Set $V = X - cl_X U_m$. Then, we have that $\{n < \omega : V \cap U_n \neq \emptyset\} \in q$. So, we may find $k < \omega$ for which $V \cap U_k \neq \emptyset$ and $m < k$, but this is a contradiction. Therefore, $x \in Z$.

(9) $\Rightarrow$ (10). This follows from Lemma 1.3.

(10) $\Rightarrow$ (4). It is evident since $P_{RK}(p) \subseteq_p \beta(\alpha)$ (by Lemma 1.2).

(2) $\Rightarrow$ (8). If $p \in U(\alpha)$ s ω_1-complete, by Lemma 2.1, then the discrete space ω is p-compact and it is not pseudocompact. $\square$

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Instituto de Matemáticas, Apartado Postal 61-3, Xangari 58089, Morelia, Michoacan, México

E-mail address: `sgarcia@matmor.unam.mx`