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THE APPROXIMATION OF CONTINUA BY $T_{1/2}$-SPACES

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Abstract

A special case of a classical result of Alexandroff states that every one dimensional metric continuum is the inverse limit of a sequence of one dimensional polyhedra, otherwise known as finite graphs. Since a compact Hausdorff space is the Hausdorff reflection of an inverse limit of finite T_0-spaces, it is natural to ask whether one dimensional metric continua are the inverse limits of connected T_0-spaces which are one dimensional in some sense. This paper shows that each one dimensional metric continuum can be characterized as the Hausdorff reflection of the limit of an inverse sequence with special bonding maps of connected finite T_0-spaces whose order or Alexandroff dimension (defined below) is one.

0. Introduction

It is known (see Example 2 of [7]) that every one dimensional metric continuum is the inverse limit of a sequence of finite (connected) graphs - that is, a connected topological space which is a

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finite union of segments which meet only at their end-points. Since a compact Hausdorff space is the Hausdorff reflection of an inverse limit of finite T_0-spaces (see [3], or see [6] for a treatment in our notation) and a chainable continuum can be represented similarly (see [5]) as the Hausdorff reflection of the inverse limit of connected ordered spaces (the *COTS* of [4]), it is natural to ask whether one dimensional metric continua are the inverse limits of connected T_0-spaces which are one dimensional in some sense. The dimension of T_0-spaces has been studied in [8] and [2] and it was shown in the former article that the (small) inductive dimension *ind* and the order dimension *odim* (one less than the length of a maximal specialization chain) coincide for Alexandroff (and hence for finite) spaces. Furthermore, it is easy to see that if X is an Alexandroff T_0-space then $ind(X) = odim(X) \leq 1$ if and only if each point of X is open or closed, that is to say, X is a $T_{1/2}$-space. The aim of this paper is to characterize one dimensional metric continua as the Hausdorff reflections of the limit of an inverse sequence of connected finite one dimensional T_0-spaces with a special type of bonding maps.

1. Approximating a One Dimensional Continuum With $T_{1/2}$-spaces

Our aim in this section is to show that every one dimensional continuum is the Hausdorff reflection of an inverse limit of connected one dimensional finite T_0-spaces. We begin with some definitions. Throughout, $\mathbb{I}$ will denote the unit interval $[0, 1]$. A *topological graph* G, is a finite set of nontrivial line segments of the form $[x, y] = \{(1-t)x + ty \mid t \in \mathbb{I}\}$ (for distinct $x, y \in \mathbb{R}^3$) whose union is connected, and so that if $[v, w], [x, y] \in G$ are distinct, then $[v, w] \cap [x, y] \subseteq \{v, w\} \cap \{x, y\}$; this union is called the *realization* of G and denoted $|G|$. Note that this definition rules out loops and distinct arcs with the same endpoints, and requires that $[v, w] \cap [x, y]$ contains at most one point when $\{v, w\}$ and $\{x, y\}$ are distinct. If $F \subseteq [v, w]$ is finite, then

$x, y \in F$ are said to be *adjacent* in F if there is no $z \in F$ such that x, y are in different components of $[v, w] \setminus \{z\}$.

Thus $|G|$ is a finite union of segments - called the *edges* of G - which meet only at one of their endpoints, which we refer to as the *vertices* of G; that is, $V(G) = \bigcup\{\{v, w\} \mid [v, w] \in G\}$. The order of a vertex v of G is the number of segments incident at v: $o(v) = |\{[w, x] \in G \mid v \in \{w, x\}\}|$. Of course, $|G|$ can be represented in many different ways as the union of a finite number of segments meeting only at their end-points - consider $[(0, 0, 0), (0, 0, 1)] = [(0, 0, 0), (0, 0, 0.5)] \cup [(0, 0, 0.5), (0, 0, 1)]$.

If G is a finite graph, then a *subdivision* of G is a finite subset P of $|G|$ which contains all the vertices of G (this is analagous to *partition* of $\mathbb{I}$, as used in [5]). Notice that if P is a subdivision of G, then for each $p \in P$, there are $v, w \in P$ such that $p \in [v, w] \in G$. Therefore there are (one or two) points of $P \cap [v, w]$ adjacent to p (in $P \cap [v, w]$); define the topological graph $G_P = \{[p, q] \mid p, q \in P, \ \exists [v, w] \in G$ such that p, q are adjacent in $[v, w]\}$.

Then $|G_P| = |G|$: $|G_P| \subseteq |G|$, for if $[p, q] \in G_P$, then for some $[v, w] \in G$, $[p, q] \subseteq [v, w]$. For the reverse inclusion, if $x \in [v, w] \in G$ then for some $p, q \in P$, x is adjacent to p in $(P \cup \{x\}) \cap [v, x]$ and to q in $(P \cup \{x\}) \cap [x, w]$; as a result, $x \in [p, x] \cup [x, q] = [p, q]$.

We say that a function $f : |G| \to |H|$ is *piecewise linear* if there exist subdivisions P and Q of G and H respectively, such that

1) For all $v \in V(G_P)$, $f(v) \in V(H_Q)$;

2) For all $[s, t] \in G_P$, either $f(s) = f(t)$ or $[f(s), f(t)] \in H_Q$ and $f : [s, t] \to [f(s), f(t)]$ is linear.

We now proceed to show that a continuous function $f : |G| \to |H|$ can be uniformly approximated by a piecewise linear function. In the sequel, $\| \, \|$ will denote a fixed norm on the Euclidean space in which $|G|$ and $|H|$ are embedded. By an interval in a graph, we mean a set of the form

$$J = [s, t) = \{rs + (1 - r)t \mid r \in [0, 1)\},$$

where s, t lie in a single edge of G, or of one of the forms $[s, t], (s, t]$, or (s, t), similarly defined. Surely an interval in G is a connected subset of G. Also, if J is an interval in G, and $f[J]$ meets more than one edge of H, then $f[J]$ contains a vertex of H; this is immediate by the connectedness of $f[J]$ and the fact that if $[v_1, w_1], [v_2, w_2]$ are edges in H, then $[v_1, w_1] \cup [v_2, w_2] \setminus \{v_1, w_1, v_2, w_2\}$ consists of two clopen subsets of $[v_1, w_1] \cup [v_2, w_2]$.

Lemma 1.1. *Let $f : |G| \to |H|$ be a continuous function. Then given $\epsilon > 0$, there is a continuous, nowhere locally constant piecewise linear function $g : |G| \to |H|$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in |G|$. Further, if f is a homeomorphism, g can also be assumed to be one.*

Proof. Given $\epsilon > 0$, let

$$0 < \gamma < \min(\{\epsilon\} \cup \{\|p - q\| \mid p, q \in V(H),\ p \neq q\})/2.$$

Since f is uniformly continuous on the compact space $|G|$, we can find $\delta > 0$ such that if $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \gamma$. Let P be a partition of G with mesh less than δ. We consider f as a map between $|G_P|$ and $|H|$.

Suppose now that $[s, t]$ is an edge of $|G_P|$; we will define a piecewise linear function $f_L : [s, t] \to |H_Q|$ such that $f_L(s) = f(s)$ and $f_L(t) = f(t)$ and such that $\|f_L(z) - f(z)\| < \epsilon$ for all $z \in [s, t]$. This suffices to prove the result, since $|G_P|$ is a finite union of arcs.

By construction, $f[[s, t]]$ can contain at most one vertex of $|H|$. If it contains no element of $V(H)$, then by the comments immediately preceding this lemma, it is contained in a single edge E of H, and we define $f_L((1-r)s+rt) = (1-r)f(s)+rf(t)$ for each $r \in [0, 1]$. f_L is linear on $[s, t]$ and for each $x \in [s, t]$, $f(x), f_L(x) \in E$ implying that $\|f(x) - f_L(x)\| \leq \gamma < \epsilon$.

Otherwise, suppose that $f[[s, t]] \cap V(H) = \{p\}$ and choose $y \in [s, t]$ such that $f(y) = p$. If we define $\mathrm{St}(p)$ to be that subset of $|H|$ which is the union of all edges incident at p, then $\mathrm{diam}(\mathrm{St}(p)) \leq 2\gamma < \epsilon$. Again by the remarks prior to this lemma, it is clear that $f[[s,t]] \subset \mathrm{St}(p)$ and hence $f(s)$ and p lie in a single edge of $|H|$, as do $f(t)$ and p. Define $f_L : [s,t] \to |H|$ by $f_L((1-2r)s + 2ry) = (1-2r)f(s) + 2rp$ for $r \in [0, \frac{1}{2}]$, $f_L(2ry + (1-2r)t) = (2-2r)p + (2r-1)f(t)$ for $r \in [\frac{1}{2}, 1]$. Then for each $x \in [s,t]$, $f(x), f_L(x) \in \mathrm{St}(p)$ and hence $\|f(x) - f_L(x)\| \leq 2\gamma < \epsilon$. Furthermore, $f_L[[s,y]]$ and $f_L[[y,t]]$ are each contained in unique edges of $|H|$ and hence f_L is piecewise linear in the sense defined above.

Finally we note that any function which is piecewise linear on $[s,t]$ can be uniformly approximated by nowhere locally constant piecewise linear functions and that if f is a homeomorphism then so is f_L and the result is proved. □

The non-obvious direction of the next theorem is now an immediate consequence of Theorem 3 of [1]:

Theorem 1.2. *A topological space X is the inverse limit of a sequence of graphs with continuous maps, $\{f_n : G_{n+1} \to G_n \mid n \in \omega\}$, if and only if, there exists a sequence of continuous piecewise linear, nowhere locally constant maps $\{g_n : G_{n+1} \to G_n \mid n \in \omega\}$, whose inverse limit is X.*

We now turn to the problem of representing each one dimensional continuum as the Hausdorff reflection of an inverse limit of $T_{1/2}$-spaces.

First, for any finite graph G, we define its *associated topological space* to be the finite $H_G = G \cup V(G)$ with the T_0 topology such that elements of G are open and the minimal neighbourhood of $v \in V(G)$ is $\{v\} \cup \bigcup\{[x,y] \in G \mid v \in \{x,y\}\}$. Also, $p_G : |G| \to H_G$ will denote the canonical quotient map. Of course, this construction gives a T_0-space associated with any subdivision P of G, by $H_P = H_{G_P}$.

Now suppose $f : |G| \to |J|$ is a continuous piecewise linear nowhere locally constant function. We will denote by N the finite set of points at which f is not differentiable or which are vertices of G. Let A be any finite subset of G and let R be a subdivision of J and then we define $B = f^{-1}[R]$; since f is finite-to-one, B is finite and we let $A \cup B \cup N = D = \{d_0, \ldots, d_m\}$.

Lemma 3 of [5] shows that it is possible to define a continuous map g from $H_D = H_{G_D}$ to $H_R = H_{J_R}$ such that:

(1) If $d_i \in D$, then $g(d_i)$ is the unique element (either a vertex or an edge) of H_R containing $f(d_i)$, and

(2) If $[d_j, d_i] \in E_D$, then $g([d_j, d_i])$ is the unique element of H_R containing $f[(d_j, d_i)]$.

We note that g is uniquely determined by the map f and the subdivisions D and R, hence we call g a (D-R)-*approximation* to f.

Suppose now that we are given an inverse sequence $\{f_n : G_{n+1} \to G_n \mid n \in \omega\}$ of finite graphs with continuous nowhere locally constant piecewise linear bonding maps.

$$\ldots \xrightarrow{f_3} G_3 \xrightarrow{f_2} G_2 \xrightarrow{f_1} G_1 \xrightarrow{f_0} G_0$$

We define an inverse sequence of one dimensional connected T_0-spaces:

$$\ldots \xrightarrow{g_3} H_3 \xrightarrow{g_2} H_2 \xrightarrow{g_1} H_1 \xrightarrow{g_0} H_0$$

as follows: For each $k \in \omega$, suppose $G_k = \{[v_j^k, w_j^k] \mid 1 \leq j \leq m_k\}$ and let Q_k denote the union of the finite sets of images of dyadic rationals of the form $i/2^k$ in each $[v_j^k, w_j^k]$, where $0 \leq i \leq 2^k$ (so $Q_k = \{(1-t)v_j^k + tw_j^k \mid [v_j^k, w_j^k] \in G_k, t = i/2^k,\ i = 0, \ldots, 2^k\}$) and let N_k denote the finite set of points at which f_k is not differentiable.

Let $R_0 = Q_0$ be the set of vertices of G and let H_0 be the T_0-space associated with the subdivision R_0. Let $R_1 = N_0 \cup f_0^{-1}[R_0] \cup Q_1$ and let H_1 be the T_0-space associated with the subdivision R_1. Now let g_0 be the (R_1-R_0)-approximation of f_0 as constructed above. Having defined $g_0, g_1, \ldots, g_{k-2}$ with domains $H_0, H_1, \ldots, H_{k-1}$ approximating the maps $f_0, f_1, \ldots f_{k-2}$, we define $R_k = N_{k-1} \cup Q_k \cup {f_k}^{-1}[R_{k-1}]$, we let g_{k-1} be the (R_k-R_{k-1})-approximation to f_{k-1} and we let H_k be the T_0-space associated with the subdivision R_k.

In what follows, for simplicity of notation, p_n will denote the natural projection $p_{H_n} : G_n \to H_n$.

Theorem 1.3. *The following diagram commutes:*

$$\begin{array}{ccccccccc} \dots & \xrightarrow{f_3} & G_3 & \xrightarrow{f_2} & G_2 & \xrightarrow{f_1} & G_1 & \xrightarrow{f_0} & G_0 \\ \dots & & \downarrow p_3 & & \downarrow p_2 & & \downarrow p_1 & & \downarrow p_0 \\ \dots & \xrightarrow[g_3]{} & H_3 & \xrightarrow[g_2]{} & H_2 & \xrightarrow[g_1]{} & H_1 & \xrightarrow[g_0]{} & H_0 \end{array}.$$

Proof. This follows directly from the definition of the maps g_k. □

Our aim now is to establish the relationship between $G = \varprojlim(G_n, f_n)$ and $H = \varprojlim(H_n, g_n)$. Throughout the rest of the paper, if (Y_n, h_n) is an inverse sequence, then we let h_{mn} denote the map $h_n \circ h_{n-1} \circ \ldots \circ h_{m-1} : Y_m \to Y_n$ if $m > n$, and denote the identity map on Y_n if $m = n$.

The proof of the next theorem is similar to that of Theorem 5 of [5].

Theorem 1.4. *G is the Hausdorff reflection of H.*

Proof. Let us denote by π_n the projection from G to G_n. By Theorem 4 of [6], it suffices to show that if we define

$$\mathcal{F}_k = \{(p_k \circ \pi_k)^{-1}[V_k] : V_k \text{ is open in } H_k\}$$

then

$$\mathcal{F} = \{\mathcal{F}_k : k \in \omega\}$$

is directed by $\subseteq$ and $\cup\mathcal{F}$ is a base for G.

That $\mathcal{F}$ is directed is clear from the definitions. So now suppose that U is open in G and $x \in U$. Then there is some $k \in \omega$ and some open set $U_k \subset G_k$ such that $x \in \pi_k^{-1}[U_k] \subset U$. Now, since for each $m \in \omega$, the subdivision R_m used in the construction of H_m is such that $R_m \supset Q_m$, it follows that we can find $m \geq k$ and an open set $V_m \subset H_m$ such that $\pi_m(x) \in p_m^{-1}[V_m] \subset f_{km}^{-1}[U_k]$ and we are done. □

2. The Hausdorff Reflection of an Inverse Limit of One Dimensional T_0-spaces

In this section we show that the Hausdorff reflection of the inverse limit of a sequence of one dimensional finite T_0 spaces with a certain type of continuous bonding maps is a one dimensional continuum. Recall that $\mathsf{n}(E)$ denotes the minimal neighbourhood of a set E; we also need further notation:

An inverse sequence of spaces $\{Y_m : m \in \omega\}$ and maps $\{h_n : Y_{n+1} \to Y_n : n \in \omega\}$, is called *separating* if whenever $x, y \in Y_n$ are closed, distinct points, then for some $m \geq n$, $h_{mn}^{-1}[x]$ and $h_{mn}^{-1}[y]$ are contained in disjoint open sets. Since there are then only a finite number of pairs of distinct closed points, it is clear that for each $n \in \omega$, there exists $m \geq n$ such that for each pair of distinct closed points $x, y \in Y_n$, $h_{mn}^{-1}[x]$ and $h_{mn}^{-1}[y]$ are contained in disjoint open sets. Now, by passing to a subsequence of the inverse spectrum (which will have the same inverse limit as the original sequence), we can assume that each of the maps h_n has the property that if $x, y \in Y_{n-1}$ are distinct closed points, then $h_n^{-1}[x]$ and $h_n^{-1}[y]$ are contained in disjoint open sets - such a function will be called a *separating map*. Thus for the purpose of finding the inverse limits of a separating sequence, we can assume that all the bonding functions are separating maps.

The following result was proved in [5]:

Theorem 2.1. *The Hausdorff reflection of the limit of a separating inverse sequence of finite T_0 spaces and continuous maps is its subspace of closed points. This subspace is a retract of the limit.*

Let us say that a one dimensional T_0-space (X, τ) is *simplicial* if whenever $\{x\} \in \tau$, $|\,\mathsf{cl}(\{x\})| = 3$, that is, there are exactly two closed points in the closure of each open point. We now show that without loss of generality, we may restrict attention to separating inverse sequences of simplicial one dimensional T_0-spaces.

First recall that the *Sierpinski space* S is the set $\{0,1\}$ with topology $\{\emptyset, \{1\}, \{0,1\}\}$. We define the *d-hedgehog with n spines* to be the quotient space obtained from n disjoint copies of the Sierpinski space by identifying the (closed) point 0 in each. (The d-hedgehog with one spine is S.) Now for each one dimensional T_0-space X, we define a space X^C as follows:

Let $B = \{u_i : 1 \leq i \leq m\}$ be a faithful enumeration of the open points of X and suppose $|\,\mathsf{cl}(u_i)| = j_i + 1$. For each i, let H_i be the d-hedgehog with j_i spines and let X^C be the disjoint union of the spaces H_i together with all singletons of $X \setminus B$. We define a topology τ on X^C as follows:

For each $u_i \in B$, let σ_i be a bijection from $\mathsf{cl}(\{u_i\}) \setminus \{u_i\}$ to the set of open points of H_i.
(i) The minimal τ-neighbourhood of $u \in H_i$ is its minimal neighbourhood in the d-hedgehog H_i.
(ii) If $v \in X \setminus B$, then the minimal τ-neighbourhood of v is $(\,\mathsf{n}(v) \cap (X \setminus B)) \cup \{\sigma_i(v) : u_i \in \mathsf{n}(v)\}$.

Now let $d : X^C \to X$ be defined by $d[H_i] = u_i$ for each $i \in \{1, \ldots, m\}$ and $d(c) = c$ for each point $c \in X \setminus B$. In essence, X^C is obtained from X by substituting a d-hedgehog with n spines in place of each open point with n closed points in its closure. We call X^C the *d-complex associated with* X and d the *d-complex projection*; X^C is clearly simplicial. Most importantly, we note that the map d is separating, for if x, y are adjacent closed points of X, then for any open $u \in X$ such that $x, y \in \mathsf{cl}(u)$, the d-hedgehog $d^{-1}[u] = H_u$ contains a closed point separating $d^{-1}[x] \cap H_u$ and $d^{-1}[y] \cap H_u$; hence $d^{-1}[x]$ and $d^{-1}[y]$ must have disjoint neighbourhoods in X^C.

Lemma 2.2. *Suppose that $g : Z \to X$ is a separating map between one dimensional T_0-spaces, X^C is the d-complex associated with X and d is the corresponding d-complex projection. There is a continuous map $e : Z \to X^C$ such that $d \circ e = g$.*

Proof. From the construction of X^C, it is clear that for each $u \in g[Z]$ (open or closed), $d^{-1}[u]$ is either an open singleton or a d-hedgehog which we denote by H_u. We define the map $e : Z \to X^C$ as follows:

If $u \in g[Z]$ is closed, then $|d^{-1}[u]| = 1$ and we define $e[g^{-1}[u]] = d^{-1}[u]$.

If $u \in g[Z]$ is open and $x \in g^{-1}[u]$, there are two cases to consider:

a) If $\mathsf{cl}(x) \subset V_u$, let $e(x) = c_u$, the unique closed point of H_u.
b) If $\mathsf{cl}(x) \not\subset V_u$, since g is continuous, for each $z \in \mathsf{cl}(x) \setminus V_u$, $g(z) = w$, for some closed $w \in \mathsf{cl}(u)$. Further, if $z_1, z_2 \in \mathsf{cl}(x) \setminus V_u$ and $g(z_1) \neq g(z_2)$, then by the continuity of g, $g(z_1)$ and $g(z_2)$ are distinct closed points in $\mathsf{cl}(u)$ whose preimages under g are not separated, contradicting the fact that g is separating. Thus all $z \in \mathsf{cl}(x) \setminus V_u$ map to the same closed $w \in \mathsf{cl}(u) \subset X$ and then, from the construction of X^C, $d^{-1}[w]$ is in the closure (in X^C) of precisely one open point $t \in H_u$ and we define $e(x) = t$.

It is immediate from the definition that e is well-defined and $d \circ e = g$; hence it only remains to show that e is continuous. However, since all spaces are finite and T_0, it suffices to show that e preserves the specialization order.

To this end, we note that if $a, b \in Z$ and $b \in \mathsf{cl}(a)$, then since g is continuous, $g(b) \in \mathsf{cl}[g(a)]$. If $g(a) = g(b)$ and this point is closed in X, then necessarily $e(a) = e(b)$ and we are done. If on the other hand $g(a) = g(b) = \alpha \in X$ and α is open, then $e(a), e(b) \in d^{-1}[\alpha] = H_\alpha$, a d-hedgehog. But then $e(b) = c_\alpha$, the unique closed point of H_α and $e(a) \in H_\alpha$ and so e preserves the specialization order.

If $g(a) \neq g(b)$, then necessarily $g(a) = \alpha$ is open and $g(b) = \gamma$ is closed. As before there are two cases to consider. If $b \in g^{-1}[\alpha]$, then $e(b) = c_\alpha$, the unique closed point of the d-hedgehog $H_\alpha = d^{-1}[\alpha]$ and since $e(a) \in H_\alpha$, $e(b) \in \mathsf{cl}[e(a)]$. If, on the other hand $b \notin g^{-1}[\alpha]$, then $e(b) = d^{-1}[\gamma]$ is adjacent to some unique open point t of the d-hedgehog H_α and since $e(a) = t$, we are done. $\square$

We note that the map e constructed above does not have to be surjective even if g is; indeed, if $u \in X$ is open and $|\,\mathsf{cl}(u)| = 2$, then $d^{-1}[u]$ is a d-hedgehog with 1 spine H and e will not map onto the closed point of H. Nor in case $|\,\mathsf{cl}(u)| > 2$ does e necessarily map $g^{-1}[u]$ onto $d^{-1}[u]$.

Now suppose $\varprojlim(X_n, g_n)$ is a separating inverse sequence of one dimensional T_0-spaces. For each $n \in \omega$, let X_n^C be the $d-$complex associated with X_n, d_n the corresponding d-complex projection and $e_n : X_{n+1} \to X_n^C$ the map constructed in the previous lemma.

Lemma 2.3. *$\varprojlim(X_n, g_n)$ is homeomorphic to $\varprojlim(X_n^C, e_n \circ d_{n+1})$ and the maps $e_n \circ d_{n+1}$ are separating.*

Proof. Each of the sequences is a subsequence of the inverse sequence

$$\ldots \xrightarrow[d_2]{} X_2 \xrightarrow[e_1]{} X_1^C \xrightarrow[d_1]{} X_1 \xrightarrow[e_0]{} X_0^C \xrightarrow[d_0]{} X_0$$

and hence have the same limit.

That the maps $e_n \circ d_{n+1}$ are separating follows immediately from the fact that the maps d_n are separating: If $x, y \in X_n^C$ are closed and distinct, then $e_n^{-1}[x]$ and $e_n^{-1}[y]$ are disjoint closed sets in X_{n+1} and hence $d_{n+1}^{-1}[e_n^{-1}[x]]$ and $d_{n+1}^{-1}[e_n^{-1}[y]]$ can be separated by open sets in X_{n+1}^C. □

Corollary 2.4. *For any separating inverse sequence of finite one dimensional T_0-spaces, there is a separating inverse sequence of simplicial one dimensional T_0-spaces which has the same inverse limit.*

If X is a one dimensional finite simplicial T_0-space, we assume that the set $F(X)$ of closed elements of X is embedded in $\mathbb{R}^3$ in such a way that if $v, w, x, y \in X$, then $[v, w] \cap [x, y] \subseteq \{v, w\} \cap \{x, y\}$. Then we can define its *associated graph,*

$$G(X) = \{[v, w] \mid v, w \in F(X), v \neq w, (\exists x)(v, w \in \mathsf{cl}(x))\},$$

and the *natural projection,* $p_X : G(X) \to X$, by

$$p_X((1-t)v + tw) = \begin{cases} v & t = 0 \\ w & t = 1 \\ x & 0 < t < 1 \end{cases}.$$

Note that p_X is surjective. Further, we define the *midpoint function* $M : X \to G(X)$, by $M(x) = .5u + .5v$, if $\mathsf{cl}(x) = \{u, x, v\}$, and $M(x) = x$ if $\{x\}$ is closed. Now suppose that X and Y are simplicial one dimensional finite T_0-spaces. If $f : X \to Y$ is continuous and $v, w \in X$ are closed points such that there exists $x \in X$ with $\mathsf{cl}(x) = \{v, x, w\}$, then we define $g : G(X) \to G(Y)$ by

1) If $f(v) \neq f(w)$, then $g((1-t)v + tw) = (1-t)M(f(v)) + tM(f(w))$;

2) If $f(v) = f(x) = f(w)$, then $g((1-t)v + tw) = (1-t)M(f(v)) + tM(f(w))$ and

3) If $f(v) = f(w) \neq f(x)$, then necessarily $f(x)$ is open and since Y is simplicial, there is some unique closed $y \neq f(v)$ in $\mathsf{cl}[f(x)]$. We then define g on $[v, w]$ by

a) $g[(1-t)v + \frac{1}{2}t(v+w)] = (1-t)f(v) + t[\frac{1}{2}f(v) + \frac{1}{2}y]$, and

b) $g[(1-t)\frac{1}{2}(v+w) + tw] = (1-t)[\frac{1}{2}f(v) + \frac{1}{2}y] + tf(v)$,

where throughout, the parameter $t \in [0,1]$. Note then that $p_Y \circ g = f \circ p_X$ and g is continuous and piecewise linear.

Suppose now that we have an inverse sequence (X_n, f_n), of simplicial, finite $T_{1/2}$-spaces and separating maps, and that $(G(X_n), g_n)$, are constructed as in the previous paragraph. Let x, y be elements of the inverse limit of the $(G(X_n), g_n)$, such that for each n, x_n and y_n are in the same edge of $G(X_n)$. We proceed to show that $x = y$.

Suppose to the contrary that $x \neq y$; then there is some $n_0 \in \omega$ such that $x_n \neq y_n$ for each $n \geq n_0$ and without loss of generality we will assume that $n_0 = 0$, and so $x_n \neq y_n$ for each

$n \in \omega$. Now fix $m \in \omega$; x_m, y_m lie in the same edge I_m of $G(X_m)$ and x_{m+1}, y_{m+1} lie in the same edge I_{m+1} of $G(X_{m+1})$. Thus $g_m[I_{m+1}] \cap I_m \neq \emptyset$ and hence from the construction of g_m one sees that $g_m[I_{m+1}] \subset I_m$. Let $D_m = p_m^{-1}[I_m] = \{d, e, h\}$ and $D_{m+1} = p_{m+1}^{-1}[I_{m+1}] = \{a, b, c\}$ where $p_j : G(X_j) \to X_j$ is the natural projection and where we assume that a, c, d and h are closed and b and e are open with $a, c \in \mathsf{cl}(b)$ and $d, h \in \mathsf{cl}(e)$. Then since $x_{m+1}, y_{m+1} \in I_{m+1} = [a, c]$, it follows that there are $s_m, t_m \in \mathbb{I}$ such that $x_{m+1} = (1 - t_{m+1})a + t_{m+1}c$ and $y_{m+1} = (1 - s_{m+1})a + s_{m+1}c$. Since the map f_m is separating, only one closed point of D_m lies in the image of D_{m+1} under the map f_m and it is easy to see that since $f_m : D_{m+1} \to D_m$ is continuous, there are essentially only three possibilities to consider:

1) f_m is constant on D_{m+1}, or

2) $f_m(a) = f_m(b) = e$ and $f_m(c)$ is closed or $f_m(b) = f_m(c) = e$ and $f_m(a)$ is closed, or

3) $f_m(a) = f_m(c)$ is closed and $f_m(b) = e$.

However, case 1) is impossible since g_m would then be constant on I_{m+1}, implying that $x_m = y_m$. If any of the four equivalent cases 2) occur, say $f_m(a) = f_m(b) = e$ and $f_m(c) = d$, then $g_m[I_{m+1}] \subset [d, M(e))]$ and hence $x_m, y_m \in [d, M(e))]$ that is to say, $x_m = (1 - t_m)d + t_m h$ and $y_m = (1 - s_m)d + s_m h$ for some $0 \leq t_m, s_m \leq \frac{1}{2}$. Finally, if either of the two equivalent cases 3) occurs, say $f_m(a) = f_m(c) = d$, then again $g_m[I_{m+1}] \subset [d, M(e))]$ and it follows that $x_m = (1-t_m)d + t_m h$ and $y_m = (1-s_m)d + s_m h$ for some $0 \leq t_m, s_m \leq \frac{1}{2}$. Thus by induction, for any $n \in \omega$, if x_{m+k}, y_{m+k} lie in the same edge of $G(X_{m+k})$ for each $1 \leq k \leq n$, then in order that x_m, y_m lie in the same edge of $G(X_m)$ and such that $x_m = (1 - t_m)d + t_m h$ and $y_m = (1 - s_m)d + s_m h$ then we must have $0 \leq s_m, t_m \leq 2^{-n}$. Since this must be true for each $n \in \omega$, we have a contradiction and hence $x = y$.

We are now in a position to prove our main theorem:

Theorem 2.5. *If (X_n, f_n) is a separating inverse sequence of simplicial, finite $T_{1/2}$-spaces, then $L = \varprojlim(G(X_n), g_n)$ is (homeomorphic to) the set of closed points of $M = \varprojlim(X_n, f_n)$.*

Proof. If we denote the natural projection from $G(X_n)$ to X_n by p_n, the projection from $M = \varprojlim(X_n, f_n)$ to X_n by π_n and the projection from $L = \varprojlim(G(X_n), g_n)$ to $G(X_n)$ by ρ_n then we can define a map $\psi : L \to M$ by

$$p_n \circ \rho_n = \pi_n \circ \psi.$$

We will prove that ψ is a homeomorphism between L and the subspace of closed points of M. For this, since L and this subspace are compact Hausdorff spaces, it suffices to show that ψ is a continuous one to one map onto the closed points of M.

To this end, we first note that ψ is continuous since it can be considered as a map into the product $\Pi\{X_j : j \in \omega\}$ and its composition with each projection $\pi_j \circ \psi = p_j \circ \rho_j$ is continuous. To show that ψ is one-to-one, suppose that $u, v \in L$ are distinct, but $\psi(u) = \psi(v)$. Then $p_n(\rho_n(u)) = p_n(\rho_m(v))$ for each $n \in \omega$ and hence $\rho_n(u)$ and $\rho_n(v)$ lie in the same edge of $G(X_n)$ for each $n \in \omega$. It follows immediately from the remarks preceding this theorem that $u = v$.

To show that $\psi(v)$ is closed in M for each $v \in L$, suppose to the contrary that $t \in \mathsf{cl}(\{\psi(v)\}) \setminus \{\psi(v)\}$. Then for each $n \in \omega$, $\pi_n(t) \in \mathsf{cl}(p_n \circ \rho_n(v))$ but for some $n_0 \in \omega$, $\pi_n(t) \neq p_n \circ \rho_n(v)$ for each $n > n_0$. Thus for each $n > n_0$, $p_n \circ \rho_n(v)$ is open and $\pi_n(t)$ is closed and $\pi_n(t) \in \mathsf{cl}(p_n \circ \rho_n(v))$. Furthermore, for $n \leq n_0$, $\pi_n(t) = \mathsf{cl}(p_n \circ \rho_n(v))$ and hence both lie in the closure of some open point of X_n. Since the maps p_n are surjective for each $n \in \omega$, for each $n \leq n_0$, we can choose $w_n \in p_n^{-1}[\pi_n(t)]$ and for each $n > n_0$, we let $w_n = p_n^{-1}[\pi_n(t)]$ and $w = (w_n)_{n\in\omega} \in L$. Then $\rho_n(v)$ and $\rho_n(w)$ lie in the same edge of $G(X_n)$ for each $n \in \omega$ and again by the remarks preceding the theorem, $v = w$; but then $\psi(v) = \psi(w) = t$, which contradicts our hypothesis.

Finally, we need to show that ψ maps onto the closed points of M. If $t \in M$ is closed, it follows that for all $s \in M$ distinct from t, there is some $n \in \omega$ such that $\pi_n(s) \notin \mathsf{cl}[\pi_n(t)]$. We consider the set $A = \cap\{(p_n \circ \rho_n)^{-1}[\mathsf{cl}(\pi_n(t))] : n \in \omega\} \subset L$. Note first that A is the intersection of a nested sequence of closed subsets of the compact T_2-space L and hence $A \neq \emptyset$. We now claim that if $r \in A$, then $\psi(r) = t$. To prove our claim, note that $r \in (p_n \circ \rho_n)^{-1}[\mathsf{cl}(\pi_n(t))]$ for each $n \in \omega$ and hence $(p_n \circ \rho_n)(r) = (\pi_n \circ \psi)(r) \in \mathsf{cl}[\pi_n(t)]$ for each n. Thus $\psi(r) = t$ and we are done. □

Combining the above result with Corollary 2.4 we obtain:

Corollary 2.6. *Each one dimensional continuum is the Hausdorff reflection of the limit of a separating inverse sequence of one dimensional T_0-spaces.*

Finally, we note that the inverse sequence g_n constructed prior to Theorem 1.3 and used in that theorem is separating and hence we have proved:

Theorem 2.7. *A topological space is a continuum of dimension one (or a singleton) if and only if it is the Hausdorff reflection of a separating inverse sequence of one dimensional T_0 spaces.*

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