Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

Topology Proceedings
Volume 24, Summer 1999, 267–294

CONTINUOUS DOMAINS WITH APPROXIMATING MAPPINGS AND THEIR UNIFORMITY

Ralph Kummetz*

Abstract

We study the interplay of order and topology in the context of approximating F-posets $(D, \leq, \mathcal{F})$. These consist of a poset $(D, \leq)$ and a directed family $\mathcal{F}$ of monotone mappings below the identity with $\sup \mathcal{F} = \mathsf{id}_D$ such that for all $f \in \mathcal{F}$ there is some $g \in \mathcal{F}$ with $f \leq g \circ g$. F-posets give rise to a uniformity whose properties are closely related to properties of $(D, \leq)$ and $\mathcal{F}$. We show that $(D, \leq)$ is a continuous dcpo such that $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$ if and only if each monotone net in D converges with respect to the uniform topology. Moreover, we prove that a pointed poset is an FS-domain if and only if it arises as an approximating F-poset whose uniform topology is compact. In this case we also obtain that the uniform topology coincides with the Lawson topology of the domain.

* Supported by DFG/Graduiertenkolleg.
Mathematics Subject Classification: 54E75, 54F05, 06B35, 06F30
Key words: Poset with approximating mappings, uniformity induced by self-maps, convergence of monotone nets, continuous domain, FS - domain

1. Introduction

When discovering T_0-topological spaces homeomorphic to their own function space and closely related to partially ordered sets, Scott [12] presented the first models of the type-free λ-calculus. In connection with this, special posets ("domains") together with their Scott topology are of great importance. The maximal cartesian closed categories both of algebraic and continuous domains were classified by Jung [6, 7]. One of them is the category of FS-domains. These are continuous dcpo's admitting a certain directed set of Scott-continuous mappings. Recently, it turned out that FS-domains have a nice topological characterization: Jung and Sünderhauf [8] showed that they appear exactly as what they call uniformly approximated spaces, see [8] for details. This characterization again uses the Scott topology of FS-domains.

In this paper we investigate a larger class of posets, which contains all FS-domains, and uniformities naturally associated with them. Such a uniformity arises from a particular family of mappings that can be used to approximate elements of the poset order-theoretically. Our main result establishes a close relationship between properties of the poset and the uniformity. As a consequence, we derive a new characterization of FS-domains in which the uniform topology plays a crucial rôle.

The objects we deal with are triples $(D, \leq, \mathcal{F})$ with $(D, \leq)$ being a partially ordered set and $\mathcal{F}$ being a directed family of monotone mappings of D with some additional properties (see Definition 2.3 below). We call them F-posets. If, furthermore, we have $\sup \mathcal{F} = \mathsf{id}_D$ with respect to the pointwise ordering of mappings, then we say that the F-poset $(D, \leq, \mathcal{F})$ is approximating. As we shall show, the sets $\{(d, e) \in D^2 \mid f(d) \leq e,\ f(e) \leq d\}$ with $f \in \mathcal{F}$ form a basis for a uniformity $\mathcal{U}_{\mathcal{F}}$ on D. We call it the F-uniformity. The induced topology is called the F-topology.

The very special case of F-posets $(D, \leq, \mathcal{F})$ in which all mappings $f \in \mathcal{F}$ are idempotent was investigated in [11], where most of the results concern algebraic dcpo's. Here we arrive at the larger class of continuous dcpo's.

Now we give a summary of our results. In the following section we study basic properties such as pseudo-metrizability and total boundedness of the F-uniformity. Several examples from analysis and domain theory are given to illustrate that F-posets occur quite naturally in certain branches of mathematics.

The main result of the third section (Theorem 3.9) relates domain-theoretic properties of approximating F-posets $(D, \leq, \mathcal{F})$ such as directed completeness and order continuity to uniform completeness and convergence properties of monotone nets. In particular, we show that $(D, \leq)$ is a continuous dcpo with $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$ if and only if each monotone net in D has a limit with respect to the F-topology.

The last section is devoted to approximating F-posets that are compact in their F-topology. We prove that in this case the Lawson topology of $(D, \leq)$ coincides with the F-topology (Theorem 4.3). Also, we obtain that FS-domains appear precisely as compact approximating F-posets with least element (Corollary 4.6).

The usual definitions concerning uniformities and topologies can be found e.g. in Bourbaki [2] and Kelley [10]. Further, for a survey on domain theory the reader is referred to Abramsky and Jung [1].

2. Definition and Basic Properties of F-posets

Let us begin by recalling basic notation. We define topologies by means of open sets. If not mentioned otherwise, no separation properties are assumed.

Throughout this paper let $(D, \leq)$ be a partially ordered set. For a subset $A \subseteq D$ and an element $d \in D$ we write $A \leq d$ if $a \leq d$ for all $a \in A$. If $A \subseteq D$ has a least element, then it is denoted by $\min A$. If D has a least element, then $(D, \leq)$ is said to be pointed. A subset $A \subseteq D$ is a lower set if $b \leq a \in A$ implies $b \in A$. A chain that is order-isomorphic to the set of natural numbers with the usual linear order is called an ω-chain. For all $d \in D$ the set $d{\downarrow} := \{e \in D \mid e \leq d\}$ is the principal ideal generated by d. Similarly, $d{\uparrow} := \{e \in D \mid d \leq e\}$ is the principal filter generated by d. An element $d \in D$ is called way below an element $e \in D$ if for all directed sets $A \subseteq D$ that have a supremum with $\sup A \geq e$ we can find some $a \in A$ with $a \geq d$. In this case we write $d \ll e$. Let $d{\Downarrow}$ denote the set of all elements of D way below d. Similarly, $d{\Uparrow}$ is the set of all $e \in D$ with $d \ll e$. A subset $B \subseteq D$ is a basis for $(D, \leq)$ if for all $d \in D$ the set $B \cap d{\Downarrow}$ is directed and has d as supremum. A poset $(D, \leq)$ is continuous if it has a basis. A dcpo is a poset in which all directed subsets have a supremum.

Let $f : D \longrightarrow D$ be a mapping. Then f is Scott-continuous if it preserves suprema of directed sets (in case these suprema exist). We say that f is below the identity if $f \leq \mathsf{id}_D$ with respect to the pointwise ordering of mappings, i.e. $f(d) \leq d$ for all $d \in D$. We call f a projection if f is a monotone and idempotent mapping below the identity. The kernel of f is the set $\ker f := \{(d, e) \in D^2 \mid f(d) = f(e)\}$. Furthermore, we define the set $B_f := \{(d, e) \in D^2 \mid f(d) \leq e,\ f(e) \leq d\}$. Clearly, B_f is a symmetric binary relation on D. We set

$$B_f(d) := \{e \in D \mid f(d) \leq e,\ f(e) \leq d\} \text{ for all } d \in D.$$

We shall mainly consider sets B_f with f being a monotone mapping below the identity. Then B_f can be illustrated as in Figure 1.

Lemma 2.1. *Let $f : D \longrightarrow D$ be a mapping and let $d \in D$. Then we have $B_f(d) = f(d){\uparrow} \cap f^{-1}[d{\downarrow}]$. If f is below the identity, then $(d, f(d)) \in B_f$ and $f(d) = \min B_f(d)$.*

Proof. Straightforward. □

Corollary 2.2. *Let $f, g : D \longrightarrow D$ be below the identity. Then $f \leq g$ if and only if $B_f \supseteq B_g$.*

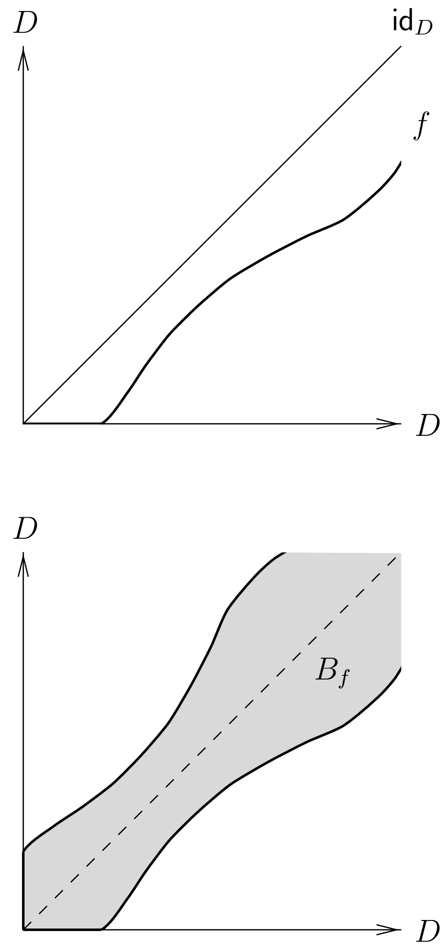

Fig. 1. The set B_f.

Proof. First let $f \leq g$ and let $(d, e) \in B_g$. Then $f(d) \leq g(d) \leq e$ and $f(e) \leq g(e) \leq d$, i.e. $(d, e) \in B_f$.

Now let $B_f \supseteq B_g$ and let $d \in D$. As $(d, g(d)) \in B_g \subseteq B_f$ (Lemma 2.1), we conclude $f(d) \leq g(d)$. □

The succeeding definition is central for this paper:

Definition 2.3. *Let $(D, \leq)$ be a poset and let $\mathcal{F}$ be a directed family of monotone mappings below the identity with the following property: For all $f \in \mathcal{F}$ there is some $g \in \mathcal{F}$ such that $f \leq g \circ g$. Then we call the triple $(D, \leq, \mathcal{F})$ an* F-poset. *It is said to be* approximating *if the pointwise supremum* $\sup \mathcal{F}$ *exists and* $\sup \mathcal{F} = \mathsf{id}_D$.

To give the reader a first impression, we mention the following two basic examples. Further examples will be presented in the sequel.

Example 2.4. The reals $\mathbb{R}$ together with their usual order can be easily turned into an approximating F-poset. For all $\varepsilon > 0$ and all $x \in \mathbb{R}$ let $f_\varepsilon(x) := x - \varepsilon$. Clearly, f_ε is a monotone mapping below the identity, $f_\varepsilon = f_{\frac{\varepsilon}{2}} \circ f_{\frac{\varepsilon}{2}}$, and $f_\delta \geq f_\varepsilon$ for all $0 < \delta \leq \varepsilon$. As $\sup_{\varepsilon > 0}(x - \varepsilon) = x$ for all $x \in \mathbb{R}$, we obtain $\underline{D_{\mathbb{R}}} := (\mathbb{R}, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ to be an approximating F-poset.

Example 2.5. Pop's as studied in [11] are special F-posets. A pop $(D, \leq, \mathcal{P})$ consists of a poset $(D, \leq)$ and a directed family $\mathcal{P}$ of projections. Idempotency of the projections implies that $(D, \leq, \mathcal{P})$ is an F-poset. Note that an F-poset $(D, \leq, \mathcal{F})$ is a pop if and only if $f \leq f \circ f$ for all $f \in \mathcal{F}$ because we always have $f \circ f \leq f$.

A Uniformity for F-posets

Every F-poset $(D, \leq, \mathcal{F})$ gives rise to a canonical uniformity on D. As $\mathcal{F}$ is directed, $\{B_f \mid f \in \mathcal{F}\}$ is a filter base by Corollary 2.2. Let $f \in \mathcal{F}$. We know that B_f is symmetric. Since f is below the identity, B_f is also reflexive. Choose some $g \in \mathcal{F}$ with $f \leq g \circ g$. Then it is straightforward to see that $B_g \circ B_g \subseteq B_f$. We obtain:

Proposition 2.6. *Let $(D, \leq, \mathcal{F})$ be an F-poset. Then $\{B_f \mid f \in \mathcal{F}\}$ is a basis for a uniformity $\mathcal{U}_{\mathcal{F}}$ on D.*

For any F-poset $\underline{D} = (D, \leq, \mathcal{F})$ we call $\mathcal{U}_{\underline{D}} := \mathcal{U}_{\mathcal{F}}$ the F-uniformity of $\underline{D}$. The induced topology $\tau_{\underline{D}}$ is the F-topology of $\underline{D}$.

If f is a projection, then it is easy to see that $B_f = \mathsf{ker}\, f$. Thus, the "pop uniformity" of a pop $(D, \leq, \mathcal{P})$, which is generated by the kernels $\mathsf{ker}\, p$ of all projections $p \in \mathcal{P}$ (cf. [11]), is precisely the F-uniformity of $(D, \leq, \mathcal{P})$.

Observe that the F-uniformity is discrete if and only if $\mathsf{id}_D \in \mathcal{F}$. Nevertheless, the F-topology can be discrete although the F-uniformity is not.

Next, we give a simple necessary and sufficient condition for $\mathcal{U}_{\mathcal{F}}$ to be pseudo-metrizable:

Proposition 2.7. *The F-uniformity of an F-poset* $(D, \leq, \mathcal{F})$ *is pseudo-metrizable if and only if* $\mathcal{F}$ *contains a greatest element or a cofinal* ω*-chain.*

Proof. First let $\mathcal{U}_{\mathcal{F}}$ be pseudo-metrizable. Then $\mathcal{U}_{\mathcal{F}}$ has a countable basis $\{A_n \mid n \in \mathbb{N}\}$. Choose some $f_1 \in \mathcal{F}$ with $B_{f_1} \subseteq A_1$. Let $n \in \mathbb{N}$ and let $f \in \mathcal{F}$ such that $B_f \subseteq A_{n+1}$. We find a mapping $f_{n+1} \in \mathcal{F}$ with $f_{n+1} \geq f, f_n$. Applying Corollary 2.2 we obtain $B_{f_{n+1}} \subseteq B_f \subseteq A_{n+1}$ and $B_{f_{n+1}} \subseteq B_{f_n}$. Thus, we have defined a $\subseteq$-decreasing sequence $(B_{f_n})_{n \in \mathbb{N}}$ with $B_{f_n} \subseteq A_n$ for all $n \in \mathbb{N}$. Let $N := \{f_n \mid n \in \mathbb{N}\}$. Let $f \in \mathcal{F}$. Then there is some $n \in \mathbb{N}$ with $A_n \subseteq B_f$, whence $B_{f_n} \subseteq A_n \subseteq B_f$ and $f_n \geq f$ by 2.2. This shows us that N is cofinal in $\mathcal{F}$. Now the assertion follows from the fact that $f_m \leq f_n$ for all $m \leq n$.

To prove the "if-part", note that $\mathcal{F}$ has a countable cofinal chain $\mathcal{C}$. Using Corollary 2.2, one easily sees that $\{B_f \mid f \in \mathcal{C}\}$ is a countable basis for $\mathcal{U}_{\mathcal{F}}$; whence $\mathcal{U}_{\mathcal{F}}$ is pseudo-metrizable. □

Note that for countable $\mathcal{F}$ without greatest element we can find a cofinal ω-chain by induction.

With the help of the following definition, which is due to Jung [7], we are able to characterize when $(D, \mathcal{U}_{\mathcal{F}})$ is a totally bounded space:

Definition 2.8. *A mapping $f : D \longrightarrow D$ is* finitely separated *from* id_D *if there is a finite set $M \subseteq D$ such that for all $d \in D$ there is some $m \in M$ with $f(d) \leq m \leq d$. Such a set M is called a* finite separating set *of f and* id_D*.*

Proposition 2.9. *Let $(D, \leq, \mathcal{F})$ be an F-poset. Then $(D, \mathcal{U}_{\mathcal{F}})$ is totally bounded if and only if each $f \in \mathcal{F}$ is finitely separated from* id_D*.*

Proof. Let $(D, \mathcal{U}_{\mathcal{F}})$ be totally bounded. Let $f \in \mathcal{F}$ and choose some $g \in \mathcal{F}$ such that $f \leq g \circ g$. Due to total boundedness there is a finite subset $M' \subseteq D$ with $D = \bigcup_{m' \in M'} B_g(m')$. Hence, for all $d \in D$ there is an element $m' \in M'$ such that $g(d) \leq m'$ and $g(m') \leq d$; whence $f(d) \leq g(g(d)) \leq g(m') \leq d$. Consequently, $M := g[M']$ is a finite separating set of f and id_D.

To prove the converse, let $f \in \mathcal{F}$ and let M be a finite separating set of f and id_D. Let $d \in D$. Then there is some $m \in M$ with $f(d) \leq m \leq d$ and thus $f(m) \leq f(d) \leq d$. This implies $d \in B_f(m)$. We infer that $D = \bigcup_{m \in M} B_f(m)$ and obtain $(D, \mathcal{U}_{\mathcal{F}})$ to be totally bounded. □

Basic Properties of the F-topology

As the sets $B_f(d)$, $f \in \mathcal{F}$, form a $\tau_{\underline{D}}$-neighbourhood basis of $d \in D$, we immediately obtain:

Remark 2.10. *A net $(d_n)_{n \in N}$ of an F-poset $(D, \leq, \mathcal{F})$ converges to some $d \in D$ with respect to the F-topology if and only if, for all $f \in \mathcal{F}$, there is an index $n_f \in N$ such that $f(d) \leq d_n$ and $f(d_n) \leq d$ for all $n \geq n_f$.*

Proposition 2.11. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset.*

(1) *For all $d \in D$ the net $(f(d))_{f \in \mathcal{F}}$ converges to d. In particular, $\bigcup_{f \in \mathcal{F}} f[D]$ is dense in D.*

(2) *Let $d \in D$. If $e = \sup_{f \in \mathcal{F}} f(d)$ exists, then $e = \min \bigcap_{f \in \mathcal{F}} B_f(d) = \min \overline{\{d\}}$.*

(3) $(D, \tau_{\underline{D}})$ *is Hausdorff if and only if, for all* $d, e \in D$, *the following holds: if* $f(d) \le e$ *and* $f(e) \le d$ *for all* $f \in \mathcal{F}$, *then* $d = e$.

Proof. (1) Let $f \in \mathcal{F}$. Then $f(d) \le g(d)$ and $f(g(d)) \le d$ for all $g \in \mathcal{F}$, $g \ge f$.

(2) It is clear from topology that $\bigcap_{f \in \mathcal{F}} B_f(d) = \overline{\{d\}}$. Let $e := \sup_{f \in \mathcal{F}} f(d)$. Then $f(e) \le f(d) \le e \le d$ for all $f \in \mathcal{F}$ and thus $e \in \bigcap_{f \in \mathcal{F}} B_f(d)$. Let $\widetilde{e} \in \bigcap_{f \in \mathcal{F}} B_f(d)$. By Lemma 2.1 we have $f(d) = \min B_f(d) \le \widetilde{e}$ for all $f \in \mathcal{F}$, whence $e \le \widetilde{e}$.

(3) is obvious because $(D, \tau_{\underline{D}})$ is Hausdorff if and only if $\bigcap_{f \in \mathcal{F}} B_f = \mathsf{id}_D$. □

The property of an F-poset to be approximating can be characterized topologically as follows:

Proposition 2.12. *Let* $\underline{D} = (D, \le, \mathcal{F})$ *be an F-poset. Then the following are equivalent:*

(i) $\underline{D}$ *is approximating.*

(ii) $(D, \tau_{\underline{D}})$ *is Hausdorff and the pointwise supremum* $\sup \mathcal{F}$ *exists.*

(iii) $\le$ *is closed in* D^2, *i.e.* $(D, \le, \tau_{\underline{D}})$ *is a partially ordered space.*

Proof. (i)→(ii): Let $d, e \in D$ such that $f(d) \le e$ and $f(e) \le d$ for all $f \in \mathcal{F}$. Then $d = \sup_{f \in \mathcal{F}} f(d) \le e = \sup_{f \in \mathcal{F}} f(e) \le d$ by (i) and thus $d = e$. Proposition 2.11(3) tells us that $(D, \tau_{\underline{D}})$ is Hausdorff.

(ii)→(i) follows from 2.11(2).

(i)→(iii): Let $d, e \in D$ and let $(d_n)_{n \in N}, (e_n)_{n \in N}$ be nets in D with $(d_n)_{n \in N} \longrightarrow d$, $(e_n)_{n \in N} \longrightarrow e$ and $d_n \le e_n$ for all $n \in N$. Let $f \in \mathcal{F}$ and choose a mapping $g \in \mathcal{F}$ with $f \le g \circ g$. There is an index $n_0 \in N$ such that $g(d) \le d_{n_0}$ and $g(e_{n_0}) \le e$. Therefore, $f(d) \le g(g(d)) \le g(d_{n_0}) \le g(e_{n_0}) \le e$. We deduce that $f(d) \le e$ for all $f \in \mathcal{F}$ and thus $d = \sup_{f \in \mathcal{F}} f(d) \le e$.

(iii)→(i): Let $d, e \in D$ such that $f(d) \leq e$ for all $f \in \mathcal{F}$. As $(f(d))_{f\in\mathcal{F}} \longrightarrow d$ (Proposition 2.11 (1)), (iii) yields $d \leq e$. Consequently, $d = \sup_{f\in\mathcal{F}} f(d)$. □

Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset and assume that the underlying uniform space $(D, \mathcal{U}_{\underline{D}})$ (topological space $(D, \tau_{\underline{D}})$, respectively) has some property E. Then, for the sake of simplicity, we say that $\underline{D}$ has property E. For instance, if $(D, \mathcal{U}_{\underline{D}})$ is totally bounded, then we say that $\underline{D}$ is totally bounded.

If not explicitly stated otherwise, all mentioned uniform (topological, respectively) properties refer to the F-uniformity (F-topology, respectively).

On a poset $(D, \leq)$ several topologies can be defined, see e.g. the compendium [5]. The upper topology of $(D, \leq)$ is generated by the subbasis $\{D\backslash d{\downarrow} \mid d \in D\}$. Dually, the lower topology is generated by $\{D\backslash d{\uparrow} \mid d \in D\}$. A subset $A \subseteq D$ is Scott-closed if it is a lower set closed under suprema of directed subsets of A (whenever these suprema exist). Complements of Scott-closed sets are Scott-open. The family of all Scott-open sets forms the Scott topology $\sigma_{(D,\leq)}$. The Lawson topology $\lambda_{(D,\leq)}$ is the join of the Scott- and the lower topology, i.e. it is generated by the subbasis $\{O \subseteq D \mid O \text{ Scott-open}\} \cup \{D \setminus d{\uparrow} \mid d \in D\}$. Now we relate the F-topology to these topologies:

Proposition 2.13. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset.*

(1) *Upper and lower topology are coarser than the F-topology: Principal ideals and principal filters are closed with respect to the F-topology.*

(2) *The Scott topology is coarser than the F-topology if and only if the Lawson topology is coarser than the F-topology.*

Proof. (1) follows immediately from Proposition 2.12 (i)→(iii). (2) is a consequence of (1) and the definition of the Lawson topology. □

For an important class of F-posets we shall show that the Lawson topology is equal to the F-topology, see Theorems 3.12 and 4.3 below.

Examples

We finish this section by giving several examples of F-posets and by investigating their topology.

Example 2.14. Let $\underline{D_{\mathbb{R}}}$ be the approximating F-poset as given in Example 2.4. Then, for all $\varepsilon > 0$, we have $B_{f_\varepsilon} = \{(x,y) \in \mathbb{R}^2 \mid |x-y| \leq \varepsilon\}$; whence the F-uniformity coincides with the Euclidean uniformity and thus the F-topology is the Euclidean topology.

Example 2.15. Consider the unit interval $[0,1]$ with the usual linear order. Let I be a non-empty set and equip $[0,1]^I$ with the product order. For all $n \in \mathbb{N}$, all finite $I_0 \subseteq I$ and all $(x_i)_{i\in I} \in [0,1]^I$ let $f_{n,I_0}((x_i)_{i\in I}) := (y_i)_{i\in I}$ with $y_i = \max\{0, x_i - \frac{1}{n}\}$ if $i \in I_0$ and $y_i = 0$ otherwise. Let $\mathcal{F} := \{f_{n,I_0} \mid n \in \mathbb{N},\ I_0 \subseteq I \text{ finite}\}$. It is routine to check that $([0,1]^I, \leq, \mathcal{F})$ is an approximating F-poset. One easily sees that $B_{f_{n,I_0}} = \{((x_i)_{i\in I}, (y_i)_{i\in I}) \in ([0,1]^I)^2 \mid |x_i - y_i| \leq \frac{1}{n} \text{ for all } i \in I_0\}$; whence $\mathcal{U}_{\mathcal{F}}$ coincides with the product uniformity of the family $([0,1], \mathcal{U}_{|.|})_{i\in I}$, where $\mathcal{U}_{|.|}$ is the Euclidean uniformity on $[0,1]$. In particular, the F-topology is the product topology on $[0,1]^I$ with $[0,1]$ carrying the Euclidean topology. Therefore, this F-poset is compact. Proposition 2.9 tells us that the mappings $f_{n,I_0} \in \mathcal{F}$ are finitely separated from $\mathsf{id}_{[0,1]^I}$. Note that $([0,1]^I, \leq)$ is an FS-domain in the sense of Jung [7], cf. also Section 4.

In the case that $|I| = 1$ we obtain the approximating F-poset $\underline{D_{[0,1]}} := ([0,1], \leq, \{f_n \mid n \in \mathbb{N}\})$ with $f_n(x) = \max\{0, x - \frac{1}{n}\}$ $(x \in [0,1],\ n \in \mathbb{N})$. Its uniformity (topology) coincides with the Euclidean uniformity (Euclidean topology) on $[0,1]$.

Example 2.16. In contrast to pop's ([11]), the mappings in $\mathcal{F}$ need not be continuous with respect to the F-topology: Consider again the unit interval $([0,1], \leq)$. For all $n \in \mathbb{N}$ let

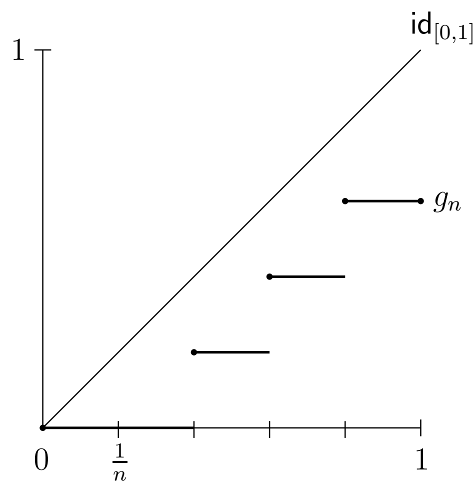

Fig. 2. The mappings g_n.

$g_n : [0,1] \longrightarrow [0,1]$ be defined as follows (cf. Figure 2): $g_1(x) := 0$ for all $x \in [0,1]$. For all $n \geq 2$ let $g_n(x) := 0$ if $x \in [0, \frac{1}{n})$, $g_n(x) := \frac{i}{n}$ if $x \in [\frac{i+1}{n}, \frac{i+2}{n})$, $i \in \{0, \ldots, n-2\}$, and $g_n(x) := \frac{n-2}{n}$ if $x = 1$. Clearly, g_n is monotone and below $\mathsf{id}_{[0,1]}$ for all $n \in \mathbb{N}$. It is easy to check that $g_n \leq g_{2n} \circ g_{2n}$ and $g_m \leq g_n$ for all $m, n \in \mathbb{N}$, $m \leq n$. Further, $\sup_{n \in \mathbb{N}} g_n = \mathsf{id}_{[0,1]}$. Therefore, $\underline{D} = ([0,1], \leq, \{g_n \mid n \in \mathbb{N}\})$ is an approximating F-poset. Let us compare $\underline{D}$ with the approximating F-poset $\underline{D_{[0,1]}} = ([0,1], \leq, \{f_n \mid n \in \mathbb{N}\})$ at the end of Example 2.15. Obviously, $g_n \leq f_n$ for all $n \in \mathbb{N}$, whence $\tau_{\underline{D}} \subseteq \tau_{\underline{D_{[0,1]}}}$ (cf. Corollary 2.2). As $\tau_{\underline{D_{[0,1]}}}$ is the Euclidean topology and therefore compact Hausdorff, we obtain $\tau_{\underline{D}} = \tau_{\underline{D_{[0,1]}}} = \lambda_{([0,1],\leq)}$.

For the following example we need some basic facts about C*-algebras. We refer the reader to the standard literature, cf. e.g. Dixmier [3], Kadison and Ringrose [9].

Example 2.17. Let A be a unital C*-algebra with identity e. For all $x \in \mathsf{A}$ let $\sigma(x)$ be the spectrum of x, i.e. $\sigma(x) := \{\lambda \in \mathbb{C} \mid x - \lambda e \text{ is not invertible}\}$. An element $x \in \mathsf{A}$ is called self-adjoint if $x = x^*$, where $*$ denotes the involution of the C*-algebra A. Let D be the real vector space of all self-adjoint elements of A. Clearly, D is closed in A with respect to the norm topology. As usual, x is said to be positive if $\sigma(x)$ is contained in the non-negative reals. The convex cone $P := \{x \in D \mid x \text{ positive}\}$ induces a partial order $\leq$ on D via $x \leq y$ if $y - x \in P$.

Next, we turn $(D, \leq)$ into an F-poset by letting $f_\varepsilon(x) := x - \varepsilon e$ for all $x \in D$ and all $\varepsilon > 0$. Obviously, this yields a monotone mapping $f_\varepsilon : D \longrightarrow D$. As $e \in P$ and $\varepsilon > 0$, we have $\varepsilon e \in P$; whence $f_\varepsilon \leq \mathsf{id}_D$. Clearly, $f_\varepsilon = f_{\frac{\varepsilon}{2}} \circ f_{\frac{\varepsilon}{2}}$. Since $f_\delta \geq f_\varepsilon$ for all $0 < \delta \leq \varepsilon$, we infer that $\underline{D}_{\mathsf{A}} := (D, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ is an F-poset.

Recall that P is closed in the norm topology $\tau_{\|.\|}$. Thus, $\leq$ is closed in $(D^2, \tau^2_{\|.\|})$. As in the proof of Proposition 2.12 (iii)$\rightarrow$(i), we derive $\underline{D}_{\mathsf{A}}$ to be approximating. Note that in case of $\mathsf{A} = \mathbb{C}$ we obtain $\underline{D}_{\mathbb{C}} = \underline{D}_{\mathbb{R}}$ as in Example 2.4.

Proposition 2.18. *The F-uniformity of $\underline{D}_{\mathsf{A}}$ coincides with the norm uniformity of the C*-algebra A restricted to D. In particular, $\underline{D}_{\mathsf{A}}$ is complete and the F-topology is precisely the restriction of the norm topology to D.*

Proof. One easily verifies that

$$B_{f_\varepsilon} = \{(x, y) \in D^2 \mid -\varepsilon e \leq x - y \leq \varepsilon e\}.$$

The sets $E_\varepsilon := \{(x, y) \in D^2 \mid \|x - y\| \leq \varepsilon\}$, $\varepsilon > 0$, form a basis for the norm uniformity of A restricted to D. Therefore, it is sufficient to show that $B_{f_\varepsilon} = E_\varepsilon$ for all $\varepsilon > 0$. In fact, we prove that $-\varepsilon e \leq z \leq \varepsilon e$ if and only if $\|z\| \leq \varepsilon$ ($z \in D$, $\varepsilon > 0$). The "if-part" follows from the well-known inequality $-\|z\|e \leq z \leq \|z\|e$. To prove the converse, we remark that in virtue of the functional calculus in C*-algebras we have $\sigma(z + \varepsilon e) = \sigma(z) + \varepsilon$ and $\sigma(-z + \varepsilon e) = -\sigma(z) + \varepsilon$. Hence, $-\varepsilon e \leq z \leq \varepsilon e$ if and only if $z + \varepsilon e, -z + \varepsilon e \in P$ if and only if $\sigma(z) \subseteq [-\varepsilon, \varepsilon]$.

Recall that $\|z\| \in \sigma(z)$ or $-\|z\| \in \sigma(z)$. Therefore, $\|z\| \in [-\varepsilon, \varepsilon]$ or $-\|z\| \in [-\varepsilon, \varepsilon]$; whence $\|z\| \leq \varepsilon$. □

We conclude this section by equipping the formal ball model of Edalat and Heckmann [4] with a canonical F-poset structure:

Example 2.19. Let (X, ρ) be a metric space and let $D := X \times \mathbb{R}_{\geq 0}$. Define a partial order on D as follows ([4]): $(x, r) \leq (y, s) \;:\Leftrightarrow\; \rho(x, y) \leq r - s$. Recall from [4] that:
(1) $(D, \leq)$ is a continuous poset.
(2) $(D, \leq)$ is a dcpo if and only if (X, ρ) is complete.
(3) $(D, \leq)$ has a countable basis if and only if (X, ρ) is separable.

Now we define for all $\varepsilon > 0$ a mapping $f_\varepsilon : D \longrightarrow D$ by $f_\varepsilon((x, r)) := (x, r + \varepsilon)$. One easily sees that f_ε is monotone and below the identity and $f_\varepsilon = f_{\frac{\varepsilon}{2}} \circ f_{\frac{\varepsilon}{2}}$. Again, $0 < \delta \leq \varepsilon$ implies $f_\delta \geq f_\varepsilon$. Therefore, $\underline{D}_{fb}(X, \rho) := (D, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ is an F-poset.

Suppose that $(x, r + \varepsilon) \leq (y, s)$ for all $\varepsilon > 0$, i.e. $\rho(x, y) \leq r + \varepsilon - s$ for all $\varepsilon > 0$. Then $\rho(x, y) \leq r - s$ and $(x, r) \leq (y, s)$. This shows us that $\underline{D}_{fb}(X, \rho)$ is approximating.

Proposition 2.20. *The F-uniformity of $\underline{D}_{fb}(X, \rho)$ equals the product uniformity of $X \times \mathbb{R}_{\geq 0}$ and, in particular, the F-topology is the corresponding product topology.*

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned} B_{f_\varepsilon} &= \{((x, r), (y, s)) \in D^2 \,|\, \rho(x, y) \leq r + \varepsilon - s,\ \rho(x, y) \leq s + \varepsilon - r\} \\ &= \{((x, r), (y, s)) \in D^2 \,|\, \rho(x, y) + s - r \leq \varepsilon,\ \rho(x, y) + r - s \leq \varepsilon\} \\ &= \{((x, r), (y, s)) \in D^2 \,|\, \rho(x, y) + |r - s| \leq \varepsilon\}. \end{aligned}$$

Hence, the assertion follows. □

It follows immediately from the previous proposition that $\underline{D}_{fb}(X, \rho)$ is complete (separable, respectively) if and only if (X, ρ) is complete (separable, respectively).

3. Continuous Domains and Convergence of Monotone Nets

In this section we establish a close relationship between domain theory and topology in terms of F-posets. First we compare directed completeness (i.e. the dcpo property) with uniform completeness (Proposition 3.2). Then we consider approximating F-posets whose underlying poset is a dcpo with the property that the supremum of any directed set A belongs to the closure of A with respect to the F-topology. We give several characterizations of these F-posets, cf. Theorem 3.9 below. Eventually, we compare the F-topology to the Lawson topology (Theorem 3.12).

We begin with a description of Cauchy nets:

Lemma 3.1. *Let $(D, \leq, \mathcal{F})$ be an F-poset and let $(d_n)_{n\in N}$ be a net in D. Then the following are equivalent:*

(i) *$(d_n)_{n\in N}$ is a Cauchy net (with respect to the F-uniformity).*

(ii) *For all $f \in \mathcal{F}$ there is an index $n_0 \in N$ with $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_0$.*

(iii) *For all $f \in \mathcal{F}$ there is an index $n_0 \in N$ with $f(d_{n_0}) \leq d_n$ and $f(d_n) \leq d_{n_0}$ for all $n \geq n_0$.*

Proof. The equivalence of (i) and (ii) follows immediately from the definition of the F-uniformity. The implication (ii)→(iii) is trivial. To show (iii)→(ii), let $f \in \mathcal{F}$ and choose a mapping $g \in \mathcal{F}$ with $f \leq g \circ g$. By (iii) we find an index $n_0 \in N$ with $g(d_{n_0}) \leq d_n$ and $g(d_n) \leq d_{n_0}$ for all $n \geq n_0$. Let $m, n \in N$, $m, n \geq n_0$. Then $f(d_m) \leq g(g(d_m)) \leq g(d_{n_0}) \leq d_n$ and, analogously, $f(d_n) \leq d_m$. □

Next, we show that directed completeness implies uniform completeness provided that all mappings in $\mathcal{F}$ are Scott-continuous. For the special case of pop's, this can be found in [11], Prop. 3.1.

Proposition 3.2. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset with $(D, \leq)$ being a dcpo. Suppose further that all mappings $f \in \mathcal{F}$ are Scott-continuous. Then $\underline{D}$ is complete.*

Proof. Let $(d_n)_{n \in N}$ be a Cauchy net in D. For all $f \in \mathcal{F}$ fix $k_f \in \mathcal{F}$ and $n_f \in N$ with $f \leq k_f \circ k_f$ and $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_f$. Let $A := \{f(d_n) \mid n \geq n_{k_f}, f \in \mathcal{F}\}$. We show that A is directed: Let $f(d_m), g(d_n) \in A$. Let $h \in \mathcal{F}$ with $h \geq k_f, k_g$. Let $l \in N$ with $l \geq n_{k_f}, n_{k_g}, n_{k_h}$. Then $f(d_m) \leq k_f(k_f(d_m)) \leq k_f(d_l) \leq h(d_l) \in A$. Similarly, $g(d_n) \leq h(d_l)$.

We prove that $(d_n)_{n \in N}$ converges to $\sup A$. Let $f \in \mathcal{F}$. Define $B := \{g(d_m) \mid m \geq n_{k_g}, n_f,\ g \in \mathcal{F}\}$. Clearly, $B \subseteq A$. Now B is cofinal in A because if $f(d_n) \in A$, then let $m \in N$ with $m \geq n_{k_{(k_f)}}, n_{k_f}, n_f$. We have $f(d_n) \leq k_f(k_f(d_n)) \leq k_f(d_m)$ because $m, n \geq n_{k_f}$ and $k_f(d_m) \in B$ since $m \geq n_{k_{(k_f)}}, n_f$. Next, for all $n \geq n_f$ we have $f[B] \leq d_n$. To see this, let $g(d_m) \in B$. Then $f(g(d_m)) \leq f(d_m) \leq d_n$ for all $n \geq n_f$. Summing up we deduce $f(\sup A) = f(\sup B) = \sup f[B] \leq d_n$ for all $n \geq n_f$. Furthermore, for all $n \geq n_{k_f}$ we have $f(d_n) \in A$, whence $f(d_n) \leq \sup A$. This yields $(d_n)_{n \in N} \longrightarrow \sup A$. □

Note that in general Scott-continuity of the mappings $f \in \mathcal{F}$ is indispensable for Proposition 3.2. A counterexample using pop's can be found in [11] (after Prop. 3.1). Example 2.4 (together with 2.14) yields an F-poset which is complete, but whose underlying poset (viz. $(\mathbb{R}, \leq)$) is not a dcpo.

During a first course in calculus the students usually become acquainted with the following easy lemma about suprema of subsets of the reals:

Let $A \subseteq \mathbb{R},\ x \in \mathbb{R}$. Then $x = \sup A$ if and only if $A \leq x$ and for all $\varepsilon > 0$ there is some $a \in A$ with $x - \varepsilon \leq a$.

Here we prove an analogous statement for approximating F-posets. For this we need a simple lemma which is also well-known in calculus:

Lemma 3.3. *Let $(D, \leq, \mathcal{F})$ be an F-poset, let $(d_n)_{n\in N}$, $(e_n)_{n\in N}$, $(x_n)_{n\in N}$ be nets in D, and let $d \in D$ with $(d_n)_{n\in N} \longrightarrow d$, $(e_n)_{n\in N} \longrightarrow d$. Further, let $d_n \leq x_n \leq e_n$ for all $n \in N$. Then $(x_n)_{n\in N} \longrightarrow d$.*

Proof. Let $f \in \mathcal{F}$. We find some $n_0 \in N$ with $f(d) \leq d_n$ and $f(e_n) \leq d$ for all $n \geq n_0$. Therefore, $f(d) \leq x_n$ and $f(x_n) \leq d$ for all $n \geq n_0$. □

Lemma 3.4. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset, let $A \subseteq D$, and let $d \in D$.*

(1) *The following are equivalent:*

   (i) $d = \sup A$ *and* $d \in \overline{A}$.

   (ii) $A \leq d$ *and for all* $f \in \mathcal{F}$ *there exists some* $a_f \in A$ *with* $f(d) \leq a_f$.

(2) *If* $A \leq d$ *and if* $(a_f)_{f\in\mathcal{F}}$ *is a net in* A *satisfying* $f(d) \leq a_f$ *for all* $f \in \mathcal{F}$, *then* $d = \sup A = \lim(a_f)_{f\in\mathcal{F}}$.

Proof. We show (1). The proof of (2) is evident by having a close look at the following conclusions for the implication (ii)→(i) in (1).

To check (i)→(ii), let $f \in \mathcal{F}$. As $d \in \overline{A}$, we find an element $a_f \in A \cap B_f(d)$; whence $f(d) \leq a_f$.

In order to prove (ii)→(i), let $f \in \mathcal{F}$ and choose a_f as in (ii). Then we have $f(d) \leq a_f \leq d$ for all $f \in \mathcal{F}$. Since $(f(d))_{f\in\mathcal{F}} \longrightarrow d$ (Proposition 2.11(1)), we conclude that $(a_f)_{f\in\mathcal{F}} \longrightarrow d$ due to Lemma 3.3. In particular, $d \in \overline{A}$. Let $A \leq e$ for some $e \in D$. As $f(d) \leq a_f \leq e$ for all $f \in \mathcal{F}$, we obtain $d = \sup_{f\in\mathcal{F}} f(d) \leq e$. □

Now we characterize approximating F-posets in which suprema of directed sets A (if they exist) lie in the closure of A:

Proposition 3.5. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent:*

(i) *For all directed subsets $A \subseteq D$ admitting a supremum we have* $\sup A \in \overline{A}$.

(ii) $f(d) \ll d$ *for all* $f \in \mathcal{F}$ *and all* $d \in D$.

In this case $(D, \leq)$ *is a continuous poset with basis* $\bigcup_{f\in\mathcal{F}} f[D]$.

Proof. (i)→(ii): Let $f \in \mathcal{F}$, let $d \in D$, and let $A \subseteq D$ be directed with $\sup A \geq d$. We infer from Lemma 3.4(1) that $f(d) \leq f(\sup A) \leq a$ for some $a \in A$.

(ii)→(i): Let $A \subseteq D$ be directed with supremum. Let $f \in \mathcal{F}$. As $f(\sup A) \ll \sup A$ by (ii), we find some $a_f \in A$ with $f(\sup A) \leq a_f$. Then $\sup A \in \overline{A}$ due to 3.4(1).

Now assume that (i) and (ii) are fulfilled and let $d \in D$, let $f_1, f_2 \in \mathcal{F}$, and let $x_1, x_2 \in D$ with $f_1(x_1) \ll d$, $f_2(x_2) \ll d$. Since $d = \sup_{f\in\mathcal{F}} f(d)$, we find some $g_1, g_2 \in \mathcal{F}$ with $f_1(x_1) \leq g_1(d)$ and $f_2(x_2) \leq g_2(d)$. Choose a mapping $h \in \mathcal{F}$ with $h \geq g_1, g_2$. Then $f_1(x_1), f_2(x_2) \leq h(d) \in \bigcup_{f\in\mathcal{F}} f[D] \cap d \twoheadleftarrow$ Hence, $\bigcup_{f\in\mathcal{F}} f[D] \cap d \twoheadleftarrow$ is directed. As $(D, \leq, \mathcal{F})$ is approximating and $\{f(d) \mid f \in \mathcal{F}\} \subseteq \bigcup_{f\in\mathcal{F}} f[D] \cap d \twoheadleftarrow$ it is clear that we have $\sup(\bigcup_{f\in\mathcal{F}} f[D] \cap d \twoheadleftarrow) = \sup_{f\in\mathcal{F}} f(d) = d$. □

The following lemma deals with the *existence* of suprema:

Lemma 3.6. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $A \subseteq D$. Then the following are equivalent:*

(i) *A has a supremum and* $\sup A \in \overline{A}$.

(ii) *There exists a convergent net $(a_f)_{f\in\mathcal{F}}$ such that $a_f \in A$ and $f[A] \leq a_f$ for all $f \in \mathcal{F}$.*

In this case we obtain $\sup A = \lim(a_f)_{f\in\mathcal{F}}$.

Proof. (i)→(ii): Due to Lemma 3.4(1) we find for all $f \in \mathcal{F}$ an element $a_f \in A$ with $f(\sup A) \leq a_f$. This yields a net $(a_f)_{f\in\mathcal{F}}$ that converges to $\sup A$ (Lemma 3.4(2)). Since f is monotone, we have $f[A] \leq f(\sup A) \leq a_f$ for all $f \in \mathcal{F}$.

(ii)→(i): Let $d := \lim(a_f)_{f\in\mathcal{F}}$. Let $a \in A$. By assumption, $f(a) \leq a_f$ for all $f \in \mathcal{F}$. As $(f(a))_{f\in\mathcal{F}} \longrightarrow a$ (Proposition 2.11(1)), we infer from Proposition 2.12 that $a \leq d$. Thus $A \leq d$. Since $(a_f)_{f\in\mathcal{F}} \longrightarrow d$, we obtain in particular that for all $f \in \mathcal{F}$ there is some $g_f \in \mathcal{F}$ with $f(d) \leq a_{g_f}$. Apply Lemma 3.4(1) to conclude $d = \sup A \in \overline{A}$. □

Concerning the existence of suprema of *directed* sets, we formulate:

Lemma 3.7. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset and let* $A \subseteq D$ *be directed. Then* A *has a supremum with* $\sup A \in \overline{A}$ *if and only if* $(a)_{a\in A}$ *is convergent. In this case we obtain* $\sup A = \lim(a)_{a\in A}$.

Proof. First let A have a supremum with $\sup A \in \overline{A}$. Lemma 3.4 tells us that there is a net $(a_f)_{f\in\mathcal{F}}$ with $a_f \in A$ and $f(\sup A) \leq a_f$ for all $f \in \mathcal{F}$. Let $f \in \mathcal{F}$. Then we have $f(a) \leq a \leq \sup A$ and $f(\sup A) \leq a_f \leq a$ for all $a \in A$ with $a \geq a_f$. Hence, $\sup A = \lim(a)_{a\in A}$.

Conversely, let $d := \lim(a)_{a\in A}$ and let $f \in \mathcal{F}$. Then, in particular, we find some $a_f \in A$ with $f(d) \leq a_f$ and $f(a') \leq d$ for all $a' \in A$ with $a' \geq a_f$. Let $a \in A$ and choose an element $a' \in A$ with $a' \geq a_f, a$. Then $f(a) \leq f(a') \leq d$. As $(D, \leq, \mathcal{F})$ is approximating, we infer $a \leq d$. Thus, $A \leq d$. Now apply 3.4(1) to obtain $d = \sup A \in \overline{A}$. □

A net $(d_n)_{n\in N}$ in D is called monotone if $m \leq n$ implies $d_m \leq d_n$ $(m, n \in N)$. A monotone net $(d_n)_{n\in N}$ gives rise to a directed set $A := \{d_n \mid n \in N\}$. It is an easy observation that $(d_n)_{n\in N}$ converges to some d if and only if $(a)_{a\in A}$ converges to d. Thus, we obtain:

Corollary 3.8. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset and let* $(d_n)_{n \in N}$ *be a monotone net in D. Then* $\{d_n \mid n \in N\}$ *has a supremum with* $\sup_{n \in N} d_n \in \overline{\{d_n \mid n \in N\}}$ *if and only if* $(d_n)_{n \in N}$ *is convergent. In this case we have* $\sup_{n \in N} d_n = \lim (d_n)_{n \in N}$.

We use the previous results to prove the main theorem of this section. This gives us several (topological, domain-theoretic) characterizations of approximating F-posets in which all monotone nets converge. It generalizes Theorem 4.8 in [11], which is a similar statement for the special case of pop's.

Theorem 3.9. *Let* $\underline{D} = (D, \leq, \mathcal{F})$ *be an approximating F-poset. Then the following are equivalent:*

(i) $(D, \leq)$ *is a dcpo and* $f(d) \ll d$ *for all* $f \in \mathcal{F}$ *and all* $d \in D$.

(ii) *Each directed subset* $A \subseteq D$ *has a supremum with* $\sup A \in \overline{A}$.

(iii) *Each monotone net is convergent with respect to the F-topology.*

(iv) $\underline{D}$ *is complete and for all directed subsets* $A \subseteq D$ *and all* $f \in \mathcal{F}$ *there exists some* $a \in A$ *with* $f[A] \leq a$.

If these conditions are fulfilled, then we also have the following:

(1) $(D, \leq)$ *is a continuous dcpo and* $\bigcup_{f \in \mathcal{F}} f[D]$ *is a basis for* $(D, \leq)$.

(2) *For all directed subsets* $A \subseteq D$ *we have* $\sup A = \lim(a)_{a \in A}$. *For all monotone nets* $(d_n)_{n \in N}$ *in D we have* $\sup_{n \in N} d_n = \lim(d_n)_{n \in N}$.

(3) *Let* $g : D \longrightarrow D$ *be a monotone mapping that is continuous with respect to the F-topology. Then g is also Scott-continuous.*

Proof. (i) and (ii) are equivalent and (1) is true due to Proposition 3.5. The equivalence of (ii) and (iii) and the validity of (2) follow from Lemma 3.7 and Corollary 3.8.

Now we prove (ii)→(iv). For the second statement let $A \subseteq D$ be directed, whence $\sup A \in \overline{A}$ by (ii). Lemma 3.6 (or Lemma 3.4) tells us that for all $f \in \mathcal{F}$ we can find some $a \in A$ with $f[A] \leq a$. To show that $\underline{D}$ is complete, let $(d_n)_{n\in N}$ be a Cauchy net. For all $f \in \mathcal{F}$ fix $k_f \in \mathcal{F}$ and $n_f \in N$ such that $f \leq k_f \circ k_f$ and $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_f$. Let $A := \{f(d_n) \mid n \geq n_{k_f},\ f \in \mathcal{F}\}$. We know from the proof of Proposition 3.2 that A is directed. By (2) we have $\sup A = \lim(a)_{a\in A}$. Let $f \in \mathcal{F}$. Then, in particular, we find some $a_0 \in A$ with $f(\sup A) \leq a_0$. Let $f_0 \in \mathcal{F}$ and $n_0 \geq n_{k_{f_0}}$ with $a_0 = f_0(d_{n_0})$. We deduce that $f(\sup A) \leq f_0(d_{n_0}) \leq k_{f_0}(d_{n_0}) \leq d_n$ for all $n \geq n_{k_{f_0}}$. Further, for all $n \geq n_{k_f}$ we have $f(d_n) \in A$, whence $f(d_n) \leq \sup A$. Consequently, $(d_n)_{n\in N}$ converges to $\sup A$.

In order to prove (iv)→(ii), let $A \subseteq D$ be directed. Because of (iv) we find a net $(a_f)_{f\in\mathcal{F}}$ with $a_f \in A$ and $f[A] \leq a_f$. Let $f \in \mathcal{F}$. Then for all $g \in \mathcal{F}$ with $g \geq f$ we have $f(a_g) \leq a_f$ (because $a_g \in A$) and $f(a_f) \leq g(a_f) \leq a_g$ (because $a_f \in A$). Lemma 3.1(iii)→(i) tells us that $(a_f)_{f\in\mathcal{F}}$ is Cauchy and thus convergent. In view of Lemma 3.6, A has a supremum with $\sup A \in \overline{A}$.

Finally, we prove (3). Let $g : D \longrightarrow D$ be monotone and continuous with respect to $\tau_{\underline{D}}$. Let $A \subseteq D$ be directed. Then $g[A]$ is directed and $\sup g[A] \leq g(\sup A)$. Due to (2) we have $g(\sup A) = g(\lim(a)_{a\in A}) = \lim(g(a))_{a\in A}$. Since $g(a) \leq \sup g[A]$ for all $a \in A$, we deduce $g(\sup A) = \lim(g(a))_{a\in A} \leq \sup g[A]$ by Proposition 2.12. □

Example 3.10. Let $D := \mathbb{R} \cup \{+\infty\}$ with $+\infty \notin \mathbb{R}$ and extend the linear order of the reals to a linear order of D in the usual way. For all $0 < \varepsilon \leq \sqrt{2}$ consider the mapping $g_\varepsilon : D \longrightarrow D,\ g_\varepsilon(d) := \min\{d - \varepsilon, \frac{1}{\varepsilon}\}$, where $(+\infty) + e := +\infty$ for all $e \in D$ (cf. Figure 3). Obviously, g_ε is monotone and below the identity. Further, one easily sees that $0 < \delta \leq \varepsilon \leq \sqrt{2}$ implies $g_\delta \geq g_\varepsilon$. We show that $g_\varepsilon \leq g_{\frac{\varepsilon}{2}} \circ g_{\frac{\varepsilon}{2}}$ for all $0 < \varepsilon \leq \sqrt{2}$.

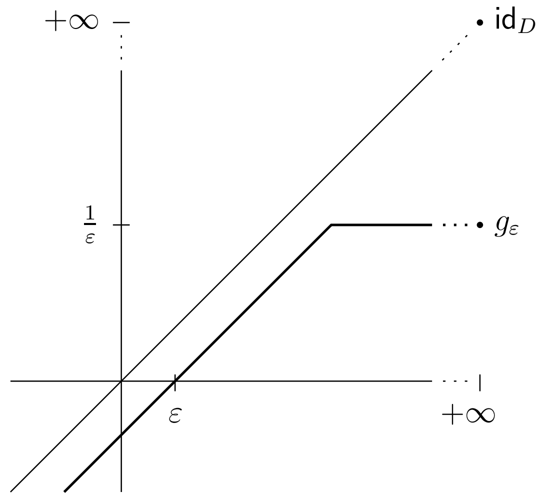

Fig. 3. The mappings g_ε.

First let $d \in D$, $d - \varepsilon \leq \frac{1}{\varepsilon}$. Then $d - \frac{\varepsilon}{2} = d - \varepsilon + \frac{\varepsilon}{2} \leq \frac{1}{\varepsilon} + \frac{\varepsilon}{2} = \frac{2+\varepsilon^2}{2\varepsilon} \leq \frac{2}{\varepsilon}$. Thus, $g_{\frac{\varepsilon}{2}}(g_{\frac{\varepsilon}{2}}(d)) = g_{\frac{\varepsilon}{2}}(d - \frac{\varepsilon}{2}) = d - \varepsilon = g_\varepsilon(d)$. Now let $d - \varepsilon \geq \frac{1}{\varepsilon}$ and $d - \frac{\varepsilon}{2} \leq \frac{2}{\varepsilon}$. Then $g_{\frac{\varepsilon}{2}}(g_{\frac{\varepsilon}{2}}(d)) = g_{\frac{\varepsilon}{2}}(d - \frac{\varepsilon}{2}) = d - \varepsilon \geq \frac{1}{\varepsilon} = g_\varepsilon(d)$. Eventually, let $d - \varepsilon \geq \frac{1}{\varepsilon}$, $d - \frac{\varepsilon}{2} \geq \frac{2}{\varepsilon}$. We deduce $g_{\frac{\varepsilon}{2}}(g_{\frac{\varepsilon}{2}}(d)) = g_{\frac{\varepsilon}{2}}(\frac{2}{\varepsilon}) = \frac{2}{\varepsilon} - \frac{\varepsilon}{2} = \frac{4-\varepsilon^2}{2\varepsilon} \geq \frac{1}{\varepsilon} = g_\varepsilon(d)$. Therefore, $\underline{D} := (D, \leq, \{g_\varepsilon \mid 0 < \varepsilon \leq \sqrt{2}\})$ is an F-poset. It is straightforward to show that $\underline{D}$ is approximating. Clearly, $\underline{D}$ fulfils the conditions of Theorem 3.9.

Additionally, we remark here that the restriction of $\mathcal{U}_{\underline{D}}$ to $\mathbb{R}$ is strictly coarser than the Euclidean uniformity. Nevertheless, the F-topology restricted to $\mathbb{R}$ coincides with the Euclidean topology. We note further that the topology of the Alexandroff one point compactification $\mathbb{R}^*$ of $\mathbb{R}$ is strictly coarser than the F-topology of $\underline{D}$, where $+\infty$ is also considered as the point in infinity of $\mathbb{R}^*$. Finally, if we restrict the topology of the two point compactification $\mathbb{R} \cup \{-\infty, +\infty\}$ to D, then we obtain precisely the F-topology.

Example 3.11. Let (X,ρ) be a complete metric space and consider the F-poset $\underline{D}_{fb}(X,\rho)$ of the formal ball model (Example 2.19). As $(x,r) \ll (y,s)$ if and only if $\rho(x,y) < r-s$ ([4], Prop. 7), we have $f((x,r)) \ll (x,r)$. Therefore, $\underline{D}_{fb}(X,\rho)$ fulfils the conditions of 3.9.

We know that in Theorem 3.9 condition (i) implies (1). In general, (1) is a strictly weaker statement than (i). Consider e.g. the unit interval $[0,1]$ with the usual order. Together with the identity map it becomes an approximating F-poset whose F-uniformity is the discrete uniformity. Clearly, $([0,1],\leq)$ is a dcpo with basis $[0,1]$, but this F-poset does not satisfy the equivalent conditions of Theorem 3.9.

Recall from Proposition 2.13 that the Lawson topology $\lambda_{(D,\leq)}$ is coarser than the F-topology of an approximating F-poset $(D,\leq,\mathcal{F})$ if and only if the Scott topology is. Here we show that this situation occurs in case that $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$. If, furthermore, all $f \in \mathcal{F}$ are finitely separated from id_D, then we even obtain $\lambda_{(D,\leq)}$ to be equal to the F-topology.

Theorem 3.12. *Let* $\underline{D} = (D,\leq,\mathcal{F})$ *be an F-poset such that* $f(d) \ll d$ *for all* $f \in \mathcal{F}$ *and all* $d \in D$*.*

(1) *If the pointwise supremum* $\sup \mathcal{F}$ *exists, then* $\underline{D}$ *is approximating if and only if* $(D,\leq)$ *is continuous and* $\lambda_{(D,\leq)} \subseteq \tau_{\underline{D}}$*.*

(2) *If* $(D,\mathcal{U}_{\underline{D}})$ *is totally bounded, then we have* $\tau_{\underline{D}} \subseteq \lambda_{(D,\leq)}$*.*

Proof. (1) First let $\underline{D}$ be approximating. Note that $(D,\leq)$ is continuous and $B(D) := \bigcup_{f\in\mathcal{F}} f[D]$ is a basis for $(D,\leq)$ by Proposition 3.5. In view of Proposition 2.13(2) it is enough to show that $\sigma_{(D,\leq)} \subseteq \tau_{\underline{D}}$. We prove that for all $d \in D$ the family $\{f(d)^{\twoheaduparrow} \mid f \in \mathcal{F}\}$ is a $\sigma_{(D,\leq)}$-neighbourhood basis of d. Clearly, $d \in f(d)^{\twoheaduparrow} \in \sigma_{(D,\leq)}$ for all $f \in \mathcal{F}$. Let $U \in \sigma_{(D,\leq)}$ with $d \in U$. As $U = \bigcup_{x\in B(D)\cap U} x^{\twoheaduparrow}$ (cf. [1], Prop. 2.3.6.), we find some $f \in \mathcal{F}$ and $e \in D$ such that $d \in f(e)^{\twoheaduparrow} \subseteq U$. Since $d = \sup_{f\in\mathcal{F}} f(d)$, there is a mapping $g \in \mathcal{F}$ with $f(e) \leq g(d)$. Let $h \in \mathcal{F}$ with $g \leq h \circ h$. Then $f(e) \leq h(h(d)) \ll h(d)$ and $h(d)^{\twoheadleftarrow} \subseteq f(e)^{\twoheadleftarrow} \subseteq U$.

Next, we show that $B_g(d) \subseteq f(d)^{\twoheadleftarrow}$ for all $d \in D$ and all $f, g \in \mathcal{F}$ with $f \leq g \circ g$. Let $e \in B_g(d)$. Then $f(d) \leq g(g(d)) \leq g(e) \ll e$ and thus $e \in f(d)^{\twoheadleftarrow}$. Clearly, this yields $\sigma_{(D,\leq)} \subseteq \tau_{\underline{D}}$.

Conversely, let $(D, \leq)$ be continuous and let $\lambda_{(D,\leq)} \subseteq \tau_{\underline{D}}$. Continuity of $(D, \leq)$ implies that $(D, \lambda_{(D,\leq)})$ is Hausdorff (cf. [1], Prop. 4.2.20.1.; [6], Theorem 4.7(i)). Consequently, the finer topology $\tau_{\underline{D}}$ is Hausdorff, too. By Proposition 2.12 $\underline{D}$ is approximating.

(2) Let $d \in D$ and let $f \in \mathcal{F}$. By applying Proposition 2.9 we find a finite separating set $M \subseteq D$ of f and id_D. Let $U := f(d)^{\twoheadleftarrow} \cap \bigcap\{D \setminus m\uparrow \mid m \in M \cap (D \setminus d\downarrow)\}$. As M is finite, we conclude $U \in \lambda_{(D,\leq)}$. By assumption we have $d \in f(d)^{\twoheadleftarrow}$. If $m \in M \cap (D \setminus d\downarrow)$, then $d \in D \setminus m\uparrow$. Therefore, $d \in U$. We prove that $U \subseteq B_f(d)$. Let $e \in U$. Clearly, $f(d) \leq e$. Let $m \in M$ such that $f(e) \leq m \leq e$. Suppose that $m \not\leq d$. Then $m \in M \cap (D \setminus d\downarrow)$ and thus $e \in D \setminus m\uparrow$; that is, $m \not\leq e$, a contradiction. Hence, $f(e) \leq m \leq d$. This yields $e \in B_f(d)$. □

4. Compact Approximating F-posets and FS-domains

This section deals with compactness of the F-topology. Its main result states that FS-domains appear precisely as compact approximating F-posets with least element.

The following theorem gives a domain-theoretic characterization of approximating F-posets to be compact with respect to the F-topology. It turns out that compact approximating F-posets are exactly those approximating F-posets satisfying the conditions of Theorem 3.9 and Proposition 2.9:

Theorem 4.1. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent:*

(i) *$\underline{D}$ is compact.*

(ii) *$(D, \leq)$ is a (continuous) dcpo, f is finitely separated from id_D and $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*

Proof. (i)→(ii): By Proposition 2.9 all $f \in \mathcal{F}$ are finitely separated from id_D. Let $A \subseteq D$ be directed. Let $f \in \mathcal{F}$. Let $M \subseteq D$ be a finite separating set of f and id_D. For all $a \in A$ choose some $m_a \in M$ with $f(a) \leq m_a \leq a$. The set $N := \{m_a \mid a \in A\}$ is finite; let $N =: \{m_1, \ldots, m_n\}$ for some $n \in \mathbb{N}$. We find elements $a_1, \ldots, a_n \in A$ with $f(a_\nu) \leq m_\nu \leq a_\nu$ for all $\nu = 1, \ldots, n$. Let $a \in A$ such that $a \geq a_1, \ldots, a_n$. Let $b \in A$ and let $\nu \in \{1, \ldots, n\}$ with $f(b) \leq m_\nu$. As $m_\nu \leq a_\nu \leq a$, we obtain $f(b) \leq a$. Thus, $f[A] \leq a$ and we see that $\underline{D}$ fulfils condition (iv) of Theorem 3.9. This proves (ii).

(ii)→(i): Apply Theorem 3.9(i)→(iv) and Proposition 2.9. □

By Lemma 2 in Jung [7] a Scott-continuous mapping $f : D \longrightarrow D$ that is finitely separated from id_D fulfils $f(d) \ll d$ for all $d \in D$. Hence, we obtain from the previous theorem:

Corollary 4.2. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset with each $f \in \mathcal{F}$ being Scott-continuous. Then $\underline{D}$ is compact if and only if $(D, \leq)$ is a (continuous) dcpo and each $f \in \mathcal{F}$ is finitely separated from id_D.*

Theorem 4.1 tells us that compact approximating F-posets satisfy the conditions of Theorem 3.12(1) and (2). As a consequence, we get:

Theorem 4.3. *The F-topology of a compact approximating F-poset $(D, \leq, \mathcal{F})$ is precisely the Lawson topology of $(D, \leq)$.*

Next, we head for FS-domains.

Theorem 4.4. *Let $(D, \leq)$ be a poset. Then the following are equivalent:*

(i) *There exists a directed family $\mathcal{F}$ such that $(D, \leq, \mathcal{F})$ is a compact approximating F-poset.*

(ii) *There exists a directed family $\widetilde{\mathcal{F}}$ consisting of Scott-continuous mappings such that $(D, \leq, \widetilde{\mathcal{F}})$ is a compact approximating F-poset.*

(iii) $(D, \leq)$ *is a (continuous) dcpo and there exists a directed family* $\mathcal{G}$ *consisting of Scott-continuous mappings finitely separated from* id_D *with* $\sup \mathcal{G} = \mathsf{id}_\mathcal{D}$*.*

Proof. (i)→(iii): From Theorem 4.1 we know that $(D, \leq)$ is a continuous dcpo. For all $f \in \mathcal{F}$ define $f^c : D \longrightarrow D$ by $f^c(d) := \sup_{x \ll d} f(x)$ $(d \in D)$. Prop. 1.12 in Jung [6] tells us that f^c is the greatest Scott-continuous mapping below f. If $f, g \in \mathcal{F}$, $f \leq g$, then $f^c \leq g^c$. Thus, $\mathcal{G} := \{f^c \mid f \in \mathcal{F}\}$ is directed. For all $d \in D$ we have $\sup_{f \in \mathcal{F}} f^c(d) = \sup_{f \in \mathcal{F}} \sup_{x \ll d} f(x) = \sup_{x \ll d} \sup_{f \in \mathcal{F}} f(x) = \sup_{x \ll d} x = d$; whence $\sup \mathcal{G} = \mathsf{id}_\mathcal{D}$. As each $f \in \mathcal{F}$ is finitely separated from id_D, this is also true for $f^c \leq f$.

(iii)→(ii): Define $\widetilde{\mathcal{F}} := \{g \circ g \mid g \in \mathcal{G}\}$. Obviously, $\widetilde{\mathcal{F}}$ is a directed family of Scott-continuous mappings below id_D and cofinal in the directed set $\{g \circ h \mid g, h \in \mathcal{G}\}$. Due to Scott-continuity we infer $\sup \widetilde{\mathcal{F}} = \sup_{g,h \in \mathcal{G}}(g \circ h) = (\sup_{g \in \mathcal{G}} g) \circ (\sup_{h \in \mathcal{G}} h) = \mathsf{id}_D \circ \mathsf{id}_D = \mathsf{id}_D$. Analogously, $\sup_{g \in \mathcal{G}}(g \circ g \circ g \circ g) = \mathsf{id}_D$. By Jung [7], Lemma 2, we know that $g \circ g \ll \mathsf{id}_D$ for all $g \in \mathcal{G}$ (because g is Scott-continuous and finitely separated from id_D). Hence, for all $g \in \mathcal{G}$ we find some $\widetilde{g} \in \mathcal{G}$ with $g \circ g \leq \widetilde{g} \circ \widetilde{g} \circ \widetilde{g} \circ \widetilde{g}$; that is, for all $f \in \widetilde{\mathcal{F}}$ there is a mapping $\widetilde{f} \in \widetilde{\mathcal{F}}$ with $f \leq \widetilde{f} \circ \widetilde{f}$. Consequently, $(D, \leq, \widetilde{\mathcal{F}})$ is an approximating F-poset. As all $g \in \mathcal{G}$ are finitely separated from id_D, this is also true for $g \circ g \leq g$. Now apply Corollary 4.2 to obtain that $(D, \leq, \widetilde{\mathcal{F}})$ is compact.

(ii)→(i) is trivial. □

The succeeding definition is due to Jung [7]:

Definition 4.5. *A pointed dcpo* $(D, \leq)$ *is an* FS-domain *if there is a directed family* $\mathcal{G}$ *of Scott-continuous mappings on* D*, each finitely separated from* id_D*, such that* $\sup \mathcal{G} = \mathsf{id}_\mathcal{D}$*.*

We just remark here that the category of FS-domains is one of the two maximal cartesian closed subcategories of the category of all pointed continuous dcpo's together with Scott-continuous mappings as morphisms, see Jung [7]; cf. also [1], Section 4.

Corollary 4.6. *Let $(D, \leq)$ be a pointed poset. Then $(D, \leq)$ is an FS-domain if and only if there is a family $\mathcal{F}$ such that $(D, \leq, \mathcal{F})$ is a compact approximating F-poset.*

Note that if $(D, \leq)$ is an FS-domain, then due to JUNG [7], Theorem 4(i), the set

$$\mathcal{F} := \{f : D \longrightarrow D \mid f \text{ Scott-continuous, } f \ll \mathsf{id}_D\}$$

induces a compact approximating F-poset $(D, \leq, \mathcal{F})$ whose F-topology coincides with the Lawson topology.

Observe that the Lawson topology of an FS-domain may be the topology of a compact approximating F-poset $\underline{D} = (D, \leq, \mathcal{F})$ where all $f \in \mathcal{F}$ are *not* Scott-continuous (and thus *not* $\tau_{\underline{D}}$-continuous, cf. Theorem 3.9(3)). Such an example is given in 2.16.

Acknowledgements

The author wishes to thank A. JUNG who suggested to extend the notion of a pop ([11]) to that of an F-poset in order to get a topological characterization of FS-domains. Further, many thanks go to M. DROSTE and D. KUSKE for all the interesting discussions on this topic and to R. HECKMANN and R. KOPPERMAN for the invitation to this conference.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

[2] N. Bourbaki. *General Topology*, Parts 1 and 2. Hermann, Paris, 1966.

[3] J. Dixmier. *C*-Algebras.* North-Holland Mathematical Library. North-Holland Publishing Company, Amsterdam, 1977.

[4] A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193:53–73, 1998.

[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin – Heidelberg – New York, 1980.

[6] A. Jung. *Cartesian Closed Categories of Domains*, volume 66 of *CWI Tracts*. Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[7] A. Jung. The classification of continuous domains. In *Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.

[8] A. Jung and P. Sünderhauf. Uniform approximation of topological spaces. *Topology and its Applications*, 83:23–38, 1998.

[9] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras*, volume I. Academic Press, New York, 1983.

[10] J.L. Kelley. *General Topology*. The University series in higher mathematics. Van Nostrand, Princeton, New Jersey, 1955.

[11] R. Kummetz. From partial orders with projections to domains. Extended abstract. In S. Brookes, A. Jung, M. Mislove, and A. Scedrov, editors, *Mathematical Foundations of Programming Semantics, Fifteenth Conference*, volume 20 of *Electronic Notes in Theoretical Computer Science*, 1999. 12 pages.

[12] D.S. Scott. Continuous lattices. In E. Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer-Verlag, 1972.

Institut für Algebra, Technische Universität Dresden, D-01062 Dresden, Germany

E-mail address: `kummetz@math.tu-dresden.de`