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COMPACT L-SPACES AND RIGHT TOPOLOGICAL GROUPS

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Abstract

The Continuum Hypothesis implies that there is a homogeneous compact L-space X. In fact, X is a group in which product is right-continuous. Under $\diamondsuit$, one can get such an X whose regular open algebra contains a Suslin tree.

1. Introduction

All spaces discussed here are Hausdorff. A space X is *hereditarily Lindelöf (HL)* iff every subspace is Lindelöf. For X compact, X is HL iff every closed subset is a G_δ. An *L-space* is a space which is HL but not hereditarily separable. By Juhász (see [8]), there are no compact L-spaces under $MA + \neg CH$. However, there are two known consistent classes of compact L-spaces. One are (compact) Suslin lines, whose existence follows from $\diamondsuit$ (Jensen), but not from CH (Jensen). The other, which requires just CH, uses a measure to ensure its hereditary Lindelöfness. This construction was discovered independently by Haydon [5], Kunen [10], and Talagrand [15]; see §5 of Negrepontis [13]. Notions such as CH, MA, and $\diamondsuit$ are covered in set theory texts [2, 7, 9].

A space X is *homogeneous* iff whenever $p, q \in X$, there is a homeomorphism φ of X with $\varphi(p) = q$. As Jensen pointed out,

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it is easy to construct homogeneous compact Suslin lines using $\diamondsuit$. However, making the CH L-space homogeneous has not been done before; we do this in Section 5. Brandsma and van Mill [1] have shown that there is enough freedom in the inductive construction (under CH) to make $\aleph_1$ non-homeomorphic versions of this L-space. It is easy to modify their construction to make a compact L-space which is rigid (i.e., the only homeomorphism is the identity).

In fact, we shall make our homogeneous X into a *right topological group*. That is, there will be a group operation $\cdot$ on X which is right-continuous (i.e., the map $x \mapsto x \cdot a$ is continuous for each $a \in X$). This implies homogeneity, since for each p, q, the map $x \mapsto xp^{-1}q$ moves p to q. Of course, one could instead make $\cdot$ left-continuous. Note that $\cdot$ cannot be made both left-continuous and right-continuous: If it were, it would then be jointly continuous by a theorem of Ellis [3] (or, see [14]); but then, since X is compact and 1^{st} countable (since it is HL), it would have to be 2^{nd} countable. It follows that the group cannot be abelian, and that the map $x \mapsto x^{-1}$ cannot be continuous.

The measure on our CH example will in fact be a Haar measure; that is, right invariant. There are a number of conditions known which imply that a compact right topological group *must* have a Haar measure; see Milnes and Pym [12]. However, such Haar measures need not always exist. In Section 6, we modify our construction to show that, assuming $\diamondsuit$, there is a compact L-space which is both a right topological group and a Suslin space (that is, there will be a Suslin tree dense in the regular open algebra of X; see Definition 6.1 and Remark 6.3). Such a space cannot support a measure. Also, since X is Suslin, X^2 will be an example of a compact right topological group which fails to have the countable chain condition. We do not know whether such an X can be a Suslin line.

The CH example has three other unusual properties. First, the group will be locally finite:

Definition 1.1. A group is *locally finite* iff every finitely generated subgroup is finite.

This definition is standard in group theory. Note that it says nothing about any topology on the group.

Second, the group action will be *scrambled*:

Definition 1.2. Let $(X, \cdot)$ be a 0-dimensional compact right topological group. A finite clopen partition, $\{P_0, \ldots, P_{n-1}\}$, is *scrambled* iff for every permutation $\sigma \in S_n$, there is an $a \in X$ such that $P_i = P_{\sigma(i)} \cdot a$ for all $i < n$. $(X, \cdot)$ is *scrambled* iff every open cover of X can be refined to a scrambled clopen partition. $(X, \cdot)$ is *super-scrambled* iff every clopen partition is scrambled.

Third, the measure will be *rational*:

Definition 1.3. Let X be compact Hausdorff and 0-dimensional. A *rational measure* on X is a finite regular Borel measure μ on X such that

- $\mu(K)$ is a positive rational for each non-empty clopen subset of K.
- For each clopen K and each rational q with $0 \leq q \leq \mu(K)$, there is a clopen $H \subseteq K$ with $\mu(H) = q$.

Remark 1.4. Every locally finite group acting on a compact space has an invariant measure (see [6], Corollary 17.9). Hence the Suslin example cannot be locally finite. It will be super-scrambled. The CH example cannot be super-scrambled, since that would contradict the invariant measure.

Sections 2, 3, and 4 contain some technical results needed for the proofs. The need for this material can best appreciated by considering the following lemma, which lists the basic properties of the CH L-space. Variants of this lemma occur in [1, 10, 13].

Lemma 1.5. *Suppose that X is compact Hausdorff, and μ is a regular Borel probability measure on X satisfying:*

1. *X has no isolated points.*
2. *$\mu(U) > 0$ for all non-empty open $U \subseteq X$.*
3. *$\mathrm{int(D)} \neq \emptyset$ whenever D is a closed G_δ subset of X with $\mu(D) > 0$.*
4. *D is 2^{nd} countable whenever D is a closed G_δ subset of X with $\mu(D) = 0$.*

Then X is an L-space, and for any $Y \subseteq X$, the following are equivalent:

a. *Y is null.*
b. *Y is meagre.*
c. *Y is nowhere dense.*
d. *Y is separable.*
e. *Y is 2^{nd} countable.*

Proof. $(c) \to (a)$: By (2), X is ccc, so there is a closed nowhere dense G_δ set K such that $Y \subseteq K$. Then $\mu(K) = 0$ by (3).

Now $(b) \to (a)$ follows immediately, and $(c) \to (b)$ is trivial.

$(a) \to (c)$: For each n, let U_n be open with $Y \subseteq U_n$ and $\mu(U_n) \leq 1/n$. By ccc, let K_n be a closed G_δ set such that $U_n \subseteq K_n$ and $K_n \backslash U_n$ is nowhere dense. By $(c) \to (a)$, we have $\mu(K_n \backslash U_n) = 0$, so that $\mu(K_n) \leq 1/n$. Let $K = \bigcap_n K_n$. Then $Y \subseteq K$ and K is a closed G_δ and null, so K is nowhere dense by (2).

The proof of $(a) \to (c)$ also showed that every null set is contained in a closed G_δ null set, so that we now have, by (4): $(a) \leftrightarrow (b) \leftrightarrow (c) \to (e) \to (d)$. Finally, for $(d) \to (c)$, it is enough to prove that every countable set D is nowhere dense. But D is null by (1) and $(b) \to (a)$, so apply $(a) \to (c)$.

X is HL by ccc and $(c) \to (e)$, and X is non-separable by $(d) \to (c)$. ☐

Properties (3) and (4) involve the closed G_δ sets, which, under CH, are easily handled in the natural construction of X as an inverse limit; see Section 5, where we prove:

Theorem 1.6. *Assume CH. Then there is a compact 0-dimensional L-space X with a measure μ and a product $\cdot : X \times X \to X$ such that:*

1. *(X, μ) satisfies the hypotheses of Lemma 1.5.*
2. *$(X, \cdot)$ is a locally finite group.*
3. *For each a, the map $x \mapsto x \cdot a$ is continuous.*
4. *$\mu(E \cdot a) = \mu(E)$ for each a and each measurable E.*
5. *μ is rational.*
6. *$(X, \cdot)$ is scrambled.*

In view of (5,6), the scrambled partitions will be precisely the partitions of X into clopen sets of the same measure. Actually, both the existence of the measure and the fact that it is rational follow from other features of X; see Remark 3.15.

As is typical of such constructions, we build X as an inverse limit. Thus, we shall have spaces X_α (for $\alpha \le \omega_1$), and projections $\pi^\beta_\alpha : X_\beta \to X_\alpha$ (for $\alpha \le \beta \le \omega_1$). X will be X_{ω_1}. The X_α, for $\alpha < \omega_1$, will be 2$^{\text{nd}}$ countable.

To achieve homogeneity, we also construct a group G_α of measure preserving homeomorphisms acting on the X_α, and we shall have $G_\alpha \le G_\beta$ whenever $\alpha \le \beta$. In particular, G_α will be acting on $X_{\alpha+1}$. Thus, the action of G_α on X_α will have to be lifted to an action of G_α on $X_{\alpha+1}$. In Section 2, we discuss some general facts about such liftings.

The construction will give us a group $G = G_{\omega_1}$ acting transitively on $X = X_{\omega_1}$, ensuring homogeneity of X. The right-continuous group operation will only appear at the very end; the action will be regular (in the sense of permutation groups), which will enable us to identify G with X (see Definition 3.1 and Lemma 3.2).

To make G act transitively, we must be able to extend a G_α to move a given point to some other given point. This is discussed in Section 3. Keeping the groups locally finite will make it easier to handle the extensions, since we may represent the group action by the action of finite groups acting on finite sets (clopen partitions of X). Keeping the measures rational allows us to get many such partitions in which all sets have the same measure; however, rationality places some additional requirements on the closed sets which are split in the construction of $X_{\alpha+1}$ from X_α; this is handled in Section 4. Most of the material in Sections 3 and 4 is not needed for the Suslin example in Section 6.

See [11, 12, 16] for a discussion of compact right topological groups in relation to topological dynamics. We may regard the set of right translations, $\{x \mapsto xa : a \in X\}$, as a flow (the discrete group X acting on the space X). If X is a compact L-space, this flow cannot be distal (since X is 1^{st} countable but not 2^{nd} countable). In fact, if the flow is scrambled, as in our examples, then it must be proximal, but this need not be true in general (take the product of our examples with the group $\{0,1\}^\omega$).

2. Actions and Liftings

We use the following notation for a group acting on a set:

Definition 2.1. If G is a group and X is any set, a (left) *action* of G on X is a map $\star : G \times X \to X$ such that the map $x \mapsto \varphi \star x$ is a permutation of X for each $\varphi \in G$, and $(\varphi\psi) \star x = \varphi \star (\psi \star x)$ whenever $\varphi, \psi \in G$ and $x \in X$. If μ is a measure on X, then the measure is $(G, \star)$ *invariant* iff φB is measurable and $\mu(\varphi B) = \mu(B)$ whenever $B \subseteq X$ is measurable and $\varphi \in G$. If X is a topological space, the action is *continuous* iff each map $x \mapsto \varphi x$ is a homeomorphism of X.

As usual, we drop the '$\star$' when the action is clear from context. X is homogeneous iff there is a group which acts transitively and continuously on X.

We now describe a general method which, given $G, X, \star$ as in Definition 2.1, constructs a Y and a projection $\pi : Y \to X$, and then lifts $\star$ to an action of G on Y. This will eventually be applied with G_α, X_α to construct $Y = X_{\alpha+1}$ and $\pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$. As in the standard L-space construction, we need to be able to control $\{x : |\pi^{-1}\{x\}| = 1\}$.

We describe this lifting first in terms of abstract sets, and then add in the group action, the topology, and the measure.

Definition 2.2. If $Y \subseteq A \times B$, then $Y_a = \{b : (a, b) \in Y\}$ for $a \in A$, and $Y^b = \{a : (a, b) \in Y\}$ for $b \in B$.

Definition 2.3 (Abstract Sets)**.** Suppose that X, I are any sets, and for $i \in I$, $F^i_0, F^i_1 \subseteq X$ with $F^i_0 \cup F^i_1 = X$. Let $M^i_x = \{\ell : x \in F^i_\ell\}$ (which is either $\{0\}$, $\{1\}$, or $\{0, 1\}$). Define $Y = Y(X, \langle F^i_\ell : i \in I, \ell < 2\rangle)$ by

$$Y = \{(x, f) \in X \times 2^I : \forall i \in I[x \in F^i_{f(i)}]\} \quad .$$

and $\pi = \pi(X, \langle F^i_\ell : i \in I, \ell < 2\rangle)$ by: $\pi : Y \to X$ and $\pi(x, f) = x$.

Note that each $Y_x = \prod_{i \in I} M^i_x \subseteq 2^I$, and that π maps Y *onto* X.

Definition 2.4 (Group Action)**.** Assume that the group G acts on X and also on I. Then G acts on 2^I by $(\varphi f)(i) = f(\varphi^{-1} i)$, and G acts on $X \times 2^I$ by $\varphi(x, f) = (\varphi x, \varphi f)$.

Lemma 2.5. *In the notation of Definitions 2.3 and 2.4, assume that $\varphi F^i_\ell = F^{\varphi i}_\ell$ for all φ, i, ℓ. Then the action of G takes Y to Y.*

Proof. Suppose $(x, f) \in Y$, so that $x \in F^i_{f(i)}$ for all i. Then $\varphi x \in \varphi F^i_{f(i)} = F^{\varphi i}_{f(i)} = F^{\varphi i}_{(\varphi f)(\varphi i)}$ for all i, so $\varphi x \in F^j_{(\varphi f)(j)}$ for all j, whence $(\varphi x, \varphi f) \in Y$. ∎

Adding in the topology, we give 2^I the usual product topology. The lifting is a way of giving a collection of sets non-empty interiors:

Lemma 2.6 (Topology)**.** *In Definition 2.3, suppose that X is a 0-dimensional compact Hausdorff space and each F^i_ℓ is closed. Then Y is closed in $X \times 2^I$ and is hence also compact Hausdorff and 0-dimensional. If $F^i_\ell \neq \emptyset$, then in Y, $\mathrm{int}(\pi^{-1}(\mathrm{F}^i_\ell)) \neq \emptyset$. If the group G acts on I, and continuously on X, then the group action on $X \times 2^I$ is continuous.*

Proof. $(F^i_\ell \times \{f : f(i) = \ell\}) \cap Y = (X \times \{f : f(i) = \ell\}) \cap Y$ is clopen in Y, and is contained in $\pi^{-1}(F^i_\ell)$. Hence, $\mathrm{int}_\mathrm{Y}(\pi^{-1}(\mathrm{F}^i_\ell)) \neq \emptyset$. □

One can also lift a measure on X to a measure on Y. This is really independent of the topology, but for simplicity, we confine ourselves to the topological situation at hand, where it is clear that one gets a regular measure.

Definition 2.7 (Measure)**.** In Definition 2.3, suppose that X is a compact 0-dimensional Hausdorff space, each F^i_ℓ is closed, and μ is a regular probability measure on X. Assume that for each $i \in I$, we have $0 < p_i, q_i < 1$, with $p_i + q_i = 1$. For each x, define a probability measure λ^i_x on M^i_x: If $M^i_x = \{0, 1\}$, then $\lambda^i_x(\{0\}) = p_i$ and $\lambda^i_x(\{1\}) = q_i$. If $M^i_x = \{\ell\}$, then $\lambda^i_x(\{\ell\}) = 1$. Let λ_x be the usual product measure on $Y_x = \prod_i M^i_x$. Then the *lifted measure* ν on Y is defined by: $\nu(E) = \int \lambda_x(E_x) d\mu(x)$.

One must verify that this integral really defines a measure. In the case that E is a basic clopen set, the integral surely exists, since we are integrating a Borel step function. Then, a measure defined on the clopen sets extends to a Baire measure, and then a regular Borel measure; see, e.g., [4], §54.

Lemma 2.8. *In Definition 2.7, $\mu = \nu\pi^{-1}$. Let G be a group which acts on X and on I. Assume that μ is G-invariant, $\varphi F^i_\ell = F^{\varphi i}_\ell$ for each i, ℓ, φ, and $p_i = p_{\varphi i}$ for each i, φ. Then ν on Y is G-invariant.*

The following example illustrates the computation of the measure, and will be useful in proving Lemma 2.12.

Example 2.9. Basic clopen sets in 2^I are of the form $N_s = \{f \in 2^I : s \subseteq f\}$, where s is a finite function, dom(s) $\subseteq$ I, and ran(s) $\subseteq \{0,1\}$. Let us compute the measure of a basic clopen set $(P \times N_s) \cap Y$ in the special case where each $F_0^i = X$. Say dom(s) $= \{\mathrm{i}_1, \dots, \mathrm{i}_\mathrm{a}, \mathrm{j}_1, \dots, \mathrm{j}_\mathrm{b}\}$ and $s(i_1) = \cdots = s(i_a) = 0$ and $s(j_1) = \cdots = s(j_b) = 1$. Let w_x^i be p_i if $x \in F_1^i$ and 1 if $x \notin F_1^i$, and let $w(x) = w_x^{i_1} w_x^{i_2} \cdots w_x^{i_a} q_{j_1} \cdots q_{j_b}$. Let $K = P \cap F_1^{j_1} \cap \cdots \cap F_1^{j_b}$. Then $\nu((P \times N_s) \cap Y) = \int_K w(x)dx$. Now consider a smaller clopen set of the special form $(P \times N_t) \cap Y$, where dom(t) $= \{\mathrm{i}_1, \dots, \mathrm{i}_\mathrm{a}, \mathrm{j}_1, \dots, \mathrm{j}_\mathrm{b}, \mathrm{k}_1, \dots, \mathrm{k}_\mathrm{b}\}$, $t(k_1) = \cdots = t(k_b) = 0$, and t extends s. Suppose also that we have each $F_1^{k_u} = F_1^{j_u}$. Then $\nu((P \times N_t) \cap Y) = p_{k_1} \cdots p_{k_b} \nu((P \times N_s) \cap Y)$.

We remark that these notions may be generalized quite a bit. We do not need these generalizations here, but they may make the numerology with the p_i, q_i seem less ad hoc. Clearly, the 0-dimensionality of X is of no fundamental importance. We may replace $2 = \{0,1\}$ by T, an arbitrary compact Hausdorff space, and replace the p_i, q_i by a regular probability measure λ^i on T. Now, instead of F_0^i, F_1^i, we have closed sets $F^i \subseteq T \times X$, and hence closed $F_t^i \subseteq X$. Now, $M_x^i = (F^i)^x \subseteq T$, and we assume that every $\lambda^i(M_x^i) > 0$. Then, in Definition 2.7, the measure λ_x^i on M_x^i is just the conditional probability: $\lambda_x^i(W) = \lambda^i(W)/\lambda^i(M_x^i)$.

Next, we show that in some cases, this lifting procedure preserves the rationality of the measure (see Definition 1.3). As a simple example, observe that the usual product (Haar) measure on $X = 2^\omega$ is not rational; however, if λ^i is a probability measure on 2 for $i \in \omega$ and λ is the product of these, then λ is rational whenever the sequence $\langle \lambda^i(\{0\}) : i \in \omega \rangle$ is redundant, as per the following definition:

Definition 2.10. The sequence $\langle p_i : i \in I \rangle$ is *redundant* iff $p_i \in \mathbb{Q} \cap (0,1)$ for every $i \in I$ and $\{i : p_i = r\}$ is infinite for every $r \in \mathbb{Q} \cap (0,1)$.

Definition 2.11. A sequence of measurable sets, $\langle F^i : i \in I\rangle$ is *rational* iff for each finite $a > 0$, each $i_1, \ldots, i_a$, and each clopen P:

$\mu(P \cap F^{i_1} \cap \cdots \cap F^{i_a}) \in \mathbb{Q}$.

If $\mu(P \cap F^{i_1} \cap \cdots \cap F^{i_a}) = 0$, then $P \cap F^{i_1} \cap \cdots \cap F^{i_a} = \emptyset$.

If μ is also rational, then every finite boolean combination of the F^i and clopen sets has rational measure. Now, using the computation in Example 2.9:

Lemma 2.12. *Suppose Y and ν are defined from μ as in Definition 2.7. Suppose also:*

μ is a rational measure on X.

Each $F^i_0 = X$.

The sequence $\langle F^i_1 : i \in I\rangle$ is rational.

For each j, if $I_j = \{i : F^i_1 = F^j_1\}$, then $\langle p_i : i \in I_j\rangle$ is redundant.

Then ν is a rational measure on Y.

3. Extending Actions

To get a right-continuous group operation on X, we shall apply the following standard definition and lemma:

Definition 3.1. An action of the group G on the set X is called:

Free iff $\forall \varphi \in G\backslash\{1\}\, \forall x \in X\, [\varphi x \neq x]$.

Faithful iff $\forall \varphi \in G\backslash\{1\}\, \exists x \in X\, [\varphi x \neq x]$.

Regular iff it is free and transitive.

The left action of a group on itself is regular. Conversely,

Lemma 3.2. *Suppose that $\star$ is a regular action of the group G on the space X. Then there are group operations $\circ$ and $\cdot$ on X such that*

$(X, \circ)$ and $(X, \cdot)$ are both isomorphic to G.

$$\{x \mapsto \varphi \star x : \varphi \in G\} = \{x \mapsto a \circ x : a \in X\} = \{x \mapsto x \cdot a : a \in X\}$$

$\circ$ is left-continuous and $\cdot$ is right-continuous.

Proof. Fix an object $1 \in X$, and let $\Gamma(\varphi) = \varphi \star 1$. Then Γ is a bijection from G onto X. Define $\circ$ by: $\Gamma(\varphi) \circ \Gamma(\psi) = \Gamma(\varphi\psi)$. Note that $\Gamma(\varphi) \circ x = \varphi \star x$. Let $x \cdot y = y \circ x$. □

In our inductive construction, the action of G_α on X_α will be free for $\alpha \leq \omega_1$ and regular for $\alpha = \omega_1$.

Definition 3.3. Let $*$ be an action of the group G on the set X. A *block system* for $(G, *)$ is a family $\Sigma \subseteq \mathcal{P}(X)\backslash\{\emptyset\}$ such that Σ is a partition of X and such that $\varphi P \in \Sigma$ for all $P \in \Sigma$. The *induced action*, $\circledast$, of G on Σ is defined by $\varphi \circledast P = \varphi P$. If Σ_1, Σ_2 are partitions, then $\Sigma_2 \lhd \Sigma_1$ (Σ_2 *refines* Σ_1) iff every set in Σ_1 is a union of sets in Σ_2.

An alternative view of actions is sometimes useful: $SYM(X)$ denotes the group of all permutations of X, and an action $*$ of G on X may be identified with a homomorphism $* : G \to SYM(X)$ (that is, $\varphi * x = (*(\varphi))(x)$). Then $*$ is faithful iff it is 1-1. Let $SYM_\Sigma(X) \leq SYM(X)$ be the subgroup consisting of those permutations φ which fix the partition Σ (that is $\varphi P \in \Sigma$ for all $P \in \Sigma$). There is an obvious homomorphism π_Σ from $SYM_\Sigma(X)$ onto $SYM(\Sigma)$. Σ is a block system for $(G, *)$ iff ran$(*) \subseteq$ SYM$_\Sigma$(X), in which case the induced action is $\circledast = \pi_\Sigma \circ * : G \to SYM_\Sigma(X)$. If $\Sigma_2 \lhd \Sigma_1$, we may view the elements of Σ_2 as points, and define $SYM_{\Sigma_1}(\Sigma_2) \leq SYM(\Sigma_2)$ and $\pi^{\Sigma_2}_{\Sigma_1} : SYM_{\Sigma_1}(\Sigma_2) \to SYM(\Sigma_1)$. This view will be useful in the following, where we represent the action of a locally finite group on a space X by its action on a sequence of finite partitions of X.

Lemma 3.4. *Assume that $\star$ is a free continuous action of the finite group G on the compact 0-dimensional Hausdorff space X. Let $\mathcal{U}$ be an open cover of X. Then:*

1. *$\mathcal{U}$ may be refined to a finite clopen partition Σ of X such that Σ is a block system for $(G, \star)$ and the induced action on Σ is free.*

2. If in addition μ is a rational $\star$-invariant measure on X, then we may obtain Σ so that all its blocks have the same measure.

Proof. First, refining $\mathcal{U}$, we may assume (since X is compact and G is finite), that $\varphi V \cap V = \emptyset$ for each $\varphi \in G \backslash \{1\}$ and each $V \in \mathcal{U}$. Refining further, we may assume that the elements of $\mathcal{U}$ are clopen and partition X. For each x, let

$$P_x = \bigcap \{\varphi V : x \in \varphi V \ \& \ V \in \mathcal{U} \ \& \ \varphi \in G\}.$$

Then $\Sigma = \{P_x : x \in X\}$ satisfies (1). For (2), choose a positive rational q such that each $\mu(P_x)/q$ is an integer, and then obtain $\Sigma' \lhd \Sigma$ satisfying (1,2), with $\mu(Q) = q$ for all $Q \in \Sigma'$. □

Definition 3.5. $\vec{\Sigma} = \langle \Sigma_n : n \in \omega \rangle$ is a *basic sequence* of partitions of the topological space X iff

Each Σ_n is a finite partition of X into clopen sets.
$\Sigma_0 \rhd \Sigma_1 \rhd \Sigma_2 \rhd \cdots$.
For some $k_n > 1$, each $P \in \Sigma_n$ is a union of k_n sets in Σ_{n+1}.
For each $k > 0$, some k_n is divisible by k.
$\bigcup_n \Sigma_n$ is a base for X.

Note if X is compact, it must be homeomorphic to the Cantor set.

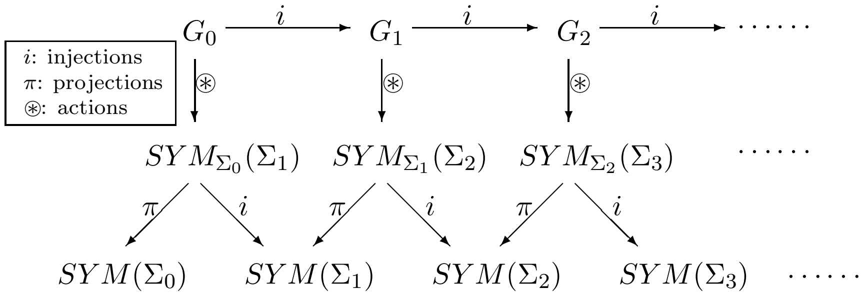
Definition 3.6. If $\vec{\Sigma}$ is a basic sequence of partitions and μ is a measure on X, then $\vec{\Sigma}$ is *μ-basic* iff for each n, all sets in Σ_n have the same measure.

Definition 3.7. If the group G acts on the space X, and $\vec{G} = \langle G_n : n \in \omega \rangle$, then $(\vec{\Sigma}, \vec{G})$ is an *action sequence* for (X, G) iff $\vec{\Sigma}$ is a basic sequence of partitions of X and:

Each G_n is a finite subgroup of G, and $G_0 \leq G_1 \leq G_2 \leq \cdots$.
$\bigcup_n G_n = G$.
Each Σ_n is a block system for G_n.

The action sequence $(\vec{\Sigma}, \vec{G})$ gives rise to the commutative diagram in Figure 1. Note that in the diagram, all the actions

Fig. 1. An Action Sequence



shown are finite groups acting on finite sets (partitions). Conversely, given such a diagram, we may retrieve the action of G on X; this action will be measure-preserving if $\vec{\Sigma}$ is μ-basic. This is important in the proof of Lemma 3.12, where we construct an action on X by defining such a diagram. We may view the action sequence as resolving the action on X into an inverse limit of an ω-sequence of actions on finite sets, in analogy to the ω_1-sequence in the proof of Theorem 1.6.

Definition 3.8. An action sequence $(\vec{\Sigma}, \vec{G})$ is

- *Eventually free* iff for all n, there is an $m \geq n$ such that the action of G_n on Σ_m is free.
- *Large* iff for each n, the action of G_n on Σ_n defines an isomorphism of G_n onto $SYM(\Sigma_n)$.

The notion of "large" is relevant to getting $(X, \cdot)$ scrambled in Theorem 1.6. By Lemma 3.4,

Lemma 3.9. *Assume that G is a countable locally finite group and $\star$ is a free continuous action of G on the compact 2^{nd} countable 0-dimensional Hausdorff space X. Also assume that μ is a rational $\star$-invariant measure on X. Then there is an eventually free μ-basic action sequence $(\vec{\Sigma}, \vec{G})$ for (X, G).*

Proof. Fix finite subgroups $\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots$ with $\bigcup_n G_n = G$. Inductively get $\{X\} = \Sigma_0 \rhd \Sigma_1 \rhd \Sigma_2 \rhd \cdots$, such that each Σ_n is a block system for G_n, the action of G_n on Σ_n is free, all sets in Σ_n have the same measure, and each set in Σ_n is a union of k_n sets in Σ_{n+1}, where $n \mid k_n$. To get Σ_{n+1}, apply Lemma 3.4 to the cover Σ_n, and then sub-divide each block into n sets of the same measure. □

Note that the measure is needed to ensure that each set in Σ_n contains the same number of subsets from Σ_{n+1}. Conversely, a basic action sequence $(\vec{\Sigma}, \vec{G})$ defines a rational $\star$-invariant measure on X. If we dropped the assumption of the measure, we could not in general get a basic action sequence. For example, there cannot be a a basic action sequence if there is a non-empty clopen set K and $\varphi_i \in G$ for $i \in \omega$ such the $\varphi_i K$ are all disjoint. This will happen if X is constructed as in Example 2.9 with each F_1^i a singleton; such a construction occurs naturally when building the Suslin space in Section 6.

Observe that we may always pass to a subsequence of $\vec{\Sigma}$, keeping the same $\vec{G}$:

Lemma 3.10. *Let* $m_0 < m_1 < m_2 < \cdots$, *and let* $\vec{\Sigma}' = \langle \Sigma_{m_n} : n \in \omega \rangle$. *Then:*
If $\vec{\Sigma}$ *is basic then* $\vec{\Sigma}'$ *is basic.*
If $\vec{\Sigma}$ *is* μ*-basic then* $\vec{\Sigma}'$ *is* μ*-basic.*
If $(\vec{\Sigma}, \vec{G})$ *is an action sequence then so is* $(\vec{\Sigma}', \vec{G})$.
If $(\vec{\Sigma}, \vec{G})$ *is eventually free, then so is* $(\vec{\Sigma}', \vec{G})$.

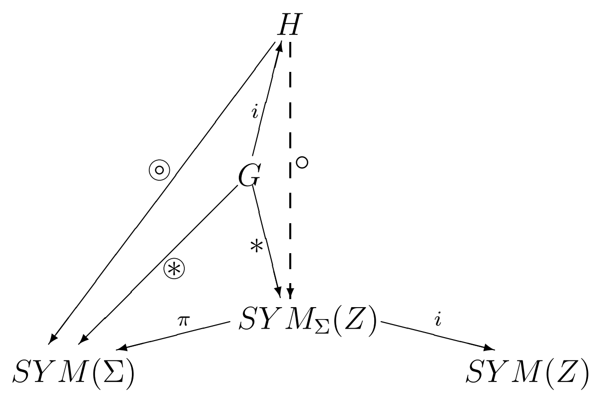
In particular, when discussing eventually free action sequences, we may always pass to a subsequence and assume that G_n acts freely on Σ'_n; except that $(\vec{\Sigma}', \vec{G})$ will not be large, even if $(\vec{\Sigma}, \vec{G})$ is.

Now, given G acting freely on X and points $a, b \in X$, we wish to extend the action to a larger group H, still acting freely, which contains an element δ moving a to b. We do this (Lemma 3.12) by obtaining $(\vec{\Sigma}, \vec{G})$ as above, and iterating Lemma 3.11

ω times. The extension process does not explicitly use the measure, although the measure was used in Lemma 3.9 to get the $(\vec{\Sigma}, \vec{G})$.

Lemma 3.11 is depicted abstractly in Figure 2. The (Z, Σ) in Lemma 3.11 will become a (Σ_{n+1}, Σ_n) in Lemma 3.12. Since there are a number of actions being discussed, we spell them out explicitly.

Fig. 2. Refining an Action



Lemma 3.11. *Assume:*

1. *G, H are groups, with $G \leq H$.*
2. *G acts on the set Z via $*$, Σ is a block system for $(G, *)$, and the induced action $\circledast$ of G on Σ is free.*
3. *For some $k > 0$, $|S| = k \cdot |H|$ for all $S \in \Sigma$.*
4. *H acts on Σ via $\circledcirc$ (which need not be free, or even faithful).*
5. *$\circledcirc$ extends $\circledast$.*
6. *$a \in A \in \Sigma$, and $b \in B \in \Sigma$, and $\varphi \circledast A \neq B$ for all $\varphi \in G$.*
7. *$\delta \in H$ and $\delta \circledcirc A = B$.*

Then, there is an action $\circ$ of H on Z such that:

1. $\circ$ *is free.*
2. Σ *is a block system for* $\circ$, *and* $\circledcirc$ *is the induced action of* H *on* Σ.
3. $\circ$ *extends* $*$.
4. $\delta \circ a = b$.

Proof. First, assume that $k = 1$. For each $S \in \Sigma$, choose a bijection Γ_S from H onto S so that whenever $\varphi \in G$ and $\varphi \circledast S = R$, we have $\varphi * \Gamma_S(\alpha) = \Gamma_R(\varphi\alpha)$ for all $\alpha \in H$; this is possible because $\circledast$ is free, so there is at most one $\varphi \in G$ with $\varphi \circledast S = R$. Also assume that $\Gamma_A(1) = a$ and $\Gamma_B(\delta) = b$; this is possible because A, B are in different $\circledast$-orbits. Then, define $\circ$ so that whenever $\beta \in H$ and $S, R \in \Sigma$ with $\beta \circledcirc S = R$, we have $\beta \circ \Gamma_S(\alpha) = \Gamma_R(\beta\alpha)$.

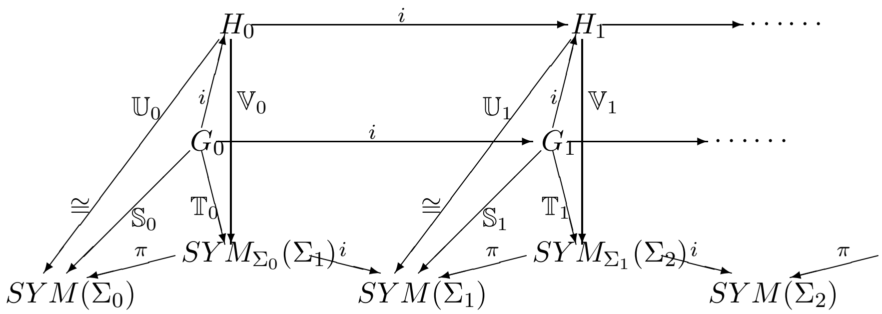
For $k > 1$, replace H by $H \times K$, where K is any group of order k, with the action only depending on H: $(\varphi, \psi)S = \varphi S$. Then, restrict $\circledcirc$ back to $H \cong H \times \{1\}$. □

Lemma 3.12. *Assume that* G *is a countable locally finite group and* $*$ *is a free continuous action of* G *on the compact* 2^{nd} *countable 0-dimensional Hausdorff space* X. *Let* $(\vec{\Sigma}, \vec{G})$ *be an eventually free action sequence for this action. Fix any* $a, b \in X$. *Then there are* $H, \circ, \vec{\Sigma}', \vec{H}$ *such that:*

$H \geq G$ *is a locally finite group acting freely on* X *via* $\circ$.
$\circ$ *extends* $*$.
$\delta \circ a = b$ *for some* $\delta \in H$.
$\vec{\Sigma}'$ *is a subsequence of* $\vec{\Sigma}$.
$(\vec{\Sigma}', \vec{H})$ *is a large eventually free action sequence for* $(X, H, \circ)$.

Proof. We may assume that $\varphi * a \neq b$ for all $\varphi \in G$, since otherwise we replace b by some b' not in the G-orbit of a. Then, passing to a subsequence, we may also assume that for each n, we have $A_n \neq B_n$ in Σ_n, with $a \in A_n$, $b \in B_n$, and $\varphi \circledast A_n \neq B_n$ for all $\varphi \in G_n$. Passing to a further subsequences, we may assume that the action of G_n on Σ_n is free, and that every set in Σ_n is the union of k_n sets in Σ_{n+1}, where $(|\Sigma_n|)! \mid k_n$. At this point, we can take $\vec{\Sigma}' = \vec{\Sigma}$; the action of each H_n on Σ_{n+1} will be free.

Fig. 3. Extending an Action Sequence



Since there are so many actions here, we denote them by $\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}$; see Figure 3. We are given $\mathbb{S}_n : G_n \to SYM(\Sigma_n)$ and $\mathbb{T}_n = \mathbb{S}_{n+1} \restriction G_n \to SYM_{\Sigma_n}(\Sigma_{n+1}) \subseteq SYM(\Sigma_{n+1})$. $\mathbb{U}_n$ and $\mathbb{V}_n$ are the corresponding actions of the H_n, which we must construct. At the same time, we shall have $\delta \in H_0 \le H_1 \le \cdots$, with $\delta \mathbb{U}_n A_n = B_n$ for all n.

To start with, $\mathbb{S}_0 : G_0 \to SYM(\Sigma_0)$ is free, so it is an injection, so we simply let $H_0 \ge G_0$ and $\mathbb{U}_0$ extend $\mathbb{S}_0$ so that $\mathbb{U}_0 : H_0 \to SYM(\Sigma_0)$ is an isomorphism. Then, fix $\delta \in H_0$ with $\delta \mathbb{U}_0 A_0 = B_0$.

Now, assume we have H_n and isomorphism $\mathbb{U}_n : H_n \to SYM(\Sigma_n)$, with $\delta \mathbb{U}_n A_n = B_n$. Then $|H_n| = (|\Sigma_n|)! \mid k_n$, so by Lemma 3.11, there is a free action $\mathbb{V}_n : H_n \to SYM(\Sigma_{n+1})$ such that Σ_n is a block system for $\mathbb{V}_n$, $\mathbb{U}_n$ is the induced action on Σ_n, and $\delta \mathbb{V}_n(A_{n+1}) = B_{n+1}$. At this point, we have injections $\mathbb{V}_n : H_n \to SYM(\Sigma_{n+1})$ and $\mathbb{S}_{n+1} : G_{n+1} \to SYM(\Sigma_{n+1})$, and these agree on G_n, so we can let H_{n+1} contain both H_n and G_{n+1} with $\mathbb{U}_{n+1} : H_{n+1} \to SYM(\Sigma_{n+1})$ an isomorphism extending both $\mathbb{V}_n$ and $\mathbb{S}_{n+1}$.

Finally, we have constructed for the H_n a diagram like that in Figure 1, so we have defined H and $\circ$. $\square$

Now, given an action of G on X and $a, b \in X$, we can construct $\vec{\Sigma}$ and then use $\vec{\Sigma}$ to extend G to an H moving a to b. If $(\vec{\Sigma}', \vec{H})$ is large, then the action of H will be scrambled:

Definition 3.13. Let $\circ$ be a continuous action of H on X. A finite clopen partition, $\{P_0, \ldots, P_{n-1}\}$ is *scrambled* iff for every $\sigma \in S_n$, there is a $\varphi \in H$ such that $\varphi \circ P_i = P_{\sigma(i)}$ for all $i < n$. The action is *scrambled* iff every open cover of X can be refined to a scrambled clopen partition.

Lemma 3.14. *Assume that G is a countable locally finite group and $*$ is a free continuous action of G on the compact 2^{nd} countable 0-dimensional Hausdorff space X. Also assume that μ is a rational $(G, *)$-invariant measure on X. Fix any $a, b \in X$. Then there is a locally finite group $H \geq G$, with an action $\circ$ on X which extends $*$, so that $\circ$ is free, scrambled, continuous, and measure-preserving, and so that $\delta \circ a = b$ for some $\delta \in H$.*

Besides being useful in the construction, the rationality of the measure is actually required by the other properties we obtain in Theorem 1.6:

Remark 3.15. *Let X be an infinite compact 0-dimensional Hausdorff space. Let G be a locally finite group which acts on X. Assume that the action is continuous, free, and scrambled. Then there is a unique G-invariant regular Borel probability measure μ on X, and this μ is rational.*

Proof. Since G is locally finite, there must be some G-invariant measure, μ (see Remark 1.4). Whenever $\Sigma = \{P_0, \ldots, P_{n-1}\}$ is a scrambled clopen partition of X, all the P_i must have the same measure, so for $A \subseteq n$, and $K = \bigcup\{P_i : i \in A\}$, we must have $\mu(K) = |A|/n$. Since for every clopen K, there is a scrambled Σ refining the cover $\{K, X \backslash K\}$, the values of μ are determined on all clopen sets, so that the G-invariant measure is unique. Clearly, $\mu(K)$ is rational for K clopen.

Now, to verify that μ is rational (Definition 1.3), fix a clopen K with $\emptyset \subsetneqq K \subsetneqq X$, and fix a rational $q \in (0,1)$. We shall find a clopen $H \subset K$ with $\mu(H) = q \cdot \mu(K)$.

Say $q = a/b$, where a, b are positive integers. Let $\Sigma = \{P_0, \ldots, P_{n-1}\}$ be a scrambled clopen partition such that $\Sigma \lhd \{K, X \backslash K\}$ and $n > b$. For each $\sigma \in S_n$, choose $\varphi_\sigma \in G$ such that $\varphi P_i = P_{\sigma(i)}$ for all $i < n$. Let G_0 be the subgroup of G generated by all the φ_σ. Then G_0 is finite (since G is locally finite), and Σ is a block system for G_0. We shall now construct H, using the fact that $n! \mid |G_0|$ (since $\pi : G_0 \to SYM(\Sigma)$ is onto).

The action of G_0 on X is free, so by Lemma 3.4, let $\Sigma' \lhd \Sigma$ be a clopen partition such that G_0 acts freely on Σ'. Then Σ' is a union of s G_0-orbits, $\Sigma' = \Omega_0 \cup \cdots \cup \Omega_{s-1}$, for some $s \geq 1$. Each $|\Omega_\ell| = |G_0|$. If $W \in \Omega_\ell$ and $W \subseteq P_i$, then $\psi W \subseteq P_i$ iff ψ is in the stabilizer $(G_0)_{P_i} = \{\chi \in G_0 : \psi P_i = P_i\}$, which has order $|G_0|/n$. Thus, exactly $|G_0|/n$ sets in Ω_ℓ are subsets of P_i. It follows that $\Sigma' = \{W_{\ell,i,j} : \ell < s, i < n, j < |G_0|/n\}$, where $W_{\ell,i,j} \in \Omega_\ell$ and $W_{\ell,i,j} \subseteq P_i$.

Let $r_\ell = \mu(\bigcup \Omega_\ell)$. Then $\sum_{\ell<s} r_\ell = 1$ and $\mu(W_{\ell,i,j}) = r_\ell/|G_0|$. Let $V_{i,j} = \bigcup_{\ell<s} W_{\ell,i,j}$, for $i < n$ and $j < |G_0|/n$. Then the $V_{i,j}$ are disjoint, $V_{i,j} \subseteq P_i$, and $\mu(V_{i,j}) = 1/|G_0| = (n/|G_0|) \cdot \mu(P_i)$. Say $q = a/b = c/(|G_0|/n)$; c is an integer because $b \mid (n-1)!$ and $(n-1)! \mid |G_0|/n$. Let $H_i = \bigcup_{j<c} V_{i,j}$. Then $\mu(H_i) = q \cdot \mu(P_i)$, so let $H = \bigcup\{H_i : P_i \subseteq K\}$. □

4. Shrinking Measurable Sets

Our result (Lemma 3.14) on extending measure-preserving actions assumes that the measure is rational, and our results on liftings of actions obtain a rational measure on Y, given a rational measure on X (Lemma 2.12), assuming that we split a rational sequence of sets (see Definition 2.11). We thus need the following result, which gives us such a rational sequence:

Lemma 4.1. *Let X be a compact 2^{nd} countable 0-dimensional Hausdorff space. Let μ be a rational measure on X. Let G be a countable locally finite group with a free continuous measure-preserving action on X. Let E be any measurable set and $\varepsilon > 0$. Then there is a closed $F \subseteq E$ such that $\mu(F) \geq (1-\varepsilon)\mu(E)$ and the sequence $\langle \varphi F : \varphi \in G\rangle$ is rational.*

The rest of this section gives a proof of this result. As in Section 3, we shall approximate the action of G by the actions of finite subgroups on finite partitions. The following definition will then be useful:

Definition 4.2. Let G be a finite group acting on X, $\emptyset \neq S \subseteq G$, and $F \subseteq X$. Then

$$\mathcal{I}(F, S, G) = \bigcap\{\varphi F : \varphi \in S\} \setminus \bigcup\{\varphi F : \varphi \in G\backslash S\} \quad .$$

If Σ is a finite clopen partition of X and μ a measure on X, then F is (Σ, G)-*rational* iff for each $P \in \Sigma$ and each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$:
$\mu(P \cap \mathcal{I}(F, S, G)) \in \mathbb{Q}$.
If $\mu(P \cap \mathcal{I}(F, S, G)) = 0$, then $P \cap \mathcal{I}(F, S, G) = \emptyset$.

Observe that $\bigcap\{\varphi F : \varphi \in S\} = \bigcup\{\mathcal{I}(F, T, G) : T \supseteq S\}$ whenever $S \in \mathcal{P}(G)\backslash\{\emptyset\}$, and this is a disjoint union. From this it is easy to see:

Lemma 4.3. *Suppose that $(\vec{\Sigma}, \vec{G})$ is an action sequence for (X, G) (see Definition 3.7), and suppose that F is (Σ_n, G_n)-rational for each n. Then the sequence $\langle \varphi F : \varphi \in G\rangle$ is rational.*

It is thus natural to prove Lemma 4.1 by successive approximation, in ω steps, getting $F_n \searrow F$, where F_n is (Σ_n, G_n)-rational. Unfortunately, $\mathcal{I}(F, S, G)$ is not a monotonic function of F. We avoid this problem by:

Definition 4.4. $F \sqsubseteq_G E$ iff $\mathcal{I}(F,S,G) \subseteq \mathcal{I}(E,S,G)$ for all $S \in \mathcal{P}(G)\backslash\{\emptyset\}$.

$\sqsubseteq_G$ is clearly transitive, and $F \sqsubseteq_G E$ implies that $F \subseteq E$. The next two lemmas show that a proof of Lemma 4.1 by successive approximation really can work:

Lemma 4.5. *If $F \sqsubseteq_H E$ and $G \leq H$ then $F \sqsubseteq_G E$.*

Proof. $\mathcal{I}(F,S,G) = \bigcup\{\mathcal{I}(F,T,H) : T \in \mathcal{P}(H)\backslash\{\emptyset\} \;\&\; T \cap G = S\}$ whenever $S \in \mathcal{P}(G)\backslash\{\emptyset\}$. □

Lemma 4.6. *If G is finite, $F_0 \sqsupseteq_G F_1 \sqsupseteq_G F_2 \sqsupseteq_G \cdots$ and $F = \bigcap_{n\in\omega} F_n$, then $F \sqsubseteq_G F_n$ for each n, and $\mathcal{I}(F,S,G) = \bigcap_{n\in\omega} \mathcal{I}(F_n,S,G)$ for each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$.*

Observe that $\varphi\mathcal{I}(F,S,G) = \mathcal{I}(F,\varphi S,G)$. Conversely,

Lemma 4.7. *Suppose that $E \subseteq X$ and G is a finite group acting on X. For each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$, choose a set $F_S \subseteq \mathcal{I}(E,S,G)$ so that $\varphi F_S = F_{\varphi S}$ for each φ, S. Define $F = \bigcup\{F_S : 1 \in S\}$. Then $F \sqsubseteq_G E$, and $\mathcal{I}(F,S,G) = F_S$ for all $S \in \mathcal{P}(G)\backslash\{\emptyset\}$.*

Observe that we may choose F_S so that $\mu(F_S)$ is any real between 0 and $\mu(\mathcal{I}(E,S,G))$. This leads to:

Lemma 4.8. *Let X be a compact Hausdorff space. Let μ be a finite non-atomic regular Borel measure on X. Let G be a finite group with a continuous measure-preserving action on X. Let Σ be a finite clopen block system for G, and assume that the induced action of G on Σ is free. Suppose that $E \subseteq X$ is measurable. For each $P \in \Sigma$ and each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$, choose a real $r_{P,S}$ such that:*

$r_{P,S} = r_{\varphi P,\varphi S}$ for each φ, P, S.

$0 \leq r_{P,S} < \mu(P \cap \mathcal{I}(E,S,G))$ whenever $\mu(P \cap \mathcal{I}(E,S,G)) \neq 0$.

$r_{P,S} = 0$ whenever $\mu(P \cap \mathcal{I}(E,S,G)) = 0$.

Then there is a closed $F \sqsubseteq_G E$ such that for each $P \in \Sigma$ and each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$:

1. *$P \cap \mathcal{I}(F, S, G)$ is closed and $\mu(P \cap \mathcal{I}(F, S, G)) = r_{P,S}$.*
2. *If $r_{P,S} = 0$ then $P \cap \mathcal{I}(F, S, G) = \emptyset$.*

Proof. We may assume that G acts transitively on Σ; if not, work on each orbit separately. Then, fix some $R \in \Sigma$; it is sufficient to obtain (1) and (2) just for $P = R$.

For each $S \in \mathcal{P}(G)\backslash\{\emptyset\}$, choose a closed $K_S \subseteq R \cap \mathcal{I}(E, S, G)$ such that $\mu(K_S) = r_{R,S}$, and take $K_S = \emptyset$ if $r_{R,S} = 0$. Now let $F_S = \bigcup\{\psi K_{\psi^{-1}S} : \psi \in G\}$. Note that $F_S \subseteq \mathcal{I}(E, S, G)$ and $\varphi(F_S) = F_{\varphi S}$. Define $F = \bigcup\{F_S : 1 \in S\}$. By Lemma 4.7, $\mathcal{I}(F, S, G) = F_S$, so that $R \cap \mathcal{I}(F, S, G) = K_S$. □

Proof of Lemma 4.1.
Assume $\mu(E) > 0$ (otherwise, let $F = \emptyset$). By Lemmas 3.9 and 3.10, let $(\vec{\Sigma}, \vec{G})$ be an action sequence for (X, G), with G_n acting freely on Σ_n. Assume that $\Sigma_0 = \{X\}$ and $G_0 = \{1\}$. Let q_n, for $n \in \omega$, be rationals with $\frac{1}{1-\varepsilon} > q_0 > q_1 > q_2 > \cdots$ and $q_n \searrow 1$. We shall find closed sets F_n, for $n \in \omega$, with $E \sqsupseteq_{G_0} F_0 \sqsupseteq_{G_1} F_1 \sqsupseteq_{G_2} F_2 \cdots$, and F will be $\bigcap_n F_n$. Also, for each n, each $P \in \Sigma_n$, and each $S \in \mathcal{P}(G_n)\backslash\{\emptyset\}$, choose a *rational* $r^n_{P,S}$ such that:

1. $(1 - \varepsilon)\mu(E) < r^0_{X,\{1\}} < q_0 r^0_{X,\{1\}} < \mu(E)$.
2. $r^n_{P,S} = r^n_{\varphi P, \varphi S}$ for each n, φ, P, S.
3. $\mu(P \cap \mathcal{I}(F_n, S, G_n)) = q_n r^n_{P,S}$ whenever $P \in \Sigma_n$ and $S \in \mathcal{P}(G_n)\backslash\{\emptyset\}$.
4. $P \cap \mathcal{I}(F_n, S) = \emptyset$ whenever $r^n_{P,S} = 0$, $P \in \Sigma_n$ and $S \in \mathcal{P}(G_n)\backslash\{\emptyset\}$.
5. $r^n_{P,S} = \sum\{r^{n+1}_{Q,T} : Q \in \Sigma_{n+1}\ \&\ Q \subseteq P\ \&\ T \in \mathcal{P}(G_{n+1})\backslash \{\emptyset\}\ \&\ T \cap G_n = S\}$ whenever $P \in \Sigma_n$ and $S \in \mathcal{P}(G_n)\backslash\{\emptyset\}$.
6. $\mu(P \cap \mathcal{I}(F_m, S, G_n)) = q_m r^n_{P,S}$ whenever $m \geq n$, $P \in \Sigma_n$ and $S \in \mathcal{P}(G_n)\backslash\{\emptyset\}$.

Of course, (3) is a special case of (6) (when $m = n$), but we view items (2-5) as being enforced during the inductive construction. Then, observe that (6) follows (since (6) for (n, m) follows from (6) for $(n+1, m)$ by using (5)). Applying (6), and letting $m \to \infty$, we have (by Lemmas 4.5 and 4.6), $\mu(P \cap \mathcal{I}(F, S, G_n)) = r^n_{P,S}$. In particular, (1) yields $\mu(F) > (1-\varepsilon)\mu(E)$. Applying (4), we see that F is (Σ_n, G_n)-rational for each n. Thus, we are done by Lemma 4.3.

For $n = 0$, choose a rational $r^0_{X,\{1\}}$ with $(1-\varepsilon)\mu(E) < r^0_{X,\{1\}} < \frac{1}{q_0}\mu(E)$, and let $F \subseteq E$ be closed with $\mu(F) = q_0 r^0_{X,\{1\}}$. Then (1-4) hold for $n = 0$.

Suppose we have F_n and the $r^n_{P,S}$ satisfying (1)-(4). *Define* $x^{n+1}_{Q,T}$ so that (a) holds, and observe that (b) and (c) follow:

a. $\mu(Q \cap \mathcal{I}(F_n, T, G_{n+1})) = q_n x^{n+1}_{Q,T}$ whenever $Q \in \Sigma_{n+1}$ and $T \in \mathcal{P}(G_{n+1}) \backslash \{\emptyset\}$.
b. $r^n_{P,S} = \sum\{x^{n+1}_{Q,T} : Q \in \Sigma_{n+1} \ \& \ Q \subseteq P \ \& \ T \in \mathcal{P}(G_{n+1}) \backslash \{\emptyset\} \ \& \ T \cap G_n = S\}$ whenever $P \in \Sigma_n$ and $S \in \mathcal{P}(G_n) \backslash \{\emptyset\}$.
c. $x^{n+1}_{Q,T} = x^{n+1}_{\varphi Q, \varphi T}$ for each φ, Q, T.

Now, since the rationals are dense, choose rationals $r^{n+1}_{Q,T}$ so that:

d. $0 < r^{n+1}_{Q,T} < \frac{q_n}{q_{n+1}} x^{n+1}_{Q,T}$ whenever $x^{n+1}_{Q,T} > 0$.
e. $r^{n+1}_{Q,T} = 0$ whenever $x^{n+1}_{Q,T} = 0$.
f. (2) above (for $n+1$) and (5) hold.

Since (d) yields $0 < q_{n+1} r^{n+1}_{Q,T} < q_n x^{n+1}_{Q,T}$, Lemma 4.8 implies that we can choose $F_{n+1} \sqsubseteq_{G_{n+1}} F_n$ so that (3) and (4) hold (for $n+1$).

5. The *CH* Construction

Proof of Theorem 1.6.

Let C be be the Cantor set, 2^ω. Let π^β_α be the natural projection from C^β onto C^α whenever $\alpha \le \beta$. Let $\pi_\alpha = \pi^{\omega_1}_\alpha$. As is typical of such L-space constructions [1, 5, 10, 13, 15], we construct $X \subseteq C^{\omega_1}$ by building inductively $X_\alpha = \pi_\alpha(X) \subseteq C^\alpha$.

For $0 < \alpha \leq \omega_1$, we construct $X_\alpha, \mu_\alpha, G_\alpha, \star_\alpha, F_\alpha$ with the following properties:

P1. X_α is closed in C^α, and has no isolated points.
P2. $X_\alpha = \pi^\beta_\alpha(X_\beta)$ whenever $\alpha \leq \beta$.
P3. Each G_α is a locally finite group, with G_α countable whenever $\alpha < \omega_1$.
P4. $G_\alpha \leq G_\beta$ whenever $\alpha \leq \beta$.
P5. $\star_\alpha$ is a free scrambled continuous action of G_α on X_α.
P6. $\varphi \star_\alpha \pi^\beta_\alpha(x) = \pi^\beta_\alpha(\varphi \star_\beta x)$ whenever $\alpha \leq \beta$, $x \in X_\beta$, and $\varphi \in G_\alpha$.
P7. F_α is a closed subset of X_α.
P8. $(\pi^{\alpha+1}_\alpha)^{-1}(F_\alpha)$ has non-empty interior in $X_{\alpha+1}$.
P9. $|(\pi^{\alpha+1}_\alpha)^{-1}(\{x\})| = 1$ for all $x \in X_\alpha \setminus \bigcup_{\varphi \in G_\alpha} \varphi F_\alpha$.
P10. Each μ_α is a $(G_\alpha, \star_\alpha)$-invariant rational probability measure on X_α.
P11. $\mu_\alpha = \mu_\beta(\pi^\beta_\alpha)^{-1}$ whenever $\alpha \leq \beta$.
P12. $\mu(F_\alpha) > 0$ and the sequence $\langle \varphi F_\alpha : \varphi \in G_\alpha \rangle$ is rational.

Now, by CH, let $B_\alpha \subseteq C^\alpha$ be closed, for $0 < \alpha < \omega$, so that whenever $B \subseteq C^{\omega_1}$ is a closed G_δ set, $B = \pi_\alpha^{-1}(B_\alpha)$ for some $\alpha < \omega_1$. We shall make sure that if $B_\alpha \cap X_\alpha$ has positive measure, it gets a non-empty interior (P13 + P8), while if $B_\xi \cap X_\xi$ is a null set, it never gets split past stage ξ (P14 + P9). Again by CH, choose $\xi_\alpha, a_\alpha, b_\alpha$, for $0 < \alpha < \omega_1$, such that $\xi_\alpha \leq \alpha$, $a_\alpha, b_\alpha \in C^{\xi_\alpha}$, and

$$\forall \xi < \omega_1 \; \forall a, b \in C^\xi \; \exists \alpha < \omega_1 [\xi = \xi_\alpha \;\&\; a = a_\alpha \;\&\; b = b_\alpha] \enspace .$$

Property P15 will make the group action transitive.

P13. If $\mu_\alpha(B_\alpha \cap X_\alpha) > 0$, then $F_\alpha \subseteq B_\alpha$.
P14. If $0 < \xi \leq \alpha$ and $\mu_\xi(B_\xi \cap X_\xi) = 0$, then $\varphi F_\alpha \cap (\pi^\alpha_\xi)^{-1}(B_\xi) = \emptyset$ for all $\varphi \in G_\alpha$.

P15. If $a_\alpha, b_\alpha \in X_{\xi_\alpha}$, then $\varphi a = b$ for some $\varphi \in G_\alpha$ and some $a, b \in X_\alpha$ such that $\pi^{\xi_\alpha}(a) = a_\alpha$ and $\pi^{\xi_\alpha}(b) = b_\alpha$.

The construction is done inductively. When $0 < \alpha < \omega_1$, we decide, in order:

a. X_α and μ_α.
b. G_α and $\star_\alpha$.
c. F_α.
d. $X_{\alpha+1}$, $\mu_{\alpha+1}$, and the action $\star_{\alpha+1}$ of G_α on $X_{\alpha+1}$.

In some of these, there are three cases: $\alpha = 1$, α a limit, and α a successor larger than 1.

(a): $X_1 = C$, and μ_1 is any rational probability measure on C. If α is a limit, then X_α and μ_α are already determined by P2 and P11. If $\alpha > 1$ is successor, then X_α and μ_α were determined in step (d) for $\alpha - 1$.

(b): Let $\widehat{G}_1 = \{1\}$. Let $\widehat{G}_\alpha$ be $\bigcup_{\delta<\alpha} G_\delta$ when α is a limit, and let $\widehat{G}_\alpha = G_{\alpha-1}$ when $\alpha > 1$ is a successor. In view of (d) and P6, the action of $\widehat{G}_\alpha$ on X_α is already determined. Then, apply Lemma 3.14 to obtain $G_\alpha \geq \widehat{G}_\alpha$ and its action on X_α so that P15 and P5 hold at α.

(c): Let $D_\alpha = B_\alpha \cap X_\alpha$. Let $H_\alpha = D_\alpha$ if $\mu_\alpha(D_\alpha) > 0$; otherwise, $H_\alpha = X_\alpha$. Let

$$E_\alpha = H_\alpha \backslash \bigcup \{\varphi((\pi^\alpha_\xi)^{-1}(D_\xi)) : 0 < \xi \leq \alpha \;\&\; \mu_\xi(D_\xi) = 0 \;\&\; \varphi \in G_\alpha\} \ .$$

Then choose $F_\alpha \subseteq E_\alpha$ by Lemma 4.1.

(d): Let $I = G_\alpha \times \omega$, and let G act on I by $\varphi(\psi, k) = (\varphi\psi, k)$. For $i \in I$, choose $p_i, q_i \in (0, 1)$ so that $p_i + q_i = 1$ and for each $\varphi \in G$, the sequence $\langle p_{(\varphi,k)} : k \in \omega\rangle$ is redundant (see Definition 2.10). Let $F^i_0 = X_\alpha$, and let $F^{(\varphi,k)}_1 = \varphi F_\alpha$. Then obtain $Y \subseteq X_\alpha \times 2^I$ and ν on Y as in Definition 2.7. ν is a rational measure by Lemma 2.12. Since I is countable, we can identify 2^I with C, giving us $X_{\alpha+1}$ and the action of G_α on $X_{\alpha+1}$.

For $\alpha = \omega_1$, we just do steps (a) and (b) to determine $X = X_{\omega_1}$ and $\mu = \mu_{\omega_1}$, with $G = \widehat{G}_{\omega_1}$. We proceed to verify the hypotheses of Lemma 1.5. (1) and (2) are clear. For (3) and (4), consider a closed G_δ set $D = B \cap X \subseteq X$, where B is a closed G_δ in X.

If $\mu(D) > 0$, then $\mathrm{int(D)} \neq \emptyset$ is guaranteed by properties P13 and P8 at stage α, where $B = \pi_\alpha^{-1}(B_\alpha)$, and hence $D = \pi_\alpha^{-1}(D_\alpha)$.

If $\mu(D) = 0$, then fix ξ such that $B = \pi_\xi^{-1}(B_\xi)$. By P14 and P9, $\pi_\xi \restriction D : D \to D_\xi$ is 1-1 on D, so that D is homeomorphic to D_ξ, and hence 2^{nd} countable.

Finally, by P5 and P15, the action of G on X is regular; P15 implies transitivity of the action because each point of X eventually ceases to be split in the inverse limit. We then we get our group operation on X by Lemma 3.2. The rest of Theorem 1.6 is clear from the construction.

6. Suslin L-spaces

The following specializes some standard definitions to the case at hand:

Definition 6.1. Let X be a compact 0-dimensional Hausdorff space. An *antichain* in X is a disjoint family $\mathcal{A}$ of non-empty clopen subsets of X; $\mathcal{A}$ is *maximal* iff $\bigcup \mathcal{A}$ is dense in X. X is *Suslin* iff X is ccc, X has no isolated points, and whenever $\mathcal{A}_n$ are maximal antichains for $n \in \omega$, $\bigcup\{\mathrm{int}(\bigcap_\mathrm{n} \mathrm{K_n}) : \forall \mathrm{n}[\mathrm{K_n} \in \mathcal{A}_\mathrm{n}]\}$ is dense in X.

So, there is a Suslin space iff there is a Suslin tree. Observe that if μ is a finite regular Borel measure on such an X, then its support is nowhere dense. Also note that X cannot be separable. X need not be HL; for example, the absolute of a Suslin space is a Suslin space. However, if there is a Suslin tree, then there is an HL Suslin space (for example, a Suslin line). We do not know whether there can be a compact Suslin line which is also a right topological group, but:

Theorem 6.2. *Assume $\diamondsuit$. Then there is a compact 0-dimensional Suslin L-space X with a group operation $\cdot : X \times X \to X$ such that:*

1. *For each a, the map $x \mapsto x \cdot a$ is continuous.*
2. *$(X, \cdot)$ is super-scrambled.*

Remark 6.3. *If X is a compact 0-dimensional HL Suslin space, then there is a Suslin tree dense in the clopen algebra of X.*

Proof. This is clear if $w(X) = \aleph_1$. Now, suppose that $w(X) \geq \aleph_2$. Since the regular open algebra is Suslin, we could find clopen sets K_α for $\alpha < \omega_2$ such that whenever $\alpha_0 < \alpha_1 < \cdots < \alpha_n$: $\bigcap_{i<n} K_{\alpha_i} \nsubseteq K_{\alpha_n}$. But then $\bigcap_{\alpha<\omega_1} K_\alpha$ would be a closed set which is not a G_δ. □

Since the group $(X, \cdot)$ cannot be locally finite (see Remark 1.4), the extension procedure described in Section 3 does not apply in proving Theorem 6.2. On the other hand, we do not have a measure to preserve, so the task is simplified a bit. We just use:

Lemma 6.4. *Assume that G is a countable group and $*$ is a free continuous action of G on the Cantor set $X \cong 2^\omega$. Fix any $a, b \in X$. Then there is a group $H \geq G$, with an action $\circ$ on X which extends $*$, such that $\circ$ is free and continuous, and such that $\sigma \circ a = b$ for some $\sigma \in H$.*

Proof. Regard G as a subgroup of $\mathcal{H}(X)$, the group of all homeomorphisms of X. Then G being free simply means that no $\varphi \in G \backslash \{1\}$ has any fixed points. We shall choose $\sigma \in \mathcal{H}(X)$ with $\sigma(a) = b$ and let H be $\langle G, \sigma \rangle$, the group generated by $G \cup \{\sigma\}$. Assume that $\varphi(a) \neq b$ for all $\varphi \in G$ (otherwise take $H = G$). In particular, $a \neq b$.

Fix a metric d on X, and use the *sup* metric on $C(X, X)$. Let $I \subseteq C(X, X)$ be the set of all σ such that $\sigma(a) = b$ and $\sigma^2 = 1$. Then I is closed in $C(X, X)$, and $I \subseteq \mathcal{H}(X)$. Since $C(X, X)$, and hence I, are separable complete metric spaces, we can proceed by a category argument:

For $n \in \omega$, A a finite subset of $G \backslash \{1\}$, and $\sigma \in I$, let $J(A, \sigma, n)$ be the set of all alternating products of σ with n or fewer elements of A – that is, σ together with all products of the forms:

$$\varphi_1 \sigma \varphi_2 \sigma \cdots \varphi_m, \quad \varphi_1 \sigma \varphi_2 \sigma \cdots \varphi_m \sigma, \quad \sigma \varphi_1 \sigma \varphi_2 \sigma \cdots \varphi_m, \quad \sigma \varphi_1 \sigma \varphi_2 \sigma \cdots \varphi_m \sigma \quad ,$$

where $1 \leq m \leq n$ and $\varphi_1, \ldots, \varphi_m \in A$. Then every element of $\langle G, \sigma \rangle \backslash G$ is in $J(A, \sigma, n)$ for some finite A, n. Let

$$U(A, n) = \{\sigma \in I : \forall \psi \in J(A, \sigma, n) \, \forall x \in X \, [\psi(x) \neq x]\} \quad .$$

All $U(A, n)$ are open in I, so we are done if we can show that they are all dense. We proceed by induction on n:

For $n = 0$: $U(A, 0) = \{\sigma \in I : \forall x \in X \, [\sigma(x) \neq x]\}$. Fix $\sigma \in I$ and $\varepsilon > 0$. Let Δ be a block system for $\{1, \sigma\}$ (see Definition 3.3) such that diam(P) $\leq \varepsilon$ for each $P \in \Delta$ and such that a, b are in different blocks of Δ. Define $\sigma' \in U(A, 0)$ with $d(\sigma', \sigma) \leq \varepsilon$ as follows. If $P, Q \in \Delta$ and $\sigma P = Q \neq P$, then $\sigma' \restriction (P \cup Q) = \sigma \restriction (P \cup Q)$. If $P \in \Delta$ and $\sigma P = P$, then partition P into two clopen sets, and let σ' exchange these sets.

For $n = 1$: Since $U(A, 0)$ is dense, it is sufficient to fix $\sigma \in U(A, 0)$ and $\varepsilon > 0$, and produce a $\sigma' \in U(A, 1)$ with $d(\sigma', \sigma) \leq \varepsilon$. Let Δ be as above, but now, since $\sigma \in U(A, 0)$, we may assume that $\sigma P \neq P$ for all $P \in \Delta$, so for some r, $\Delta = \{P_i : i < r\} \cup \{Q_i : i < r\}$, where $\sigma P_i = Q_i$. Say $a \in P_0$, so that $b \in Q_0$. σ' will also exchange each P_i, Q_i, which guarantees $d(\sigma', \sigma) \leq \varepsilon$. Since $\varphi(a) \neq b$ and $\varphi(b) \neq a$, we may also assume that $Q_0 \cap \varphi P_0 = P_0 \cap \varphi Q_0 = \emptyset$ for each $\varphi \in A$. Now, $J(A, \sigma', 1)$, contains, besides σ', elements of the forms $\sigma' \varphi, \varphi \sigma', \sigma' \varphi \sigma'$, for $\varphi \in A$. To make sure that these have no fixed points, it is sufficient to ensure that $\sigma'(x) \neq S(x) := \{\varphi(x) : \varphi \in A\} \cup \{\varphi^{-1}(x) : \varphi \in A\}$. for each $x \in X$. This is satisfied for $x \in P_0 \cup Q_0$ by taking $\sigma' \restriction (P_0 \cup Q_0) = \sigma \restriction (P_0 \cup Q_0)$; in particular, $\sigma'(a) = b$, so that $\sigma' \in \mathcal{I}$. For $i > 0$, $\sigma' \restriction P_i$ can be an arbitrary homeomorphism onto Q_i such that $\sigma'(x)$ avoids the finite set $S(x)$.

To prove $U(A, n+1)$ is dense, where $n \geq 1$: Let $B = A \cup \{\varphi\varphi' : \varphi, \varphi' \in A \;\&\; \varphi\varphi' \neq 1\}$. Applying induction, we may assume that $U(B, n)$ is dense, so it is sufficient to fix $\sigma \in U(B, n)$ and $\varepsilon > 0$, and produce a $\sigma' \in U(A, n+1)$ with $d(\sigma', \sigma) \leq \varepsilon$. By compactness of X, we may fix $\delta > 0$ such that whenever $\sigma' \in I$, $d(\sigma', \sigma) \leq \delta$, $\psi \in J(B, \sigma', n)$, and $x \in X$, we have $d(\psi(x), x) > \delta$. Let $\Delta = \{P_i : i < r\} \cup \{Q_i : i < r\}$ be as above, but now assume that each block has diameter $\leq \min(\delta, \epsilon)$. Again, σ' will exchange each P_i, Q_i and will agree with σ on $P_0 \cup Q_0$. Note that σ' will then automatically be in $U(B, n)$ and hence in $U(A, n)$.

Furthermore, note that to get $\sigma' \in U(A, n+1)$, it is sufficient to ensure that $\sigma'\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n(x) \neq x$ whenever $x \in X$ and $\varphi_0, \varphi_1, \ldots, \varphi_n \in A$. To see this, consider the other types of products in $J(A, \sigma', n+1)$. If $\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n\sigma'(x) = x$, then we would have $\sigma'\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n(y) = y$, where $y = \sigma'(x)$. Likewise, if $\sigma'\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n\sigma'(x) = x$, then we would have $\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n(y) = y$. Finally, if $\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n(x) = x$, then we would have $\sigma'\varphi_1\sigma'\varphi_2\sigma' \cdots \varphi_n\varphi_0(y) = y$, where $y = \varphi_0^{-1}(x)$. This contradicts either $\sigma' \in U(B, n)$ (when $\varphi_n\varphi_0 \neq 1$) or $\sigma' \in U(A, n-1)$ (when $\varphi_n\varphi_0 = 1$).

Now, given any x, let $i(x)$ be the $i < r$ such that $x \in P_i \cup Q_i$. So, we always have $i(\sigma'(x)) = i(x)$ Note that for $\varphi_1, \ldots, \varphi_n \in A$, the $n+1$ numbers $i(x)$, $i(\sigma'\varphi_n(x)), \ldots,$ $i(\sigma'\varphi_1 \cdots \sigma'\varphi_n(x))$ are all distinct: If not, then for some $m < \ell \leq n$ and some $y \in X$, we would have $i(\sigma'\varphi_m \cdots \sigma'\varphi_{\ell-1}\sigma'\varphi_\ell(y)) = i(y)$. Then for ψ equal either $\sigma'\varphi_m \cdots \sigma'\varphi_{\ell-1}\sigma'\varphi_\ell$ or $\varphi_m \cdots \sigma'\varphi_{\ell-1}\sigma'\varphi_\ell$, we would have y and $\psi(y)$ in the same block, so that $d(\psi(y), y) \leq \delta$, contradicting our choice of δ and Δ.

Thus, if we do have $\sigma'\varphi_0\sigma'\varphi_1\sigma' \cdots \varphi_n(x) = x$, there will be some ℓ such that $i := i(\sigma'\varphi_\ell \cdots \sigma'\varphi_n(x)) > i(\sigma'\varphi_m \cdots \sigma'\varphi_n(x))$ for all $m \neq \ell$. If $y = \sigma'\varphi_\ell \cdots \sigma'\varphi_n(x)$, we would then have $\sigma'(y) = \varphi_\ell \cdots \sigma'\varphi_n\sigma'\varphi_0 \cdots \varphi_{\ell-1}(y)$. Since all the intermediate values used in computing $\varphi_\ell \cdots \sigma'\varphi_n\sigma'\varphi_0 \cdots \varphi_{\ell-1}(y)$ lie in $\bigcup_{j<i} P_j \cup Q_j$, we can easily avoid these equalities by defining $\sigma' \restriction P_i \cup Q_i$ by induction on $i < r$: Let $\sigma' \restriction P_0 \cup Q_0 = \sigma \restriction P_0 \cup Q_0$.

For $i > 0$: Let τ_i agree with σ' on $\bigcup_{j<i} P_j \cup Q_j$ and with σ on $\bigcup_{j \geq i} P_j \cup Q_j$. Let $\sigma' \restriction P_i$ be a homeomorphism onto Q_i such that for $x \in P_i$:

$$\sigma'(x) \notin \{\psi(x) : \psi \in J(A, \tau_i, n+1) \text{ or } \psi^{-1} \in \mathrm{J}(\mathrm{A}, \tau_{\mathrm{i}}, \mathrm{n}+1)\} \quad .$$

Then $\sigma' \restriction Q_i$ is just the inverse of $\sigma' \restriction P_i$. □

Remark 6.5. Under CH (or MA), one can use this to construct a right topological group operation on the Cantor set with no Haar measure. Just start with a countable group acting freely on X so that some clopen set is moved to a proper subset of itself, and then extend this action $2^{\aleph_0}$ times to get a free transitive action.

Lemma 6.6. *Assume that G is a countable group and $*$ is a free continuous action of G on the Cantor set $X \cong 2^\omega$. Then there is a countable group $H \geq G$, with an action $\circ$ on X which extends $*$, such that $\circ$ is free, continuous, and super-scrambled.*

Proof. We follow the notation in the proof of Lemma 6.4. Since there are only countably many partitions into clopen sets, it is sufficient to fix a finite clopen partition, $\{P_0, \ldots, P_{r-1}\}$, with $r \geq 1$, and extend G to an $H = \langle G, \sigma' \rangle$, where $\sigma' P_i$ is P_i for $i \geq 2$ and P_{1-i} for $i < 2$. Fix $a \in P_0$ and $b \in P_1$ with a, b in different G-orbits. Fix $\sigma \in I$ such that σ exchanges P_0, P_1 and is the identity on P_i for $i \geq 2$. Choose a metric d on X such that $d(x, y) > 1$ whenever $x \in P_i$, $y \in P_j$, and $i \neq j$. Then choose $\sigma' \in I$ with $\langle G, \sigma' \rangle$ free and $d(\sigma', \sigma) \leq 1$. □

The rest of the proof of Theorem 6.2 is now like the standard $\diamondsuit$ construction of a Suslin tree, modified to capture antichains in X rather than antichains in the tree. We follow the same basic notation as in the proof of Theorem 1.6 in Section 5. We again obtain $X \subseteq C^{\omega_1}$, where C is the Cantor set.

Definition 6.7. For K clopen in C^{ω_1}, let supt(K) be the least $\alpha < \omega_1$ such that $K = (\pi_\alpha)^{-1}\pi_\alpha(K)$. If $\mathcal{B}$ is a family of clopen subsets of C^{ω_1} and $\alpha < \omega_1$, then

$$\mathcal{B} \restriction \alpha = \{\pi_\alpha(K) : K \in \mathcal{B} \ \& \ \mathrm{supt(K)} \le \alpha\} \quad .$$

If $X \subseteq C^\alpha$ and $\mathcal{B}$ is a family of clopen subsets of C^α, then

$$\mathcal{B} \cap X = \{K \cap X : K \in \mathcal{B}\} \quad .$$

Lemma 6.8. *Suppose that $\mathcal{B}$ is a family of non-empty clopen subsets of C^{ω_1}, X is closed in C^{ω_1}, and $\mathcal{B} \cap X$ is a maximal antichain in X. Then*

$$\{\xi < \omega_1 : (\mathcal{B} \restriction \xi) \cap \pi_\xi(X) \text{ is a maximal antichain in } \pi_\xi(X)\}$$

is club in ω_1.

Definition 6.9. A $\diamondsuit$-*sequence* is a sequence $\{\mathcal{B}_\xi : \xi < \omega_1\}$ such that each $\mathcal{B}_\xi$ is a family of non-empty clopen subsets of C^ξ and $\{\xi : \mathcal{B} \restriction \xi = \mathcal{B}_\xi\}$ is stationary whenever $\mathcal{B}$ is any family of non-empty clopen subsets of C^{ω_1}.

It is easily seen that the usual definitions of $\diamondsuit$ imply that there is such a $\diamondsuit$-sequence.

Proof of Theorem 6.2.
Fix a diamond sequence $\{\mathcal{B}_\xi : \alpha < \omega_1\}$. Choose $\xi_\alpha, a_\alpha, b_\alpha$ exactly as in in Section 5. Now construct $X_\alpha, G_\alpha, \star_\alpha, F_\alpha$ satisfying:

Q1. X_α is closed in C^α, and has no isolated points.
Q2. $X_\alpha = \pi^\beta_\alpha(X_\beta)$ whenever $\alpha \le \beta$.
Q3. Each G_α is a group, with G_α countable whenever $\alpha < \omega_1$.
Q4. $G_\alpha \le G_\beta$ whenever $\alpha \le \beta$.
Q5. $\star_\alpha$ is a free continuous super-scrambled action of G_α on X_α.
Q6. $\varphi \star_\alpha \pi^\beta_\alpha(x) = \pi^\beta_\alpha(\varphi \star_\beta x)$ whenever $\alpha \le \beta$, $x \in X_\beta$, and $\varphi \in G_\alpha$.
Q7. $F_\alpha = \{p_\alpha\}$ for some $p_\alpha \in X_\alpha$.

Q8. $(\pi^{\alpha+1}_\alpha)^{-1}(F_\alpha)$ has non-empty interior in $X_{\alpha+1}$.

Q9. $|(\pi^{\alpha+1}_\alpha)^{-1}(\{x\})| = 1$ for all $x \in X_\alpha \setminus \bigcup_{\varphi\in G_\alpha} \varphi F_\alpha$.

Q10. $\varphi p_\alpha \in U$ for all $\varphi \in G_\alpha$ and all $U \in \mathcal{U}_\alpha$; where $\mathcal{U}_\alpha$ is the family of all $\bigcup((\pi^\alpha_\xi)^{-1}(\mathcal{B}_\xi \cap X_\xi))$ such that $\xi \le \alpha$ and $(\pi^\alpha_\xi)^{-1}(\mathcal{B}_\xi \cap X_\xi)$ is a maximal antichain in X_α.

Q11. If $a_\alpha, b_\alpha \in X_{\xi_\alpha}$, then $\varphi a = b$ for some $\varphi \in G_\alpha$ and some $a, b \in X_\alpha$ such that $\pi^{\xi_\alpha}(a) = a_\alpha$ and $\pi^{\xi_\alpha}(b) = b_\alpha$.

The induction uses the same sequence of steps (a)(b)(c)(d) as in Section 5, deleting mention of the measure. In (b), use Lemmas 6.4 and 6.6 instead of Lemma 3.14. In (c), use the Baire Category Theorem to choose $p_\alpha \in \bigcap\{\varphi U : U \in \mathcal{U}_\alpha \ \&\ \varphi \in G_\alpha\}$. In (d), again use $I = G_\alpha \times \omega$; if we took $I = G_\alpha$, then $X_{\alpha+1}$ would contain isolated points.

Next, note that whenever $\mathcal{B}_\xi \cap X_\xi$ is a maximal antichain in X_ξ, induction on α shows that $(\pi^\alpha_\xi)^{-1}(\mathcal{B}_\xi \cap X_\xi)$ is a maximal antichain in X_α for all $\alpha \ge \xi$. The usual $\diamondsuit$ argument now shows that X is ccc: Let $\mathcal{B} \cap X$ be a maximal antichain in X, and assume that $K \cap X \neq \emptyset$ for all $K \in \mathcal{B}$. Fix $\xi < \omega_1$ such that $\mathcal{B} \restriction \xi = \mathcal{B}_\xi$ and $(\mathcal{B} \restriction \xi) \cap X_\xi$ is a maximal antichain in X_ξ. Then $\mathcal{B} = (\pi_\xi)^{-1}(\mathcal{B}_\xi)$, so that $\mathcal{B}$ is countable. Also, $X \setminus \bigcup \mathcal{B}$ is 2^{nd} countable, since π_ξ is 1-1 on $X \setminus \bigcup \mathcal{B}$. It follows that every nowhere dense set is 2^{nd} countable, so that X is HL.

Finally, to see that X is Suslin, let $\mathcal{A}_n$, for $n \in \omega$, be maximal antichains. So, $\mathcal{A}_n = (\pi_{\xi_n})^{-1}(\mathcal{B}_{\xi_n} \cap X_{\xi_n})$ for some $\xi_n < \omega_1$. Let $\alpha = \sup_n \xi_n$. For each n, choose $K_n \in \mathcal{B}_{\xi_n}$ such that $p_\alpha \in (\pi^\alpha_{\xi_n})^{-1}(K_n)$. Then $(\pi_{\xi_n})^{-1}(K_n) \in \mathcal{A}_n$, and $\emptyset \neq \mathrm{int}((\pi_\alpha)^{-1}\{\mathrm{p}_\alpha\}) \subseteq \bigcap_{\mathrm{n}}(\pi_{\xi_{\mathrm{n}}})^{-1}(\mathrm{K}_{\mathrm{n}})$.

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