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ON THE RECONSTRUCTION OF LOCALLY CONVEX SPACES FROM THEIR GROUPS OF HOMEOMORPHISMS

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Abstract

Let X and Y be normal locally convex spaces that have a nonempty open set which intersect every straight line in a bounded set, and let $H(X)$, $H(Y)$ denote the groups of self-homeomorphisms of X and Y respectively. Our main goal is to prove the following reconstruction theorem. If there is an isomorphism φ between $H(X)$ and $H(Y)$, then there exists a homeomorphism π between X and Y such that for every $h \in H(X)$, $\varphi(h) = \pi \circ h \circ \pi^{-1}$.

1. Introduction

A classical theorem of Hölder [H] asserts that every automorphism of the symmetric group on the finite set S with n elements is inner if $n \neq 6$.

Gerstenhaber [G1] showed that this is true for any infinite set S.

Apparently Gerstenhaber was the first who proposed to investigate systematically the problem of reconstruction of general topological spaces from their autohomeomorphisms groups. His work [G2] contains a new proof of the Fine-Schweigert result

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[FS] that every automorphism of the group of homeomorphisms of the unit interval onto itself is inner. He formulated explicitely the reconstruction problem for manifolds.

Let us mention that among topological spaces one can easily find nonhomeomorphic spaces that have the same homeomorphisms group; take, e.g., closed segment $[0,1]$ and open interval $(0,1)$.

The reconstruction problem for Euclidean manifolds was solved by Whittaker [W]. McCoy [M] extended Whittaker's results to Hilbert cube manifolds.

The second author of this article in the paper [R1] developed a general approach which allowed him to solve affirmatively the reconstruction problem for various classes of topological spaces.

In particular Rubin obtained two strengthenings of the result for Euclidean manifolds. He proved the reconstruction theorem for the class K^{norm} of all spaces which are a manifold based on some normed linear space. He also proved the reconstruction theorem for the class K^{poly} of all polyhedra in which every point is an accumulation point of its orbit.

The goal of this work is to present a complete proof of the following result.

Theorem 1.1. *Let $\mathcal{V}_N$ be the class of* <u>*normal*</u> *locally convex topological vector spaces over $\mathbb{R}$ that have a nonempty straight line bounded open set, and let K_N be the class of manifolds over $\mathcal{V}_N$. Then K_N is faithful.*

Without the assumption of normality Theorem 1.1 was claimed to be true in [R1], but unfortunately the arguments of [R1] contain a gap.

We explain the notions that appear in Theorem 1.1.

<u>A faithful class.</u> Let $H(X)$ denote the group of self-homeomorphisms of a topological space X. A class K of topological spaces is said to be faithful, if for every $X, Y \in K$ and an isomorphism φ between $H(X)$ and $H(Y)$ there is a

homeomorphism π between X and Y such that for every $h \in H(X)$, $\varphi(h) = \pi \circ h \circ \pi^{-1}$.

A straight line bounded set. Let X be real vector space. A subset $A \subseteq X$ is said to be straight line bounded, if for every straight line L in X, $A \cap L$ is a bounded subset of L.

A manifold over a class of topological spaces. Let $\mathcal{V}$ be a class of topological spaces, and X be a topological space. X is said to be a manifold over $\mathcal{V}$, if for every $x \in X$ there are $Y \in \mathcal{V}$, an open $U \subseteq Y$ and a map $\pi : cl(U) \to X$ such that π is a homeomorphism between $Dom(\pi)$ and $Rng(\pi)$, $x \in \pi(U)$, $Rng(\pi)$ is closed in X, and $\pi(U)$ is open in X.

In §2 we prove (Theorem 2.6(a)) that a certain class of topological spaces (so-called fully excellent spaces) is faithful. In §3 we prove (Theorem 3.1) that every member of K_N is fully excellent. The proof of Theorem 1.1 thus follows from the main theorems of §2 and §3.

We call a locally convex space an LCS. The central object of our research is the class of LCS that have a nonempty straight line bounded open set. Shortly LCS satisfying this property are denoted by SLB-spaces. In §4 we shall see that the class of normed spaces is properly contained in the class of normal SLB-spaces.

This paper is intended to be read independently of [R1]. Lemma 3.39 from [R1] is formulated and proved correctly for normed vector spaces. However, the claim of 3.51(b1) that the proof of Lemma 3.39 can be transferred to SLB-spaces, is wrong. It remains unknown whether or not a nonempty straight line bounded absolutely convex open set in an SLB-space is excellent. In fact, we suspect that for non-normal spaces the answer is negative.

The proof of Theorem 2.6(a) relies on a certain reconstruction theorem for complete Boolean algebra of regular open sets. This reconstruction theorem is stated below as Theorem 1.4.

Notations 1.2. For a topological space X and a subset $A \subseteq X$ let $int(A)$, $cl(A)$ and $bd(A)$ denote the interior, closure and boundary of A respectively. Recall that $U \subseteq X$ is regular open, if $int(cl(U)) = U$. The Boolean algebra of regular open subsets of X is denoted by $Ro(X)$. As usual, the partial ordering of the Boolean algebra $Ro(X)$ is set inclusion.

Definition 1.3. (a) Let $H(X)$ denote the group of self-homeomorphisms of X. Let $G \leq H(X)$, and let $\mathcal{R} \subseteq Ro(X)$ be closed under G, that is, for every $g \in G$ and $U \in \mathcal{R}$, $g(U) \in \mathcal{R}$. The structure $HR(X, G, \mathcal{R})$ is defined as follows

$$HR(X, G, \mathcal{R}) = \langle \mathcal{R}, G; \leq, \circ, Ap \rangle,$$

where $\leq$ is the inclusion relation on $\mathcal{R}$; $\circ$ is the composition in G and Ap is the application function from $G \times \mathcal{R}$ to $\mathcal{R}$. That is, $Ap(h, V) \stackrel{def}{=} h(V)$. Let $\mathcal{L}_{HR}$ denote the language of $HR(X, G, \mathcal{R})$.

(b) For $g, h \in H(X)$ we denote $g^h = h \circ g \circ h^{-1}$, and

$$var(g) \stackrel{def}{=} int(cl(\{x \in X \mid g(x) \neq x\})).$$

If $G \leq H(X)$, then $VAR(X, G) \stackrel{def}{=} \{var(g) \mid g \in G\}$.

Note that $VAR(X, G)$ is closed under G. To see this just notice that $h(var(g)) = var(g^h)$.

(c) Let X be a Hausdorff space and $G \leq H(X)$. We say that $\langle X, G \rangle$ is *a topological local movement system* (TLMS), if for every nonempty open subset $U \subseteq X$ there is $g \in G \setminus \{Id\}$ such that $g\lceil(X \setminus U) = Id$.

Part (b) of Theorem 1.4 below is a corollary of Part (a). The reader who wishes to avoid the definition of an interpretation may skip the first part.

Theorem 1.4. *(a) The class*

$$\{HR(X, G, VAR(X, G)) \mid \langle X, G \rangle \text{ is a TLMS}\}$$

is strongly first order interpretable in the class

$$\{G \mid \text{there exists } X \text{ such that } \langle X, G \rangle \text{ is a TLMS}\}.$$

(b) For $i = 1, 2$ let $\langle X_i, G_i \rangle$ be a TLMS. Let $\varphi : G_1 \to G_2$ be an isomorphism. Then there exists a unique isomorphism ψ between $HR(X_1, G_1, VAR(X_1, G_1))$ and $HR(X_2, G_2, VAR(X_2, G_2))$ that extends φ.

The notion of a first order interpretation is defined in ([R1], Definition 1.1). The definition of a strong first order interpretation is given in ([R3], Definition 1.3(f)).

A proof of the above theorem can be found in ([R2], Theorem 2.10). A special case which covers the spaces dealt with in this note, appears in ([R1], Section 2). See in particular, Theorem 2.14. The proof in [R2] is shorter and cleaner.

The results of this work leave the following questions open.

Problem 1.5. *Let K be the class of locally convex topological vector spaces over $\mathbb{R}$ that have a nonempty straight line bounded open set. Is K faithful?*

Problem 1.6. *Let K be the class of metrizable locally convex topological vector spaces over $\mathbb{R}$. Is K faithful?*

2. Excellent Sets

The notions of an excellent set and of an excellently structured space were defined in ([R1], Definitions 3.11 and 3.13). The definitions presented below are a slight weakening of the notion. This weakening seems to be needed in order to apply it to the class $\mathcal{V}_N$.

For a topological space X we denote by G a subgroup of $H(X)$, and by $\mathcal{R}$ a dense subset of $Ro(X)$ such that $\mathcal{R}$ is closed under G. Density of $\mathcal{R}$ means that every nonempty open subset of X contains a nonempty member of $\mathcal{R}$. In what follows R, S, T are open subsets of X, and U, V, W denote members of $\mathcal{R}$.

Definitions 2.1 and 2.3 rely only on X and G and not on $\mathcal{R}$.

Definition 2.1. (a) Let $R \cong S$ stands for the fact that there is $h \in G$ such that $h(R) = S$.

(b) We say that an open set T is a small set, if $T \neq \emptyset$, and for every nonempty open $S \subseteq T$ there is $T' \cong T$ such that $T' \subseteq S$.

(c) If T is a small set, then the small component of T is defined by $sc(T) = \bigcup\{T' \mid T' \cong T\}$.

Let $SC(X, G) \stackrel{def}{=} \{sc(T) \mid T \text{ is a small set}\}$ and $S(X, G) \stackrel{def}{=} \bigcup\{S \mid S \text{ is a small set}\}$.

(d) Let T and S be nonempty open sets. We say that T is strongly small in S and denote this by $T \prec S$, if for every nonempty open subset $R \subseteq S$ there is $h \in G$ such that $h(T) \subseteq R$ and $var(h) \subseteq S$.

Proposition 2.2. *(a) If T is small and $\emptyset \neq S \subseteq T$, then S is small and $sc(S) = sc(T)$.*

(b) Let $S \in SC(X, G)$, and suppose that there are a nonempty open T_0 and a small set S' such that $cl(T_0) \subseteq S' \subseteq S$. If $T \subseteq S$ is small, then $cl(T)$ is contained in a small set.

(c) Suppose that for every $x \in X$ and a small $T \not\ni x$ there is a nonempty open $T_0 \subseteq T$ such that $x \notin cl(T_0)$. If T is strongly small in S, then $cl(T) \subseteq S$.

(d) If X is a regular space, then the conclusions of Parts (b) and (c) hold for X and G.

Proof. The proposition is trivial. Note that if X is a regular space, then the assumptions of (b) and (c) hold for X and G. So Part (d) follows.

We prove Part (c). Suppose that T is strongly small in S, then $T \subseteq S$. Suppose by contradiction that $x \in cl(T) \setminus S$.

So $x \notin T$. Since T is strongly small in S, T is small. So there is a nonempty open set $T_0 \subseteq T$ such that $x \notin cl(T_0)$. There is $h \in G$ such that $var(h) \subseteq S$ and $h(T) \subseteq T_0$. Hence $x \in cl(T)$ but $h(x) = x \notin cl(h(T))$. A contradiction. $\square$

Definition 2.3. Let $T \subseteq X$ be open. T is called excellent, if the following holds.
(1) T is small.
(2) For every nonempty open S: if $cl(S) \subseteq T$, then S is strongly small in T.
(3) For every $x \in bd(T)$ and small $S \subseteq sc(T)$ there is $S' \cong S$ such that $S' \subseteq$T, $x \in cl(S')$ and $cl(S') \setminus \{x\} \subseteq T$.
(4) For every open set S: if $|cl(S) \cap cl(T)| \leq 1$, then there is $T' \cong T$ such that $T \subseteq T'$, $T' \cap S = \emptyset$ and $cl(T) \setminus cl(S) \subseteq T'$.

Definition 2.4. (a) Let $S \in SC(X, G)$. We say that S is excellently structured with respect to X, G and $\mathcal{R}$, if the following holds.
(1) For every $x \in S$, an open $T \ni x$ and a small $V \in \mathcal{R}$: if $V \subseteq S$, then there is $V' \cong V$ such that $x \in V' \subseteq cl(V') \subseteq T$. (Note that by Proposition 2.2(a), if X is regular, then (1) holds).
(2) For every $x \in S$, and an open $T \ni x$ there are $U, V \in \mathcal{R}$ such that U is excellent, $U \subseteq T$, $U \cap V = \emptyset$, and $cl(U) \cap cl(V) = \{x\}$.

(b) We say that $\langle X, G, \mathcal{R}\rangle$ is excellent, if for every $S \in SC(X, G)$, S is excellently structured. $\langle X, G, \mathcal{R}\rangle$ is called fully excellent, if it is excellent and $X = S(X, G)$.

(c) Let $HR(X, G, \mathcal{R})$ denote the structure $\langle \mathcal{R}, G; \leq, \circ, Ap\rangle$, where $\leq$ is the inclusion relation on $\mathcal{R}$, $\circ$ is the composition in G and Ap is the application function from $G \times \mathcal{R}$ to $\mathcal{R}$. That is, $Ap(h, V) \stackrel{def}{=} h(V)$. Let $\mathcal{L}_{HR}$ denote the language of $HR(X, G, \mathcal{R})$.

(d) Let $MRS(X, G, \mathcal{R})$ denote the structure
$$\langle S(X, G), \mathcal{R}, G; \leq, \circ, Ap, \varepsilon\rangle,$$
where ε is the belonging relation between $S(X, G)$ and $\mathcal{R}$. That is, $\langle x, V\rangle \in \varepsilon$, if $x \in V$.

(e) Let

$$K^{EXC} \stackrel{def}{=} \{\langle X, G, \mathcal{R}\rangle \mid \langle X, G, \mathcal{R}\rangle \text{ is excellent}\};$$

$$K^{EXC}_{HR} \stackrel{def}{=} \{HR(X, G, \mathcal{R}) \mid \langle X, G, \mathcal{R}\rangle \in K^{EXC}\};$$

$$K^{EXC}_{MRS} \stackrel{def}{=} \{MRS(X, G, \mathcal{R}) \mid \langle X, G, \mathcal{R}\rangle \in K^{EXC}\}.$$

Proposition 2.5. *Suppose that $\langle X, G, \mathcal{R}\rangle$ is fully excellent. Let $G \le G_1 \le H(X)$, and $\mathcal{R} \subseteq \mathcal{R}_1 \subseteq Ro(X)$ be such that $\mathcal{R}_1$ is closed under G_1. Then $\langle X, G_1, \mathcal{R}_1\rangle$ is excellent.*

Proof. Trivial. □

The aim of this section is to prove the following theorem.

Theorem 2.6. *(a) Let K^{FEXC} be the class of all Hausdorff spaces X such that $\langle X, H(X), VAR(X, H(X))\rangle$ is fully excellent. Then K^{FEXC} is faithful.*

Actually we shall prove the following more informative statement which implies Part (a) above.

(b) K^{EXC}_{MRS} is first order strongly interpretable in K^{EXC}_{HR}.

Theorem 2.6 will be used in the following way. We shall prove that if X is a manifold over a normal straight line bounded LCS, then

$$\langle X, H(X), VAR(X, H(X))\rangle$$

is excellent and $S(X, H(X)) = X$. So the class of such spaces is faithful.

Lemma 2.7. *(a) Suppose that $\langle X, G, \mathcal{R}\rangle \in K^{EXC}$. If $S \subseteq X$ is a small set, then S is not clopen.*

(b) Suppose that $\langle X, G, \mathcal{R}\rangle \in K^{EXC}$. If $T \subseteq X$ is small, then $cl(T)$ is contained in a small set.

(c) Suppose that $\langle X, G, \mathcal{R}\rangle \in K^{EXC}$. If T and S are open subsets of X, and T is strongly small in S, then $cl(T) \subseteq S$.

(d) There are first order formulas $\varphi_S(U)$, $\varphi_{SS}(U,V)$ *and* $\varphi_{eqsc}(U,V)$ *in* $\mathcal{L}_{HR}$ *which in* $HR(X,G,\mathcal{R})$ *express respectively the facts that* U *is small, that* $U \prec V$ *and that* $sc(U) = sc(V)$.

Let $\varphi_0(U,V) \stackrel{def}{\equiv} \forall W((U \prec W) \to (W \cap V \neq \emptyset))$. *Let* $\varphi_1(U,V)$ *be the formula that says:*
(1) U *is small;*
(2) $U \cap V = \emptyset$*;*
(3) for every small U_1*: if* $sc(U_1) = sc(U)$*, then there is* $U_2 \cong U_1$ *such that* $U_2 \subseteq U$ *and* $\varphi_0(U_2,V)$.

Let $\langle X,G,\mathcal{R}\rangle \in K^{EXC}$.

(e) Suppose that $U,V \in \mathcal{R}$ *and* $cl(U) \cap cl(V) \neq \emptyset$. *Then* $HR(X,G,\mathcal{R}) \models \varphi_0[U,V]$.

(f) If $HR(X,G,\mathcal{R}) \models \varphi_1[U,V]$*, then* $cl(U) \cap cl(V) \neq \emptyset$.

(g) If $U,V \in \mathcal{R}$*,* U *is excellent,* $U \cap V = \emptyset$*, and* $cl(U) \cap cl(V) \neq \emptyset$*, then* $HR(X,G,\mathcal{R}) \models \varphi_1[U,V]$.

Proof. (a) Let S be a small set. By Definition 2.4(a)(2), there is an excellent set $W \subseteq sc(S)$ such that $bd(W) \neq \emptyset$. By the smallness of S and Definition 2.3(3), there is $S' \cong S$ such that $bd(S') \cap bd(W) \neq \emptyset$. So S is not clopen.

(b) Let $T \subseteq X$ be a small set and $x \in T$. By Clause 1 of Definition 2.4(a), there is a small V such that $x \in V \subseteq cl(V) \subseteq T$. So $cl(V)$ is contained in a small set. Hence the conclusion of Proposition 2.2(b) holds.

(c) Let $x \in X$ and T be a small set not containing x. Then by Clause 1 of Definition 2.4(a), there is $V \in \mathcal{R} \setminus \{\emptyset\}$ such that $cl(V) \subseteq T$. So $x \notin cl(V)$. Thus the conclusion of Proposition 2.2(c) holds.

(d) The proof is trivial.

(e) Let $cl(U) \cap cl(V) \neq \emptyset$, and suppose that $U \prec W$. By Part (c), $cl(U) \subseteq W$. Hence $W \cap cl(V) \neq \emptyset$, and so $W \cap V \neq \emptyset$. This means that $HR(X,G,\mathcal{R}) \models \varphi_0[U,V]$.

(f) To the contrary suppose that $cl(U) \cap cl(V) = \emptyset$. We are going to show that $HR(X,G,\mathcal{R}) \models \neg\varphi_1[U,V]$. We may assume that U is small. Hence there is $U_1 \in \mathcal{R}$ such that U_1

is excellent, and $sc(U_1) = sc(U)$. Let $U_2 \cong U_1$ be such that $U_2 \subseteq U$. Hence $U_2 \cap V = \emptyset$. By Definition 2.3(4), there is $W \cong U_2$ such that $cl(U_2) \subseteq W$ and $W \cap V = \emptyset$. Since W is excellent, and by Definition 2.3(2), $U_2 \prec W$. This shows that $HR(X, G, \mathcal{R}) \models \neg\varphi_0[U_2, V]$. We have shown that there is no $U_2 \cong U_1$ such that $U_2 \subseteq U$ and $HR(X, G, \mathcal{R}) \models \varphi_0[U_2, V]$. So $HR(X, G, \mathcal{R}) \models \neg\varphi_1[U, V]$. Obtaining contradiction says that $cl(U) \cap cl(V) \neq \emptyset$.

(g) Let U and V satisfy to all conditions of (g). Conjuncts (1) and (2) of φ_1 hold trivially. We prove that conjunct (3) holds. Let $U_1 \in \mathcal{R}$ be a small set such that $sc(U_1) = sc(U)$. Let $x \in cl(U) \cap cl(V)$. By Definition 2.3(3), there is $U_2 \cong U_1$ such that $U_2 \subseteq U$ and $x \in cl(U_2)$. So $cl(U_2) \cap cl(V) \neq \emptyset$. It thus follows from Part (e) that $HR(X, G, \mathcal{R}) \models \varphi_0[U, V]$. Hence conjunct (3) of φ_1 holds. □

Let $\mathcal{R}^{\cap} = \{U^1 \cap U^2 \mid U^1, U^2 \in \mathcal{R}\}$. Note that if $\langle X, G, \mathcal{R}\rangle$ belongs to K^{EXC}, so is $\langle X, G, \mathcal{R}^{\cap}\rangle$.

In the next three lemmas we carefully distinguish between satisfaction of formulas in $HR(X, G, \mathcal{R}^{\cap})$ and satisfaction of formulas in $HR(X, G, \mathcal{R})$. In order to avoid this technical complication one can assume that $\mathcal{R}$ is closed under finite intersections. To obtain such an $\mathcal{R}$ one can close the original $\mathcal{R}$ under intersection, obtaining the set $cl^{\cap}(\mathcal{R})$. Indeed, $cl^{\cap}(\mathcal{R})$ is reconstructible from $\mathcal{R}$, but this reconstruction uses formulas of the second order logic. So using $cl^{\cap}(\mathcal{R})$ one loses the first order interpretability of X in $H(X)$. Apart of this, the subsequent lemmas can be read without reference to $HR(X, G, \mathcal{R}^{\cap})$.

Let $\vec{U}$ denote $\langle U^1, U^2\rangle$.

Lemma 2.8. *(a) For every first order formula $\varphi(U_1, \ldots, U_n, \vec{h})$ in $\mathcal{L}_{HR}$ there is a first order formula $\varphi^{\cap}(\vec{U}_1, \ldots, \vec{U}_n, \vec{h})$ in $\mathcal{L}_{HR}$ such that for every $\langle X, G, \mathcal{R}\rangle \in K^{EXC}$, $\vec{U}_1, \ldots, \vec{U}_n \in \mathcal{R}^2$ and $\vec{h} \in G^k$:*
$HR(X, G, \mathcal{R}^{\cap}) \models \varphi[U_1^1 \cap U_1^2, \ldots, U_n^1 \cap U_n^2, \vec{h}]$ iff $HR(X, G, \mathcal{R}) \models \varphi^{\cap}[\vec{U}_1, \ldots, \vec{U}_n, \vec{h}]$.

Let $\varphi_2(U,V)$ be the following formula in $\mathcal{L}_{HR}$.

$$\varphi_2(U,V) \equiv \varphi_1(U,V) \wedge (\forall U_1,U_2)\left(\left(\bigwedge_{i=1}^{2}(U_i \subseteq U) \wedge \bigwedge_{i=1}^{2}\varphi_0(U_i,V)\right) \rightarrow \varphi_0(U_1,U_2)\right).$$

Let $\langle X,G,\mathcal{R}\rangle \in K^{EXC}$.

(b) If $U,V \in \mathcal{R}$ and $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U,V]$, then $|cl(U) \cap cl(V)| = 1$ and $cl(U) \cap cl(V) \subseteq S(X,G)$.

(c) If $U,V \in \mathcal{R}$, U is excellent and $|cl(U) \cap cl(V)| = 1$, then $HR(X,G,\mathcal{R}) \models \varphi_2[U,V]$.
(Note that since $\langle X,G,\mathcal{R}^{\cap}\rangle \in K^{EXC}$, also $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U,V]$.)

(d) If $U,V \in \mathcal{R}$, $U_1 \in \mathcal{R}^{\cap}$, $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U,V]$, $U_1 \subseteq U$ and $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_0[U_1,V]$, then $|cl(U_1) \cap cl(V)| = 1$.

(e) For every $x \in S(X,G)$ there are $U,V \in \mathcal{R}$ such that $\{x\} = cl(U) \cap cl(V)$ and $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U,V]$.

Proof. (a) The proof is trivial.

(b) Let $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U,V]$. Since U is small and by Lemma 2.7(b), $cl(U) \subseteq S(X,G)$. So the second statement of Part (b) holds.

By Lemma 2.7(f), $cl(U) \cap cl(V) \neq \emptyset$. Suppose by contradiction that $x_1 \neq x_2$ and $x_1,x_2 \in cl(U) \cap cl(U)$. By Lemma 2.7(b), $cl(U)$ is contained in a small set. Hence $x_1,x_2 \in sc(U)$. Let T_1 and T_2 be disjoint neighborhoods of x_1, x_2 respectively, and let $W \in \mathcal{R}$ be an excellent set such $W \subseteq sc(U)$. By Definition 2.4(a)(1), there are $W_1,W_2,W_3 \cong W$ such that $x_i \in W_i \subseteq cl(W_i) \subseteq T_i$, $i = 1,2$, and $cl(W_1) \subseteq W_3 \subseteq T_1$. By Definition 2.3(2), $W_1 \prec W_3$. For $i = 1,2$ let $U_i = U \cap W_i$. So $U_i \in \mathcal{R}^{\cap}$. Clearly, $x_i \in cl(U_i) \cap cl(V)$. Hence by Lemma 2.7(e), $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_0[U_i,V]$. Since $U_1 \subseteq W_1 \prec W_3$, $U_1 \prec W_3$. Also, $W_3 \cap U_2 = \emptyset$. So $HR(X,G,\mathcal{R}^{\cap}) \models \neg\varphi_0[U_1,U_2]$. This shows that $HR(X,G,\mathcal{R}^{\cap}) \models \neg\varphi_2[U,V]$. A contradiction, so Part (b) is proved. (c) Let U and V be as in (c), and let $\{x\} = cl(U) \cap cl(V)$. By Lemma 2.7(a), $U \cap V = \emptyset$, hence by Lemma 2.7(g), $HR(X,G,\mathcal{R}) \models \varphi_1[U,V]$.

Let $W \subseteq U$ be such that $HR(X, G, \mathcal{R}) \models \varphi_0[W, V]$. We show that $x \in cl(W)$. By Definition 2.3(4), there is $U' \cong U$ such that $U' \cap V = \emptyset$ and $U' \supseteq cl(U) \setminus cl(V) = cl(U) \setminus \{x\}$. Due to $HR(X, G, \mathcal{R}) \models \varphi_0[W, V]$ we get $W \not\prec U'$. By the excellence of U' and Definition 2.3(2), $cl(W) \not\subseteq U'$. But $cl(W) \setminus cl(V) \subseteq cl(U) \setminus cl(V) \subseteq U'$. Hence $cl(W) \cap cl(V) \not\subseteq U'$. So $\emptyset \neq cl(W) \cap cl(V) \subseteq cl(U) \cap cl(V) = \{x\}$. Hence $x \in cl(W)$.

It follows that if for $i = 1, 2$, $W_i \subseteq U$ and $HR(X, G, \mathcal{R}) \models \varphi_0[W_i, V]$, then $cl(W_1) \cap cl(W_2) \neq \emptyset$, and so by Lemma 2.7(e), $HR(X, G, \mathcal{R}) \models \varphi_0[W_1, W_2]$. Thus $HR(X, G, \mathcal{R}) \models \varphi_2[U, V]$.

(d) Let U,V and U_1 be as in (d). Suppose by contradiction that $cl(U_1) \cap cl(V) = \emptyset$. Let $\{x\} = cl(U) \cap cl(V)$. Since U is small, and $\langle X, \mathcal{R}, G \rangle$ is excellent, there is an excellent $W' \in \mathcal{R}$ such that $W' \subseteq sc(U)$. By Definition 2.4(a)(1), there is $W \cong W'$ such that $x \in W \subseteq cl(W) \subseteq X \setminus cl(U_1)$. By Definition 2.3(4), there is $W_1 \cong W$ such that $W_1 \cap U_1 = \emptyset$ and $cl(W) \subseteq W_1$. By Definition 2.3(2), $W \prec W_1$. Let $U_2 = W \cap U$. Hence $U_2 \in \mathcal{R}^{\cap}$. Since $U_2 \subseteq W \prec W_1$, $U_2 \prec W_1$. Also $W_1 \cap U_1 = \emptyset$. So we have

(i) $HR(X, G, \mathcal{R}^{\cap}) \models \neg\varphi_0[U_2, U_1]$.

Since $x \in W$ and $x \in cl(U)$, $x \in cl(U \cap W) = cl(U_2)$. Hence $x \in cl(U_2) \cap cl(V)$. So by Lemma 2.7(e), we have

(ii) $HR(X, G, \mathcal{R}^{\cap}) \models \varphi_0[U_2, V]$.

Also, it is assumed that

(iii) $HR(X, G, \mathcal{R}^{\cap}) \models \varphi_0[U_1, V]$.

Lastly,

(iv) $U_1, U_2 \subseteq U$.

It follows that $HR(X, G, \mathcal{R}^{\cap}) \models \neg\varphi_2[U, V]$. A contradiction. So Part (d) is proved.

(e) The claim of (e) follows trivially from Definition 2.4(a)(2) and Part (c). □

Pairs $\langle U, V \rangle \in \mathcal{R}^2$ that satisfy in $HR(X, G, \mathcal{R}^{\cap})$ the formula φ_2 will represent points of $S(X, G)$. Part (e) of the last lemma shows that every member of $S(X, G)$ is indeed represented by such a pair. We have still to show that the following facts are expressible in $HR(X, G, \mathcal{R}^{\cap})$.

(1) $\langle U_1, V_1 \rangle$ and $\langle U_2, V_2 \rangle$ represent the same point.

(2) The point represented by $\langle U, V \rangle$ belongs to W.

Lemma 2.9. *Let*

$$\varphi_3(U_1,V_1,U_2,V_2) \equiv$$
$$(\forall U_1' \subseteq U_1)(\forall U_2' \subseteq U_2)\left(\left(\bigwedge_{i=1}^{2}\varphi_0(U_i',V_i)\right) \to \varphi_0(U_1',U_2')\right).$$

Let $\langle X,G,\mathcal{R}\rangle \in K^{EXC}$. Then for every $U_1,V_1,U_2,V_2 \in \mathcal{R}$: if for $i=1,2$, $HR(X,G,\mathcal{R}^\cap) \models \varphi_2[U_i,V_i]$, then $HR(X,G,\mathcal{R}^\cap) \models \varphi_3[U_1,V_1,U_2,V_2]$ iff $cl(U_1)\cap cl(V_1) = cl(U_2)\cap cl(V_2)$.

Proof. Suppose that for $i=1,2$, $HR(X,G,\mathcal{R}^\cap) \models \varphi_2[U_i,V_i]$.
Proof of $\Rightarrow$: Assume that $HR(X,G,\mathcal{R}^\cap) \models \varphi_3[U_1,V_1,U_2,V_2]$, and suppose by contradiction that $cl(U_1)\cap cl(V_1) = \{x_1\} \neq \{x_2\} = cl(U_2)\cap cl(V_2)$. Let $W_1,W_2 \in \mathcal{R}$ be excellent sets such that $x_i \in W_i$ for $i=1,2$, and $cl(W_1)\cap cl(W_2)=\emptyset$. Let $U_i' = U_i \cap W_i$. Hence $U_i' \in \mathcal{R}^\cap$. By Definition 2.3(4), there is $W_3 \cong W_1$ such that $cl(W_1) \subseteq W_3$ and $W_3 \cap U_2' = \emptyset$. By Definition 2.3(2), $U_1' \prec W_3$, and hence

(i) $HR(X,G,\mathcal{R}^\cap) \models \neg\varphi_0[U_1',U_2']$.

For $i=1,2$, $x_i \in cl(U_i)\cap W_i$, and hence $x_i \in cl(U_i')$. Also $x_i \in V_i$. So by Lemma 2.7(e), hence

(ii) for $i=1,2$, $HR(X,G,\mathcal{R}^\cap) \models \varphi_0[U_i',V_i]$.

Also

(iii) for $i=1,2$, $U_i' \subseteq U_i$.

So $HR(X,G,\mathcal{R}^\cap) \models \neg\varphi_3[U_1,V_1,U_2,V_2]$, a contradiction.

Proof of $\Leftarrow$: Assume that $cl(U_1)\cap cl(V_1) = cl(U_2)\cap cl(V_2) = \{x\}$. For $i=1,2$ let $U_i' \in \mathcal{R}^\cap$ be such that $U_i' \subseteq U_i$ and $HR(X,G,\mathcal{R}^\cap) \models \varphi_0[U_i',V_i]$. By Lemma 2.8(d), $x \in cl(U_i')\cap V_i$. In particular, $x \in cl(U_1')\cap cl(U_2')$. So by Lemma 2.7(e), $HR(X,G,\mathcal{R}^\cap) \models \varphi_0[U_1',U_2']$.

We have thus shown that $HR(X,G,\mathcal{R}^\cap) \models \varphi_3[U_1,V_1,U_2,V_2]$. The proof of the lemma is complete. □

We denote $\langle U,U\rangle$ by $\overleftrightarrow{U}$.

Lemma 2.10. *Let*

$$\varphi_4(U,V,W) \equiv (\exists U_1, V_1, W_1)((U_1 \prec W_1 \subseteq W)\wedge \varphi_2^{\cap}(\overleftrightarrow{U}_1, \overleftrightarrow{V}_1) \wedge \varphi_3^{\cap}(\overleftrightarrow{U}_1, \overleftrightarrow{V}_1, \overleftrightarrow{U}, \overleftrightarrow{V})).$$

Let $\langle X, G, \mathcal{R}\rangle \in K^{EXC}$*. Suppose that* $U, V, W \in \mathcal{R}$ *and that* $HR(X,G,\mathcal{R}) \models \varphi_2^{\cap}[\overleftrightarrow{U}, \overleftrightarrow{V}]$*. Then* $HR(X,G,\mathcal{R}) \models \varphi_4[U,V,W]$ *iff* $cl(U) \cap cl(V) \subseteq W$*.*

Proof. Suppose that $U,V,W \in \mathcal{R}$ and that $HR(X,G,\mathcal{R}) \models \varphi_2^{\cap}[\overleftrightarrow{U}, \overleftrightarrow{V}]$.

Proof of $\Leftarrow$*:* Suppose that $cl(U) \cap cl(V) = \{x\} \subseteq W$. Since $\langle X, G, \mathcal{R}\rangle$ is excellent, and by Definition 2.4(a)(1), there exists $W_1 \in \mathcal{R}$ such that $x \in W_1 \subseteq W$. By Definition 2.4(a)(1), there is $W_2 \cong W_1$ such that $x \in W_2 \subseteq cl(W_2) \subseteq W_1$. By Definition 2.4(a)(2), there are $U_1, V_1 \in \mathcal{R}$ such that U_1 is excellent, $U_1 \subseteq W_2$ and $cl(U_1) \cap cl(V_1) = \{x\}$. By the choice of W_1 and Definition 2.3(2),

(i) $$U_1 \prec W_1 \subseteq W.$$

By Lemma 2.8(c), $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_2[U_1, V_1]$. So by Lemma 2.8(a),

(ii) $$HR(X,G,\mathcal{R}) \models \varphi_2^{\cap}[\overleftrightarrow{U}_1, \overleftrightarrow{V}_1].$$

By Lemma 2.9, $HR(X,G,\mathcal{R}^{\cap}) \models \varphi_3[U_1,V_1,U,V]$, and hence by Lemma 2.8(a),

(iii) $$HR(X,G,\mathcal{R}^{)} \models \varphi_3^{\cap}[\overleftrightarrow{U}_1, \overleftrightarrow{V}_1, \overleftrightarrow{U}, \overleftrightarrow{V}].$$

We have shown that $HR(X,G,\mathcal{R}^{)} \models \varphi_4[U,V,W]$.

Proof of $\Rightarrow$*:* Suppose that $HR(X,G,\mathcal{R}^{)} \models \varphi_4[U,V,W]$. Let U_1, V_1 and W_1 be as assured by φ_4. By Lemma 2.8(a) and (c), there is $y \in X$ such that $\{y\} = cl(U_1) \cap cl(V_1)$, and by Lemmas 2.9 and 2.8(a), $cl(U) \cap cl(V) = cl(U_1) \cap cl(V_1) = \{y\}$.

By Proposition 2.7(c), and since $U_1 \prec W_1$ we have $cl(U_1) \subseteq W_1$. Hence $y \in W_1 \subseteq W$. That is, $cl(U) \cap cl(V) \subseteq W$. □

Proof of Theorem 2.6: (b) The following are the main formulas in the interpretation.

$$\varphi_U(U,V) \equiv \varphi_2^{\cap}(\overleftrightarrow{U},\overleftrightarrow{V}),$$

$$\varphi_{Eq}(U_1,V_1,U_2,V_2) \equiv \varphi_3^{\cap}(\overleftrightarrow{U_1},\overleftrightarrow{V_1},\overleftrightarrow{U_2},\overleftrightarrow{V_2}),$$

and

$$\varphi_\varepsilon(U,V,W) \equiv \varphi_4(U,V,W).$$

φ_U represents $S(X,G)$. This follows from Lemma 2.8(b), (e) and (a). φ_{Eq} represents equality. This is a consequence of Lemmas 2.9 and 2.8(a). φ_ε represents the belonging relation between members of $S(X,G)$ and members of $\mathcal{R}$.

(a) For $i = 1,2$ let $X_i \in K^{FEXC}$, and let $\varphi : H(X_1) \to H(X_2)$ be an isomorphism. Let $H_i = H(X_i)$ and $\mathcal{R}_i = VAR(X_i,H_i)$. It follows from Definitions 2.3(2) and 2.4(a)(1) that $\langle X_i,H_i\rangle$ is a TLMS. So by Theorem 1.4(b), there is an isomorphism ψ between $HR(X_1,H_1,\mathcal{R}_1)$ and $HR(X_2,H_2,\mathcal{R}_2)$ such that ψ extends φ.

By Part (b), ψ can be extended to an isomorphism χ between
$MRS(X_1,H_1,\mathcal{R}_1)$ and $MRS(X_2,H_2,\mathcal{R}_2)$. $S(X_i,H_i) = X_i$. It follows that χ is an isomorphism between $\langle X_1,\mathcal{R}_1,H_1;\le,\circ,Ap,\varepsilon\rangle$ and
$\langle X_2,\mathcal{R}_2,H_2;\le,\circ,Ap,\varepsilon\rangle$. Let $\tau = \chi\lceil X_1$. By Definition 2.4(a)(1), $\mathcal{R}_i$ is a base for X_i. This implies that τ is a homeomorphism between X_1 and X_2.

The facts that χ preserves the function Ap and that $\chi \supseteq \varphi$ imply that for every $h \in H_1$, $\varphi(h) = h^\tau$. So Part (a) is proved. □

3. Straight Line Bounded Locally Convex Spaces

The following statement plays the crucial role in the proof of Theorem 1.1.

Theorem 3.1. *Every manifold over the class of normal SLB-spaces is fully excellent. More precisely, if X is a manifold over the class of normal SLB-spaces, then $\langle X, H(X), VAR(X, H(X))\rangle$ is fully excellent.*

It is claimed in [R1] that the class of manifolds over the SLB-spaces is faithful. The proof given in [R1] was phrased for manifolds over the normed spaces, and we stated there that almost the same proof works for SLB-spaces. This claim happens to be wrong. An attempt to adapt the proof given in [R1] to SLB-spaces, fails in Lemma 3.39(d), (f) and (g) of [R1]. Whereas, Part (f) can be proved in a different way without any extra assumptions, and the proofs that we presently know for Parts (d) and (g) require the assumption of normality. We conjecture that the claims of (d) and (g) imply the normality of the SLB-space in question. Note that actually we use countable paracompacthess of LCS. It turns out that for LCS this is equivalent to normality. It remains open (Problem 1.5) whether the class of all SLB-spaces is faithful.

For basic facts on LC spaces see [K]. $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of natural numbers respectively.

Let Y be an LCS containing a straight line bounded neighbourhood of 0. Then Y has a local base at 0 consisting of straight line bounded absolutely convex open sets. Throughout the paper the latter sets briefly will be called balls. Also, by a ball with center at $x \in Y$ we mean a set $x + B$, where B is a ball. The Minkowski functional based on a ball B is defined as follows

$$\|x\|_B = \inf(\{\lambda \mid \lambda > 0 \text{ and } x \in \lambda B\}) \ .$$

It is known that $\| \ \|_B$ is a continuous norm on Y and $\frac{y}{\|y\|_B} \in bd(B)$ for every $y \in Y \setminus \{0\}$.

Locally convex spaces that contain a ball are called here SLB-spaces. It is easy to see that SLB-spaces are exactly those LC spaces which admit a coarser normed topology.

Our first very simple observation says in particular that each point of an SLB-space is G_δ.

Claim 3.2. *For each point x of an SLB-space Y there exists a decreasing sequence $\{E_i\}_{i\in I\!N}$ of closed balls with center at x such that $\bigcap_{i\in I\!N} E_i = \{x\}$.*

Proof. Let B be a ball, define $U_i = \frac{1}{i+1}B + x$ and $E_i = cl(U_i)$. $\square$

The first three parts of the following proposition are proved in the same way as for normed spaces.

Proposition 3.3. *Let Y be an SLB-space and $B \subseteq Y$ be a ball. Then*

(a) For every $0 < \alpha \le \beta < 1$ there is $h \in H(Y)$ such that $var(h) = B$ and $h(\beta B) = \alpha B$.

(b) For every $0 < \alpha < 1$, $v \in B \setminus \{0\}$ and $0 < \beta < 1 - \|v\|_B$ there is $h \in H(Y)$ such that $var(h) = B$, $h(0) = v$ and $h(\alpha B) = v + \beta B$.

(c) Let C be a ball such that $cl(C) \subseteq B$. Then there is $h \in H(Y)$ such that $var(h) \subseteq 2B$ and $h(B) = C$.

Proof.

(a) Let $g \in H([0, \infty))$ be the the following piecewise linear function.
(i) $g\lceil[0, \beta]$ is linear, $g(0) = 0$ and $g(\beta) = \alpha$;
(ii) $g\lceil[\beta, 1]$ is linear, $g(\beta) = \alpha$ and $g(1) = 1$;
(iii) $g\lceil[1, \infty) = Id$.
Let $h(0) = 0$, and $h(y) = g(\|y\|_B) \cdot \frac{y}{\|y\|_B}$. It is obvious that h is as required.

(b) By Part (a), we may assume that $\alpha = \beta$. Let $g : [0, \infty) \to [0, 1]$ be the following piecewise linear function.
(i) $g\lceil[0, \alpha] = 1$;
(ii) $g\lceil[\alpha, 1]$ is linear, $g(\alpha) = 1$ and $g(1) = 0$;
(iii) $g\lceil[1, \infty) = 0$.
Let $h(y) = y + g(\|y\|_B) \cdot v$. It is easy to see that h is as required.

(c) Let $g : (1, \infty) \times [0, \infty) \to [0, \infty)$ be defined as follows. Denote $g(s, t)$ by $g_s(t)$. For every $s \in (1, \infty)$, g_s is a piecewise linear function such that
(i) $g_s \lceil [0, \frac{1}{2}] = Id$;
(ii) $g_s \lceil [\frac{1}{2}, \frac{s}{2}]$ is a linear function taking $\frac{1}{2}$ to $\frac{1}{2}$ and $\frac{s}{2}$ to 1;
(iii) $g_s \lceil [\frac{s}{2}, s]$ is a linear function taking $\frac{s}{2}$ to 1 and s to s;
(iv) $g_s \lceil [s, \infty) = Id$.

Note that since $cl(C) \subseteq B$, for every $y \in Y$, $\|y\|_B < \|y\|_C$. Let h be defined as follows: $h(0) = 0$, and if $y \neq 0$, then

$$h(y) = g\left(\frac{\|y\|_C}{\|y\|_B}, \|y\|_C\right) \cdot \frac{y}{\|y\|_C} .$$

It is left to the reader to check that h is as required. □

We recall the definitions of lower and upper semicontinuity.

Definition 3.4. A real function f defined on a topological space Y is called upper (lower) semicontinuous if for every $x \in Y$ and $\epsilon > 0$, there is a neighborhood U of x such that for every $y \in U$, $f(y) < f(x) + \epsilon$ (respectively $f(y) > f(x) - \epsilon$).

Let us demonstrate how lower and upper semicontinuous functions arise naturally on arbitrary topological vector space Y. Let $S \subseteq Y \setminus \{0\}$ be any set. For a point $x \in S$ we consider the half-line $\{\lambda x \mid \lambda \geq 0\}$ and denote its upper and lower parts by $x^{\uparrow} = \{\lambda x \mid \lambda > 1\}$, and $x^{\downarrow} = \{\lambda x \mid 0 \leq \lambda < 1\}$ respectively.

(1) Let $c \in (0, 1]$ be any constant. For a set F that satisfies $F \cap x^{\uparrow} = \emptyset$ for each $x \in S$, we define the function $f_1 : S \to [c, 1]$ as follows

$$f_1(x) = \sup(\{\lambda \geq 0 \mid \lambda x \in F\} \cup \{c\}) .$$

(2) Let $c \geq 1$ be any constant. For a set F that satisfies $F \cap x^{\downarrow} = \emptyset$ for each $x \in S$, we define the function $f_2 : S \to [1, c]$ as follows

$$f_2(x) = \inf(\{\lambda \geq 0 \mid \lambda x \in F\} \cup \{c\}) .$$

Lemma 3.5. *Let F be a closed subset of a topological vector space Y. Then f_1 and f_2 defined above are upper and lower semicontinuous functions respectively.*

Proof. We prove the claim for f_1, in the case of f_2 the arguments are analogous. Fix $x \in S$, and suppose, to the contrary, that $\epsilon' > 0$ be a counter-example to the claim. That is, for every neighborhood U of x there exists $y \in U \cap S$ such that $f_1(y) \geq f_1(x) + \epsilon'$. Let

$$\epsilon = \sup(\{\epsilon' \mid \epsilon' \text{ is a counter-example to the claim}\}),$$

and let $\alpha = f_1(x) + \epsilon$. So $\langle \alpha, x\rangle$ has the following property: for every open interval J containing α and every neighborhood U of x there are $\lambda \in J$ and $y \in U$ such that $\lambda y \in F$. That is, $\langle \alpha, x\rangle \in cl(\{\langle \mu, y\rangle \mid \mu y \in F\})$. By the continuity of the mapping $\langle \mu, y\rangle \mapsto \mu y$ and the closedness of F, $\alpha x \in F$. But $\alpha = f_1(x) + \epsilon > f_1(x)$. A contradiction, so the claim is proved. □

The following assertion will be needed in the proof that nonempty balls in a general SLB-space possess the property (3) of excellent sets (Definition 2.3).

Lemma 3.6. *Let $\langle Y, \tau\rangle$ be an SLB-space, and $f : Y \to [a, b]$ be an upper semicontinuous function. Then for each $x_0 \in Y$ there is a continuous function $g : Y \to [a, b]$ such that $g(x_0) = f(x_0)$, and for every $x \in Y \setminus \{x_0\}$, $g(x) > f(x)$ or $g(x) = f(x) = b$.*

Proof. Without loss of generality we may assume that $[a, b] = [0, 1]$. Let $\alpha_0 = f(x_0)$. If $\alpha_0 = 1$, then the function with the constant value 1 is as required. Suppose that $\alpha_0 < 1$. According to Claim 3.2, there is a decreasing sequence $\{E_i\}_{i \in I\!N}$ of closed balls with center at x_0 such that $\bigcap_{i \in I\!N} E_i = \{x_0\}$.

We define by induction a sequence $\{U_n\}_{n \in I\!N}$ of open balls with center at x_0 such that $Y = U_0 \supseteq S_1 \supseteq U_1 \supseteq S_2 \supseteq U_2 \supseteq \ldots$, where $S_n \stackrel{def}{=} cl(U_n)$. Suppose that U_{n-1}, $n \geq 1$ has been defined. Put $V_n = \{x \in Y \mid f(x) < \alpha_0 + \frac{1}{n}\}$. By upper semicontinuity of

f, the set V_n is open. Let U_n be a ball with center at x_0 such that $cl(U_n) \subseteq V_n \cap U_{n-1} \cap E_n$.

The set U_i is an open ball with center at x_0, therefore the set $W_i \overset{def}{=} U_i - x_0$ is the unit ball of some normed topology σ_i. Let σ be the topology on Y generated by $\bigcup_{i \geq 1} \sigma_i$. Then $\tau \supseteq \sigma$ and $\langle Y, \sigma \rangle$ is a metrizable LCS (cf. [K], 15.11.1).

Note that U_n is convex and open for both topologies τ and σ, therefore, by ([K], 16.4.7), the closures $cl(U_n)$ with respect to both topologies coincide with $acl(U_n)$, and then S_n remains closed in $\langle Y, \sigma \rangle$ for every $n \in \mathbb{N}$.

Recall that for a subset A of a real vector space Y the algebraic hull of A in Y is defined to be

$$acl_Y(A) = \bigcup \{ cl_L(A \cap L) \mid L \text{ is a straight line in } Y \},$$

where $cl_L(A \cap L)$ denotes the usual topological closure in the line L.

Let $A_1 = \bigcup_{i \in \mathbb{N}} (S_{2i} \setminus U_{2i+1})$ and $A_2 = \bigcup_{i \in \mathbb{N}} (S_{2i+1} \setminus U_{2i+2})$. Denote by $K_1 \overset{def}{=} A_1 \cup \{x_0\}$ and $K_2 \overset{def}{=} A_2 \cup \{x_0\}$. It is easy to show that K_1 and K_2 are closed in Y and

(∗) $K_1 \cup K_2 = A_1 \cup A_2 \cup \{x_0\} = Y$.

Define functions f_ℓ on K_ℓ, $\ell = 1, 2$ as follows

(1) $f_1 \lceil (S_0 \setminus U_1)$ is a constant function with the value 1;

(2) For every $i \in \mathbb{N}$, $f_1 \lceil (S_{2i} \setminus U_{2i+1})$ is a constant function with the value $\min(\alpha_0 + \frac{1}{2i}, 1)$;

(3) $f_1(x_0) = \alpha_0$.

The function f_2 is defined similarly:

(4) For every $i \in \mathbb{N}$, $f_2 \lceil (S_{2i+1} \setminus U_{2i+2})$ is a constant function with the value $\min(\alpha_0 + \frac{1}{2i+1}, 1)$;

(5) $f_2(x_0) = \alpha_0$.

Now we are going to prove that f_1 and f_2 are continuous on $\langle K_1, \sigma \rangle$ and $\langle K_2, \sigma \rangle$, respectively. Firstly, note that the sets $S_{2i} \setminus U_{2i+1}$ and $K_1 \setminus (S_{2i} \setminus U_{2i+1})$ are disjoint and they are closed with respect to the topology σ. Since being a constant function, f_1 is σ-continuous on $S_{2i} \setminus U_{2i+1}$. For every $i \in \mathbb{N}$, U_{2i+1} is a σ-neighborhood of x_0, and for every $x \in U_{2i+1} \cap K_1$, $f_1(x_0) = \alpha_0 \leq f_1(x) \leq \alpha_0 + \frac{1}{2i+2}$. So f_1 is σ-continuous at x_0. Analogously, f_2 is σ-continuous.

We next check the fact

(∗∗) for every $x \in A_\ell$, $f_\ell(x) > f(x)$ or $f_\ell(x) = f(x) = 1$.

We show this for f_1, a similar argument holds for f_2. If $f(x) = 1$, then $x \in S_0 \setminus U_1$, so $f_1(x) = 1$. Now let $f(x) < 1$. If $x \in S_{2i} \setminus U_{2i+1}$, then $f_1(x) = \min(\alpha_0 + \frac{1}{2i}, 1) > f(x)$.

By the metrizability of $\langle Y, \sigma\rangle$, for every $\ell = 1, 2$, choose a σ-continuous function g_ℓ from Y to $[0, 1]$ extending f_ℓ. Let $g = \max(g_1, g_2)$. So g is σ-continuous. Since $\tau \supseteq \sigma$, g is τ-continuous. It follows from (∗) and (∗∗) that for every $x \in Y \setminus \{x_0\}$, $g(x) > f(x)$ or $g(x) = f(x) = 1$. By the construction, $f_1(x_0) = f_2(x_0) = \alpha_0$, then $g(x_0) = \alpha_0$. So g satisfies all required conditions. □

Apparently Lemma 3.6 is well known for metrizable spaces. Nevertheless we can't omit the assumption that Y is an SLB-space in the next lemmas. The reconstruction question for the class of all metrizable locally convex spaces is still unsolved (Problem 1.6).

We now prove Lemma 3.39(f) of [R1] for general SLB-spaces.

Lemma 3.7. *Let Y be an SLB-space, and $B \subseteq Y$ be a ball. Then for every $x_0 \in bd(B)$ and a nonempty closed subset $F \subseteq B$, there is $h \in H(Y)$ such that (1) $var(h) \subseteq 2B$; (2) $h(F) \subseteq cl(B)$; and (3) $bd(B) \cap h(F) = \{x_0\}$.*

Proof. Making use of Proposition 3.3(a) and (b), without loss of generality we may assume that $\frac{5}{8}x_0 \in F \subseteq \frac{5}{8}x_0 + \frac{1}{8}B$. Hence $F \subseteq \frac{3}{4}B \setminus cl(\frac{1}{2}B)$.

Let $S = bd(B)$. Define $f : S \to [\frac{1}{2}, 1]$ as follows

$$f(x) = \sup(\{\lambda \mid \lambda x \in F\} \cup \{\tfrac{1}{2}\}).$$

It follows easily from Lemma 3.5 that f is an upper semicontinuous function.

Note that, in view of our assumptions, $\alpha_0 \overset{def}{=} f(x_0) > \frac{1}{2}$, and for every $x \in S$, $f(x) < \frac{3}{4}$.

Let $f^* : Y \to I\!R$ be defined by $f^*(0) = \frac{1}{2}$, and for every $y \neq 0$

$$f^*(y) = \max(\min(\|y\|_B, 1) \cdot f(\tfrac{y}{\|y\|_B}), \tfrac{1}{2})\,.$$

The following holds for f^*:
(i) f^* extends f;
(ii) $f^* : Y \to [\frac{1}{2}, \frac{3}{4})$;
(iii) f^* is an upper semicontinuous function.

Let g^* be the function obtained by applying Lemma 3.6 to f^*, x_0 and $[\frac{1}{2}, \frac{3}{4}]$. Note that since $\frac{3}{4} \notin Rng(f^*)$, for every $x \in Y \setminus \{x_0\}$, $g^*(x) > f^*(x)$. Let $g = g^* \restriction S$. Then g has the following properties.
(1) $g : S \to [\frac{1}{2}, \frac{3}{4}]$;
(2) g is continuous;
(3) For every $x \in S \setminus \{x_0\}$, $g(x) > f(x)$;
(4) $g(x_0) = f(x_0)$.

Let $k : [\frac{4}{3}, 2] \times [0, \infty) \to [0, \infty)$ be defined as follows. Denote $k(s, t)$ by $k_s(t)$. For every $s \in [\frac{4}{3}, 2]$, $k_s(t)$ is a piecewise linear increasing function such that
(i) $k_s \restriction [0, \frac{1}{4}] = Id$;
(ii) $k_s \restriction [\frac{1}{4}, \frac{1}{2}]$ is a linear function taking $\frac{1}{4}$ to $\frac{1}{4}$ and $\frac{1}{2}$ to $s \cdot \frac{1}{2}$;
(iii) $k_s \restriction [\frac{1}{2}, \frac{3}{4}]$ is a linear function taking $\frac{1}{2}$ to $s \cdot \frac{1}{2}$ and $\frac{3}{4}$ to $s \cdot \frac{3}{4}$;
(iv) $k_s \restriction [\frac{3}{4}, 2]$ is a linear function taking $\frac{3}{4}$ to $s \cdot \frac{3}{4}$ and 2 to 2;
(v) $k_s \restriction [2, \infty) = Id$.

Clearly, the function k is continuous. Note that for every $s \in [\frac{4}{3}, 2]$, $k_s \in H([0, \infty))$.

Let $h : Y \to Y$ be defined as follows: $h(0) = 0$, and for every $y \neq 0$,

$$h(y) = k\left(\frac{1}{g(\frac{y}{\|y\|_B})}, \|y\|_B\right) \cdot \frac{y}{\|y\|_B} \ .$$

The mapping h is well-defined since $\frac{1}{g\left(\frac{y}{\|y\|_B}\right)} \in [\frac{4}{3}, 2]$.

Also, for every $y \in Y$, $h \restriction \{ty \mid t \geq 0\} \in H(\{ty \mid t \geq 0\})$. So h is a bijection of Y. The functions k, g, and $\| \ \|_B$ are continuous, hence h is continuous at every $y \in Y \setminus \{0\}$. It follows that h is continuous on Y since $h \restriction \frac{1}{4}B = Id$.

The inverse mapping has the following form: $h^{-1}(0) = 0$, and for every $y \neq 0$

$$h^{-1}(y) = k_1 \left(\frac{1}{g(\frac{y}{\|y\|_B})}, \|y\|_B \right) \cdot \frac{y}{\|y\|_B} \ ,$$

where $k_1(s,t) \stackrel{def}{=} k_s^{-1}(t)$. It is obvious that the definition of k_1 is similar to k, so k_1 is continuous. Therefore the same argument that shows the continuity of h apllies for h^{-1}. So $h \in H(Y)$.

Clearly, (1) $var(h) \subseteq 2B$, since $k(s,t) = t$ for every $t \geq 2$. Also, $z_0 \stackrel{def}{=} \alpha_0 x_0 \in F$ and $h(z_0) = x_0$.

Let $x \in F \setminus \{z_0\}$. If $x = \lambda z_0$, then $\lambda < 1$, and thus $h(x) = \alpha_0 x = \lambda x_0 \in B$. So $h(x) \notin S$. Otherwise, let $u = \frac{x}{\|x\|_B}$, $\beta = \max(\{\lambda \mid \lambda x \in F\})$ and $z = \beta x$. Clearly, $\beta \geq 1$. Then $f(u) = \|z\|_B$, and since $\frac{1}{2} \leq \|x\|_B < \frac{3}{4}$, $h(x) = \frac{1}{g(u)}\|x\|_B u$. Because $u \in S$ and

$$\frac{1}{g(u)}\|x\|_B < \frac{1}{f(u)}\|x\|_B \beta = \frac{1}{f(u)}\|z\|_B = 1$$

it follows that $h(x)$ has the form αu, where $\alpha < 1$. Since $u \in S$ we have $h(x) \in B$, and, as a result, (2) $h(F) \subseteq B \cup \{x_0\}$, (3) $S \cap h(F) = \{x_0\}$. We have shown that h is as required in the lemma. □

The proofs of our next lemmas strongly rely on the normality of locally convex spaces under consideration. The reasonings are based very essentially on the observation that every normal LCS is a countably paracompact topological space [T]. The reader can also recover this proof from ([A], p. 37). Recall that a topological space is countably paracompact if every countable open cover can be refined by a locally finite open cover.

The role of countable paracompactness is expressed by the following classical theorem of Dowker (cf. [E], 5.2.8, 5.5.20).

Theorem 3.8. *(Dowker) The following conditions are equivalent for a normal topological space X.*
(1) X is countably paracompact;
(ii) $X \times [0,1]$ is normal;
(iii) Suppose that $f : X \to I\!\!R$ and $g : X \to I\!\!R$ are respectively lower and upper semicontinuous functions, and for every $x \in X$, $g(x) < f(x)$. Then there exists a continuous function $h : X \to I\!\!R$ such that for every $x \in X$, $g(x) < h(x) < f(x)$.

Now, under the assumption of normality, we prove a weakening of Lemma 3.39(g) from [R1]. The fact proved in this lemma will be used in order to show that an open convex set in a normal SLB-space has property (4) of excellent sets (Definition 2.3).

Lemma 3.9. *Let Y be a normal SLB-space, and $B \subseteq Y$ be a ball. Then for every nonempty closed set $F \subseteq Y$ with the properties $F \cap B = \emptyset$ and $|cl(B) \cap F| \leq 1$, there is $h \in H(Y)$ such that (1) $var(h) \subseteq 2B$; (2) $cl(B) \setminus F \subseteq h(B)$; and (3) $h(B) \cap F = \emptyset$.*

Proof. Let $S = bd(B)$. We define $f : S \to [1, 1\frac{1}{2}]$ as follows

$$f(x) = \inf(\{\lambda \mid \lambda x \in F\} \cup \{1\tfrac{1}{2}\}).$$

It follows easily from Lemma 3.5 that f is a lower semicontinuous function.

Let $S_1 = S \setminus F$. There are two possibilities: either $S_1 = S$, or for some $x_0 \in S$, $S_1 = S \setminus \{x_0\}$. Since, by Claim 3.2, every point of Y is G_δ, in any case S_1 is an F_σ set in Y. But F_σ subset of a normal countably paracompact space is countably paracompact due to the result by Zenor (cf. [E], 5.5.16). Hence S_1 is also normal and countably paracompact.

Clearly, the restriction $f \restriction S_1$ is lower semicontinuous. Then, for every $x \in S_1$, $1 < f(x)$. Therefore, another application of Dowker theorem provides us with a continuous function $g_1 : S_1 \to I\!\!R$ such that for every $x \in S_1$, $1 < g_1(x) < f(x)$.

Now, we construct a new function $g : S \to [1, 1\frac{1}{2}]$. If $S_1 = S$, define $g = g_1$. Otherwise, let again $S_1 = S \setminus \{x_0\}$. Then we put

$g(x_0) = 1$ and for every $x \in S_1$ define

$$g(x) = 1 + \min(1, \|x - x_0\|_B) \cdot (g_1(x) - 1).$$

It is easy to check that the constructed function g has the following properties.

(i) g is continuous;

(ii) for every $x \in S_1$, $1 < g(x) < f(x) \leq 1\frac{1}{2}$;

We also know that if $\{x_0\} = S \cap F$, then

(iii) $g(x_0) = 1$.

Let $k : [1, 1\frac{1}{2}] \times [0, \infty) \to [0, \infty)$ be defined as follows. Denote $k(s,t)$ by $k_s(t)$. For every $s \in [1, 1\frac{1}{2}]$, $k_s(t)$ is a piecewise linear increasing function such that:

(i) $k_s \restriction [0, \frac{1}{2}] = Id$;

(ii) $k_s \restriction [\frac{1}{2}, 1]$ is a linear function taking $\frac{1}{2}$ to $\frac{1}{2}$ and 1 to s;

(iii) $k_s \restriction [1, 2]$ is a linear function taking 1 to s and 2 to 2;

(iv) $k_s \restriction [2, \infty) = Id$.

Clearly, k is continuous. Note that for every $s \in [1, 1\frac{1}{2}]$, $k_s \in H([0, \infty))$.

Let $h : Y \to Y$ be defined as follows: $h(0) = 0$, and for every $y \neq 0$

$$h(y) = k\left(g(\frac{y}{\|y\|_B}), \|y\|_B\right) \cdot \frac{y}{\|y\|_B} \ .$$

Since $Rng(g) \subseteq [1, 1\frac{1}{2}]$, h is well-defined. Also, for every $y \in Y$, $h \restriction \{ty \mid t \geq 0\} \in H(\{ty \mid t \geq 0\})$. So h is a bijection of Y. The functions k, g, and $\| \ \|_B$ are continuous, hence h is continuous at every $y \in Y \setminus \{0\}$. It follows that h is continuous on Y since $h \restriction \frac{1}{2}B = Id$.

The inverse mapping has the following form: $h^{-1}(0) = 0$, and for every $y \neq 0$

$$h^{-1}(y) = k_1\left(g(\frac{y}{\|y\|_B}), \|y\|_B\right) \cdot \frac{y}{\|y\|_B},$$

where $k_1(s,t) \stackrel{def}{=} k_s^{-1}(t)$. It is obvious that k_1 is defined similarly to k, hence the arguments that show the continuity of h^{-1} are as for h. So $h \in H(Y)$.

Since for every $s \in [1, 1\frac{1}{2}]$ and $t \geq 2$, $k(s,t) = t$, we have (1) $var(h) \subseteq 2B$.

Let prove (3). If $\|y\|_B \leq \frac{1}{2}$ then $h(y) = y$, so $h(\frac{1}{2}B) = \frac{1}{2}B$. Take any y with $\frac{1}{2} \leq \|y\|_B \leq 1$. If $\frac{y}{\|y\|_B} \in S_1$ then $g(\frac{y}{\|y\|_B}) < 1$ and

$$\|h(y)\|_B = k\left(g(\frac{y}{\|y\|_B}), \|y\|_B\right) \leq g(\frac{y}{\|y\|_B}) < f(\frac{y}{\|y\|_B}).$$

This proves $h(y) \notin F$, because by definition, $f(\frac{y}{\|y\|_B}) \cdot \frac{y}{\|y\|_B} \in F$. If $\frac{y}{\|y\|_B} = x_0$ then $g(\frac{y}{\|y\|_B}) = 1$ and, as it is easy to see, $h(y) = y$. In both cases we proved (3) $h(B) \cap F = \emptyset$.

It remains to show (2). Let again $\frac{1}{2} \leq \|y\|_B \leq 1$. Note that

$$\|h^{-1}(y)\|_B = k_s^{-1}(\|y\|_B), \text{ where } s = g(\frac{y}{\|y\|_B}).$$

If $\frac{y}{\|y\|_B} \in S_1$ then $s > 1$. Because k_s^{-1} is an increasing function and $k_s^{-1}(s) = 1$ we obtain that $k_s^{-1}(\|y\|_B) < 1$, in other words, $h^{-1}(y) \in B$. If $\frac{y}{\|y\|_B} = x_0$ then $h^{-1}(y) = y$. Thus, (2) $cl(B) \setminus F \subseteq h(B)$ is proved, and we have shown that h is as required in the lemma. □

We now prove Lemma 3.39(d) from [R1]. Here again the proof relies on the normality of Y. This lemma is, in some sense, dual to the previous Lemma 3.9.

Lemma 3.10. *Let Y be a normal SLB-space and $B \subseteq Y$ be a ball. Then for every closed $F \subseteq B$ there is $h \in H(Y)$ such that (1) $var(h) \subseteq B$, and (2) $h(F) \subseteq \frac{1}{2}B$.*

Proof. Let $S = bd(B)$. Define $f : S \to [\frac{1}{4}, 1]$ as follows

$$f(x) = \sup(\{\lambda \mid \lambda x \in F\} \cup \{\tfrac{1}{4}\}).$$

Again, it follows easily from Lemma 3.5 that f is an upper semicontinuous function on S.

For every $x \in S$, $f(x) < 1$. By the theorem of Dowker, there is a continuous function $g : S \to (\frac{1}{4}, 1)$ such that for every $x \in S$, $f(x) < g(x) < 1$.

We define $k : (\frac{1}{4}, 1) \times [0, \infty) \to [0, \infty)$. Denote $k(s, t)$ by $k_s(t)$. For every $s \in (\frac{1}{4}, 1)$, $k_s(t)$ is a piecewise linear increasing function such that
(i) $k_s \restriction [0, \frac{1}{4}] = Id$;
(ii) $k_s \restriction [\frac{1}{4}, s]$ is a linear function taking $\frac{1}{4}$ to $\frac{1}{4}$ and s to $\frac{1}{2}$;
(iii) $k_s \restriction [s, 1]$ is a linear function taking s to $\frac{1}{2}$ and 1 to 1;
(iv) $k_s \restriction [1, \infty) = Id$.
Clearly, k is continuous. Note that for every $s \in (\frac{1}{4}, 1)$, $k_s \in H([0, \infty))$.

Let $h : Y \to Y$ be defined as follows: $h(0) = 0$, and for every $y \neq 0$,

$$h(y) = k\left(g(\frac{y}{\|y\|_B}), \|y\|_B\right) \cdot \frac{y}{\|y\|_B} \ .$$

Since $Rng(g) \subseteq (\frac{1}{4}, 1)$, h is well-defined. Also, for every $y \in Y$, $h \restriction \{ty \mid t \geq 0\} \in H(\{ty \mid t \geq 0\})$. So h is a bijection of Y. The functions k, g, and $\| \ \|_B$ are continuous, hence h is continuous at every $y \in Y \setminus \{0\}$. It follows that h is continuous on Y since $h \restriction \frac{1}{4}B = Id$.

Exactly as in the previous lemmas we prove that h^{-1} is continuous. It means that $h \in H(Y)$.

(1) $var(h) \subseteq B$ holds since for every $s \in (\frac{1}{2}, 1)$ and $t \geq 1$, $k(s, t) = t$.

Let $y \in F$ and $\|y\|_B \geq \frac{1}{4}$. Then, $\|y\|_B < g(\frac{y}{\|y\|_B}) < 1$, and, according to the definition of $k(s, t)$, we have $\|h(y)\|_B \leq \frac{1}{2}$, in other words, (2) $h(F) \subseteq \frac{1}{2}B$. We have shown that h is as required in the lemma. ☐

Definition 3.11. Let X be a manifold over the class of normal SLB-spaces. Let Y be a normal SLB-space, $B \subseteq Y$ be a nonempty ball, and $\pi : cl(B) \to X$ be such that π is a homeomorphism between $Dom(\pi)$ and $Rng(\pi)$, $Rng(\pi)$ is closed in X, and $\pi(B)$ is open in X. Then π is called a cellular map, $\pi(B)$ is called the cell of π, and $\pi(\frac{1}{2}B)$ is called the half-cell of π.

Proof of Theorem 3.1: The proposition below constitutes a proof of the theorem.

Proposition 3.12. *Let X be a manifold over the class of normal SLB-spaces. Let $G = H(X)$ and $\mathcal{R} = VAR(X, H(X))$. Then*

(a) Every half-cell is small in $\langle X, G\rangle$.

(b) $X = S(X, G)$.

(c) Every half-cell is excellent in $\langle X, G\rangle$.

(d) Every member of $SC(X, G)$ is excellently structured in $\langle X, G, \mathcal{R}\rangle$.

Proof. (a) It follows from Proposition 3.3(a), (b) and (c).

(b) This part follows from Part (a) and the fact that every $x \in X$ belongs to a half-cell.

(c) Let U be the half-cell of π, and suppose that Y is a normal SLB-space, $B \subseteq Y$, $Dom(\pi) = cl(2B)$ and $U = \pi(B)$. We check that U satisfies properties $(1) - (4)$ from the definition of excellent sets.

Note that if $g \in H(Y)$ and $var(g) \subseteq 2B$, then $g^{\pi} \cup (Id\restriction(X \setminus \pi(2B))) \in H(X)$. Hence we may assume that $X = 2B$, $U = B$ and $G = \{g \in H(Y) \mid var(g) \subseteq 2B\}$.

<u>Property (1)</u> was proved in Part (a).

<u>Property (2)</u> follows from Lemma 3.10 and Proposition 3.3.

<u>Property (3)</u> follows from Lemma 3.7.

<u>Property (4)</u> follows from Lemma 3.9.

(d) As in Part (c), we may assume that $X = 2B$ and $G = \{g \in H(Y) \mid var(g) \subseteq 2B\}$.

By Proposition 3.3(a), $B \in \mathcal{R}$. So property (1) of definition 2.4(a) follows from regularity of X.

To prove property (2) of Definition 2.4(a), notice that for every $x_0 \in 2B$ and open $T \ni x_0$ there is $C \cong B$ such that $C \subseteq T$ and $x_0 \in bd(C)$. By property (3) of excellent sets and the fact (proved in Part (c)) that C is excellent, there is $C' \cong C$ such that $C' \subseteq C$ and $cl(C') \cap bd(C) = \{x_0\}$. So $U \stackrel{def}{=} C'$ and $V \stackrel{def}{=} 2B \setminus cl(C)$ satisfy the requirements of Definition 2.4(a)(2). $\square$

Proof of Theorem 1.1: It follows trivially from Theorems 2.6(a) and 3.1. $\square$

4. Examples

Evidently, every normed space is a normal SLB-space. Let Y be a strict topological inductive limit of a sequence of normed spaces E_n such that E_n is a proper closed subspace of E_{n+1} for every n. Then Y is a normal non-metrizable SLB-space ([K], §19.4). In particular, the class of normal SLB-spaces contains all LCS wcich are strict topological inductive limits of a sequence of Banach spaces. Note that latter spaces are both barrelled and bornological LCS ([K], 27.1.4, 28.2.2).

Here we construct an example of an SLB-space that is not normal and show that, in general, normal SLB-spaces are not necessarily barrelled or bornological LCS.

We shall use the basic properties of the free locally convex space $L(X)$, where X is a completely regular topological space. The space $L(X)$ is characterized by the following axioms: (1) X is identified with a closed subspace of $L(X)$ that forms a Hamel basis in $L(X)$; (2) every continuous mapping f from X to a locally convex space E lifts to a continuous linear operator $\bar{f} : L(X) \to E$.

By $C_p(X)$ we denote the space $C(X)$ of all continuous real functions on X equipped with the topology of pointwise convergence. $C_k(X)$ stands for $C(X)$ equipped with the topology of uniform convergence on compact subsets of X.

The canonical duality between $L(X)$ and $C_p(X)$ is compatible with the topologies of these spaces and allows us to identify $C_p(X)$ with the weak dual of $L(X)$, and to identify the weak dual of $C_p(X)$ with the space $L(X)$ in the weak topology.

It is well known also that if $X = (X, \rho)$ is a metric space, then there exists a continuous norm on $L(X)$ (so-called the Graev extention of the metric ρ). For the convenience of less experienced readers we give an explicit formula of this extention of ρ from X to the closed hyperplane $L_0(X) = \{\sum a_x x : \sum a_x = 0\} \subset L(X)$ (see [U]). Let

$$u = \sum_{i=1}^{m} a_i x_i - \sum_{j=1}^{n} b_j y_j \ \in \ L_0(X),$$

where $a_i, b_j \geq 0$. Then the norm on $L_0(X)$ is defined as follows

$$\|u\| = \min\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\rho(x_i, y_j) \mid c_{ij} \geq 0,\ \sum_{j=1}^{n} c_{ij} = a_i,\ \sum_{i=1}^{m} c_{ij} = b_j\right\}.$$

A topological space X is called *submetrizable* if there exists a one-to-one continuous mapping of X onto a metrizable space.

Proposition 4.1. *The free locally space $L(X)$ is an SLB-space iff X is a submetrizable topological space.*

Proof. If $L(X)$ is an SLB-space, then $L(X)$ itself is a submetrizable topological space and hence its subspace X has the same property. In the opposite direction, let $f : X \to M$ be an one-to-one mapping, where M is a metrizable space. Denote by E the space $L(M)$ equipped with the constructed above norm. Then $f : X \to E$ is a continuous mapping and it lifts to a continuous linear one-to-one operator $\bar{f} : L(X) \to E$. The preimage of the unit ball of E is a ball in $L(X)$. □

For every compact X, the space $L(X)$ is a Lindelöf topological space and hence normal, being a countable union of compact subspaces. So $L(X)$ is a normal SLB-space for every metrizable compact X. On the other hand, if X is a non-normal submetrizable space (for instance, the square of the Sorgenfrey line Z ([E], 2.3.12)), then $L(X)$ is a non-normal SLB-space. Note that $L(Z)$ contains a closed copy of Z^2 and therefore $L(Z)$ itself is also a non-normal SLB-space.

Our last assertion should be considered as a part of mathematical folklore.

Proposition 4.2. *If X is an infinite compact space, then $L(X)$ is not barrelled, and $L(X)$ is not a bornological LCS.*

Proof. The Mackey topology (see [K], §21.4) $\tau\langle L(X), C(X)\rangle$ on $L(X)$ coincides with the topology of uniform convergence on convex compact subsets of $C_p(X)$. The latter topology is strictly finer than the initial topology of $L(X)$, because according to the Raikov's theorem [R] the initial topology of $L(X)$ can be described as the topology of uniform convergence on compact subsets of $C_k(X)$ (see also [F], [U]). It follows that $L(X)$ is not a Mackey space and hence neither barrelled nor a bornological LCS. □

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