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NONCLASSICAL TECHNIQUES FOR MODELS OF COMPUTATION

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Abstract

After surveying recent work and new techniques in domain theoretic models of spaces, we introduce a new topological concept called recurrence, and consider some of its applications to the model problem.

1. Introduction

To every compact metric space X, we may assign a continuous dcpo, or *domain,* by forming the collection $\mathbf{U}X$ of all nonempty closed subsets of X, and then ordering them under reverse inclusion. A continuous dcpo is a partially ordered set carrying intrinsic notions of approximation and completeness. As such, it also carries an intrinsic topology, known as the Scott topology. In the case of the *upper space* $\mathbf{U}X$, as it is called, one finds that X may be recovered from $\mathbf{U}X$ via

$$X \simeq \max \mathbf{U}X = \{\{x\} : x \in X\} \subseteq \mathbf{U}X$$

where the topology on $\max \mathbf{U}X \subseteq \mathbf{U}X$ is the relative Scott topology inherited from $\mathbf{U}X$. Because of this homeomorphism, we say that $\mathbf{U}X$ is a model of the space X.

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In [4], Abbas Edalat used only this link to the classical world to define a generalization of the Riemann integral for bounded functions $f : X \to \mathbb{R}$ on a compact metric space X. Once it was observed that X could be represented as the maximal elements of the continuous dcpo $\mathbf{U}X$, all other details were handled using domain theory. Naturally, if all we need to do to define an integral is represent a space in this manner, one wonders about exactly which spaces have models.

This paper is intended to be entirely self-contained. It is about the model problem above and it introduces many new techniques that have been used to make substantial progress on the question. Among them, we find a little descriptive set theory, the relationship between the Baire property and topological completeness, one of Choquet's infamous games, and a new idea, a rare form of denseness called recurrence.

In the next section, we tell the reader what a domain is and discuss the Scott topology in detail, pointing out that while it is only a T_0 topology, it still has many properties that may be taken advantage of. In the next, we consider the model question and review the work which has been done on it. This leads to the hierarchy for countably based models: There are four classes of models, two of them are completely solved, one is well-understood, and the last, the class of all countably based models, remains an enigma. However, after introducing recurrence, we gain a lot of insight about spaces with models, and actually end up learning something about them in general: They are all Baire spaces.

2. Background

In this section we review some basic ideas and results that will be needed throughout the course of this paper.

2.1. Domain Theory

2.1.1. Order

The reader unfamiliar with the basics of domain theory will find [1] valuable. We touch on certain basic aspects which are not quoted very often.

Definition 2.1. A partially ordered set $(P, \sqsubseteq)$ is a set P together with a binary relation $\sqsubseteq \subseteq P^2$ which is

(i) reflexive: $(\forall x \in P)\, x \sqsubseteq x$,

(ii) antisymmetric: $(\forall x, y \in P)\, x \sqsubseteq y \;\&\; y \sqsubseteq x \Rightarrow x = y$, *and*

(iii) transitive: $(\forall x, y, z \in P)\, x \sqsubseteq y \;\&\; y \sqsubseteq z \Rightarrow x \sqsubseteq z$.

We refer to partially ordered sets as *posets.*

Definition 2.2. A *least element* in a poset $(P, \sqsubseteq)$ is an element $\bot \in P$ such that $\bot \sqsubseteq x$ for all $x \in P$. Such an element is unique and is called a *bottom* element. An element $x \in P$ is *maximal* if $(\forall y \in P)\, x \sqsubseteq y \Rightarrow x = y$. The set of maximal elements in a poset is written $\max P$. Similarly, one has the notions of *greatest element* and *minimal element.*

Definition 2.3. Let $(P, \sqsubseteq)$ be a poset. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of a subset $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$.

A poset $(P, \sqsubseteq)$ is abbreviated to P, just as in topology, where one writes X for the topological space (X, τ).

Definition 2.4. In a poset $(P, \sqsubseteq)$, $a \ll x$ iff for all directed subsets $S \subseteq P$ which have a supremum,

$$x \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, a \sqsubseteq s.$$

We say that a is an approximation of x whenever $a \ll x$. The set of all approximations of x is written $\mathop{\downarrow\!\!\downarrow} x$. An element $x \in P$ is *compact* if $x \ll x$. The set of compact elements in a poset P is written $K(P)$.

Definition 2.5. A poset P is *continuous* if $\twoheaddownarrow x$ is directed with supremum x for all $x \in P$.

It is usually easier to find a basis for a poset.

Definition 2.6. A subset B of a poset P is a *basis* for P if $B \cap \twoheaddownarrow x$ contains a directed subset with supremum x, for each $x \in P$.

Lemma 2.7. *A poset is continuous iff it has a basis.*

Definition 2.8. A poset is *algebraic* if its compact elements form a basis. A poset is ω-*continuous* if it has a countable basis.

Continuity provides a definite notion of *approximation* for posets.

Proposition 2.9 (Zhang [24]). *Continuous posets have the interpolation property:* $x \ll y \Rightarrow (\exists z)\, x \ll z \ll y$.

A useful form of *completeness* is offered by a dcpo.

Definition 2.10. A poset is a *dcpo* if every directed subset has a supremum.

Domains possess both approximation and completeness.

Definition 2.11. A *domain* is a continuous poset which is also a dcpo. A domain is also called a *continuous dcpo.*

2.1.2. The Topological Aspect

One of the interesting things about a domain is that its order-theoretic structure is rich enough to support the derivation of intrinsically defined topologies. The most important of these is the Scott topology.

Definition 2.12. A subset U of a poset P is *Scott open* if

(i) U is an upper set: $x \in U$ & $x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: For every directed $S \subseteq P$ which has a supremum,

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on P is called the Scott topology. It is denoted σ_P.

Proposition 2.13 (Zhang [24])**.** *The collection* $\{ \mathop{\uparrow\!\!\uparrow} x : x \in P \}$ *is a basis for the Scott topology on a continuous poset* P*.*

Unless explicitly stated otherwise, all topological statements about posets are made with respect to the Scott topology.

Proposition 2.14. *A function* $f : P \to Q$ *between posets is continuous iff*

(i) f *is monotone:* $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$*.*

(ii) f *preserves directed suprema: For every directed* $S \subseteq P$ *which has a supremum,*

$$\bigsqcup f(S) \text{ exists } \& \ f(\bigsqcup S) = \bigsqcup f(S).$$

Definition 2.15. The *Lawson topology* on a continuous poset P has as a basis all sets of the form $\mathop{\uparrow\!\!\uparrow} x \setminus \uparrow F$ where $x \in P$ and $F \subseteq P$ is finite.

Proposition 2.16 (Jung [14])**.** *The Lawson topology on a continuous dcpo is compact iff it is Scott compact and the intersection of any two Scott compact upper sets is Scott compact.*

Definition 2.17. A *Scott domain* is a continuous dcpo with least element $\perp$ in which each pair of elements bounded from above has a supremum.

Notice that we have not required Scott domains to be ω-algebraic.

Proposition 2.18. *Every Scott domain has compact Lawson topology.*

Domains with compact Lawson topology have a property that will interest us later on.

Proposition 2.19. *If D is a domain with compact Lawson topology, then the relative Scott and Lawson topologies on* $\max D$ *agree.*

2.2. Topology

We review topological ideas that have proven to be indispensable in the study of models.

2.2.1. Locally Compact Sober Spaces

The standard reference on locally compact sober spaces is [13].

Definition 2.20. A subset of a space is *compact* if each of its open covers has a finite subcover.

Definition 2.21. A topological space X is *locally compact* if it has a basis of compact neighborhoods, that is, given a point $x \in X$ and an open set $U \subseteq X$ containing x, there is a compact set $K \subseteq X$ and an open set $V \subseteq X$ such that $x \in V \subseteq K \subseteq U$.

Definition 2.22. The *saturation* of a set K in a topological space X is

$$\text{sat}\, K = \bigcap \{U : K \subseteq U^{open}\}.$$

A subset $K \subseteq X$ is *saturated* if $K = \text{sat}\, K$, that is, if it is the intersection of all open sets containing it.

All sets in a T_1 space are saturated. On the other hand, the saturated subsets of a poset with its Scott topology are merely the upper sets.

Lemma 2.23. *Let X be a topological space and $K \subseteq X$. Then*

(i) *The set* sat K *is saturated.*

(ii) *The set K is compact iff* sat K *is compact.*

Corollary 2.24. *A locally compact space has a basis of compact saturated neighborhoods.*

Locally compact spaces on their own are difficult to understand.

Definition 2.25. A closed subset of a space is *irreducible* if it is nonempty and cannot be written as the union of two closed proper subsets. A T_0 space is *sober* if every irreducible closed set is the closure of a unique point.

Domains topologically are locally compact sober spaces [1].

Theorem 2.26. *The Scott topology on a continuous dcpo is locally compact and sober.*

More generally, the Scott topology on a continuous poset is locally compact, in which case sobriety can then be viewed as a form of completeness.

Proposition 2.27 (Zhang [24])**.** *A continuous poset is a dcpo iff its Scott topology is sober.*

However, outside the realm of locally compact spaces, sobriety is probably best understood as being an appropriate substitute for the Hausdorff axiom. This is clearly illustrated by the *Hofmann-Mislove theorem.*

Theorem 2.28 (Hofmann-Mislove [13])**.** *In a sober space, the class of nonempty compact saturated sets is closed under filtered intersections.*

As every Hausdorff space is sober and all sets in a T_1 space are saturated, an easy corollary of Theorem 2.28 is that the filtered intersection of nonempty compact sets in a Hausdorff space is nonempty and compact. And so we see that sobriety is a *useful* generalization of the Hausdorff axiom.

Anyone familiar with locally compact sober spaces knows the value of the Hofmann-Mislove theorem. However, for all of its popularity, one of its most basic implications was missed until very recently: It asserts the topological completeness of locally compact sober spaces.

2.2.2. The Choquet Phenomenon

One of Choquet's original motivations for introducing the spaces we call Choquet complete (called *strong Choquet* in [15]) was to provide an elegant and unified approach to the classical Baire category arguments of analysis. Despite the success of the idea, it remains largely unknown to many. The Choquet phenomenon is this: Not only are the Baire spaces of analysis Choquet complete, but so too are the non-Hausdorff spaces of theoretical computer science, i.e., domains in their Scott topology.

Definition 2.29. Let (X, τ) be a space and $\tau_* = \{(U, x) : x \in U \in \tau\}$. (X, τ) is *Choquet complete* if there is a sequence $(a_n)_{n\geq 1}$ of functions

$$a_n : \tau_*^n \to \tau$$

such that

(i) For each $((U_1, x_1), \cdots, (U_n, x_n)) \in \mathrm{dom}(a_n)$,

$$x_n \in a_n((U_1, x_1), \cdots, (U_n, x_n)) \subseteq U_n,$$

and

(ii) For any sequence (V_n, x_n) in τ_* with $V_{n+1} \subseteq a_n((V_1, x_1), \cdots, (V_n, x_n))$, for all $n \geq 1$, we have

$$\bigcap_{n\geq 1} V_n \neq \emptyset.$$

By definition, the function a_n maps nonempty open sets to nonempty open sets. Choquet complete spaces possess abstract notions of the two fundamentals of computation: (i) *approximation* and (ii) *completeness.*

Theorem 2.30. *We have the following standard facts:*

(i) *A Choquet complete space is Baire.*

(ii) *A locally compact Hausdorff space is Choquet complete.*

(iii) *A metric space is Choquet complete iff it is completely metrizable.*

(iv) *A G_δ subset of a Choquet complete space is Choquet complete.*

A proof of (iv) appears in [9], while the others are all due to Choquet [2]. Interestingly, Choquet's completeness includes the most well-known form of completeness in topology.

Corollary 2.31. *Every Čech-complete space is Choquet complete.*

Proof. A Čech-complete space is a G_δ subset of a compact Hausdorff space. The result now follows by applying Prop. 2.30(ii) and then Prop. 2.30(iv). □

Notice, however, that while Čech-completeness requires spaces to be Tychonoff, Choquet completeness allows for the possibility of no separation whatsoever. The importance of this is made certain by the next result, a straightforward generalization of one first given in [21].

Theorem 2.32. *Every locally compact sober space is Choquet complete.*

Proof. Let (X, τ) be locally compact sober. Define the approximation scheme

$$a : \{(U, x) : x \in U \in \tau\} \to \tau$$

as follows: Given an open set U and a point $x \in U$, use Corollary 2.24 to choose an open set V and a compact *saturated* set K such that

$$x \in V \subseteq K \subseteq U.$$

Then set $a(U, x) = V$. In this way, we know that for all $(U, x) \in \mathrm{dom}(a)$, there is a compact saturated set K with $x \in a(U, x) \subseteq K \subseteq U$.

Finally, given elements $(U_n, x_n) \in \mathrm{dom}(a)$ with $U_{n+1} \subseteq a(U_n, x_n) \subseteq U_n$, for all $n \geq 1$, we immediately obtain a decreasing sequence of nonempty compact saturated sets (K_n) with

$$\bigcap_{n \geq 1} U_n = \bigcap_{n \geq 1} K_n.$$

But this intersection is nonempty by the Hofmann-Mislove theorem.

Setting $a_n((U_1, x_1), \cdots, (U_n, x_n)) = a(U_n, x_n)$ finishes the proof. □

The last result gives a new and simple proof of the following.

Corollary 2.33. *A locally compact sober space is Baire.*

Proof. All Choquet complete spaces are Baire. □

We have several times now informally referred to domains as being “complete.” The attentive reader will have noticed that until now this statement has been without topological justification.

Corollary 2.34. *The Scott topology on a continuous dcpo is Choquet complete.*

Proof. A continuous dcpo in its Scott topology is a locally compact sober space. □

Thus, complete metric spaces, locally compact Hausdorff spaces and domains in their Scott topology all possess the same notion of topological completeness.

2.2.3. The Baire Property

Another topological idea whose importance in domain theory has been undervalued is the Baire property: A space is *Baire* if the intersection of countably many open dense sets is dense.

Definition 2.35. A space is *completely Baire* if all of its closed subsets are Baire.

The terminology "completely Baire" is from Kechris [15]. In [10], they are called *Baire spaces in the strong sense,* where a proof of the following may be found.

Proposition 2.36. *If X is a completely Baire space, then every G_δ subset of X is completely Baire.*

The relevance to domains is as follows.

Proposition 2.37. *A locally compact sober space is completely Baire.*

Proof. If C is a closed subset of a locally compact sober space X, then C is locally compact and sober in its relative topology, as each of these properties are hereditary for closed sets. But then C is Baire by Corollary 2.33. □

In fact, sometimes the topological completeness of a second countable space is *equivalent* to its being completely Baire, provided one carries a weak additional assumption (like local compactness, for example).

Theorem 2.38 (Hofmann [12]). *For a second countable, locally compact space X, the following are equivalent:*

(i) *The space X is sober.*

(ii) *Every closed subset of X is Baire.*

Here is another example from descriptive set theory that we will have the opportunity to use later.

Definition 2.39. A space is *analytic* if it is separable, metrizable and is the continuous image of a Polish space.

For example, if (X, τ_1) is a separable metric space, and there is a Polish topology τ_2 on X such that $\tau_1 \subseteq \tau_2$, then (X, τ_1) is an analytic space.

Theorem 2.40 (Σ_1^1 Determinacy [15]). *An analytic space is Polish iff all of its closed subsets are Baire.*

The notation "Σ_1^1 Determinacy," a convention borrowed from [15], means that the last result depends on the assumption of *analytic determinacy,* the principle that all analytic games on $\mathbb{N}$ are determined.

3. Models of Spaces

We review the notion of a model in domain theory, consider the classical examples, and discuss the major results in the area.

3.1. The Definition of Model

The purpose of a model is to isolate a topological space within a domain in a useful way.

Definition 3.1. A *model* of a space X is a continuous dcpo D together with a homeomorphism

$$\phi : X \to \max D$$

where $\max D$ carries its relative Scott topology inherited from D.

Here are some examples.

Example 3.2. A model of the real line. The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \ \&\ a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{I}\mathbb{R}$ is $\bigcap S$, while the approximation relation is characterized by $I \ll J \Leftrightarrow J \subseteq \text{int}(I)$. A countable basis for $\mathbf{I}\mathbb{R}$ is given by $\{[p, q] : p, q \in \mathbb{Q} \ \&\ p \leq q\}$.

The domain $\mathbf{I}\mathbb{R}$ is called the *interval domain* and it is a model of the real line since $\max \mathbf{I}\mathbb{R} = \{[x] : x \in \mathbb{R}\} \simeq \mathbb{R}$.

There is also an elegant model of the irrationals using partial mappings on the naturals.

Definition 3.3. A *partial function* $f : X \rightharpoonup Y$ between sets X and Y is a function $f : A \to Y$ defined on a subset $A \subseteq X$. We write $\text{dom}\, f = A$ for the *domain* of a partial map $f : X \rightharpoonup Y$.

Example 3.4. A model of the irrationals. The set of partial mappings on the naturals

$$[\mathbb{N} \rightharpoonup \mathbb{N}] = \{\ f \mid f : \mathbb{N} \rightharpoonup \mathbb{N} \textit{ is a partial map }\}$$

becomes an ω-algebraic dcpo when ordered by extension

$$f \sqsubseteq g \Leftrightarrow \text{dom}\, f \subseteq \text{dom}\, g \ \&\ f = g \text{ on } \text{dom}\, f.$$

The supremum of a directed set $S \subseteq [\mathbb{N} \rightharpoonup \mathbb{N}]$ is $\bigcup S$, under the view that functions are certain subsets of $\mathbb{N} \times \mathbb{N}$, while the approximation relation is

$$f \ll g \Leftrightarrow f \sqsubseteq g \ \&\ \text{dom}\, f \text{ is finite}.$$

The maximal elements of $[\mathbb{N} \rightharpoonup \mathbb{N}]$ are the *total functions,* that is, those functions f with $\operatorname{dom} f = \mathbb{N}$.

It is easy to see that as a space in its relative Scott topology we have

$$\max\,[\mathbb{N} \rightharpoonup \mathbb{N}] \simeq \mathbb{N}^{\mathbb{N}}$$

where $\mathbb{N}$ has the discrete topology. Of course, $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R} \setminus \mathbb{Q}$, so we have a model of the irrationals.

Example 3.5. A model of the Cantor set. The collection of functions

$$\Sigma^\infty = \{\, s \mid s : \{1, \ldots, n\} \to \{0,1\}, 0 \leq n \leq \infty \,\}$$

is also an ω-algebraic dcpo under the extension order

$$s \sqsubseteq t \Leftrightarrow |s| \leq |t| \;\&\; (\,\forall\, 1 \leq i \leq |s|\,)\; s(i) = t(i),$$

where $|s|$ is written for the cardinality of dom s. The supremum of a directed set $S \subseteq \Sigma^\infty$ is $\bigcup S$, while the approximation relation is

$$s \ll t \Leftrightarrow s \sqsubseteq t \;\&\; |s| < \infty.$$

The extension order in this special case is usually called the *prefix* order. The elements $s \in \Sigma^\infty$ are called *strings* over $\{0, 1\}$. The quantity $|s|$ is called the *length* of a string s. The *empty string* ε is the unique string with length zero. It is the least element $\perp$ of Σ^∞.

We call Σ^∞ the *Cantor set model* since $\max \Sigma^\infty = \{s : |s| = \infty\}$ is homeomorphic to the Cantor set.

Example 3.6. A model for locally compact Hausdorff spaces. If X is a locally compact Hausdorff space, its upper space

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \textit{ is compact}\}$$

ordered under reverse inclusion

$$A \sqsubseteq B \Leftrightarrow B \subseteq A$$

is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$ and the approximation relation is $A \ll B \Leftrightarrow B \subseteq \mathrm{int}(A)$. The *upper space* is a model of X because $\max \mathbf{U}X = \{\{x\} : x \in X\} \simeq X$.

Example 3.7. A model for complete metric spaces. Given a metric space (X, d), the *formal ball model* [5]

$$\mathbf{B}X = X \times [0, \infty)$$

is a poset when ordered via

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s.$$

The approximation relation is characterized by

$$(x, r) \ll (y, s) \Leftrightarrow d(x, y) < r - s.$$

The poset $\mathbf{B}X$ is continuous. However, $\mathbf{B}X$ is a dcpo iff the metric d is complete. In addition, $\mathbf{B}X$ has a countable basis iff X is a separable metric space. Finally, $\max \mathbf{B}X = \{(x, 0) : x \in X\} \simeq X$, so $\mathbf{B}X$ is a model of X.

A conceptually simpler model of complete metric spaces is given in [21]. However, we have listed the formal ball model here because of its elegance.

From these examples, we see that the spaces of interest in mathematics, classically speaking, all have models. The *model question* in domain theory calls for the characterization of precisely those spaces which possess a model.

3.2. Countably Based Models

The Scott topology on a continuous dcpo D is second countable iff D is ω-continuous [8].

Definition 3.8. A *countably based model* of a space X is a model

$$(D, \phi : X \simeq \max D)$$

in which the continuous dcpo D is ω-continuous.

Lawson proved that a certain *subset* of countably based models capture exactly the Polish spaces.

Theorem 3.9 (Lawson [17]). *For a topological space X, the following are equivalent:*

(i) *The space X is Polish.*

(ii) *There is an ω-continuous dcpo D whose relative Scott and Lawson topologies on $\max D$ agree such that $X \simeq \max D$.*

In [18], the author asked whether or not every Polish space had a model by a Scott domain. The reason was that earlier Flagg and Kopperman [7] had proven it for zero-dimensional Polish spaces.

Theorem 3.10 (Ciesielski, Flagg & Kopperman [3]). *Every Polish space is homeomorphic to the maximal elements of an ω-continuous Scott domain.*

The converse, in view of Prop. 2.19, follows from Lawson's theorem. Finally, we confront the real issue at hand: Which metric spaces have countably based models?

Theorem 3.11 (Martin [21]). *The maximal elements of an ω-continuous dcpo are regular iff Polish.*

As the formal ball model $\mathbf{B}X$ provides a countably based model for any Polish space X, we arrive at the following.

Corollary 3.12. *A regular space has a countably based model iff it is Polish.*

In addition, Lawson's theorem is now a trivial consequence of the proof of the last corollary: If the relative Scott and Lawson topologies on $\max D$ agree, then $\max D$ is regular in its relative Scott topology. But more importantly, we learn something about the nature of countably based domains in general.

Corollary 3.13. *There is no countably based model of the rationals.*

There is a simpler way to explain our inability to model the rationals that we will see later. Perhaps a better example of a space without a countably based model, one for which Theorem 3.11 must be used, is given by a subset of the real line which is completely Baire but not Polish ([15], p.161).

3.3. Measurement

The set $[0,\infty)^*$ is the domain of nonnegative reals in their opposite order.

Definition 3.14. A Scott continuous map $\mu : P \to [0,\infty)^*$ on a continuous poset P *induces the Scott topology near* $X \subseteq P$ iff for all $x \in X$, if (x_n) is a sequence in $\mathop{\downarrow\!\!\downarrow} x$ with $\lim \mu x_n = \mu x$, then (x_n) is directed with supremum x.

If $\mu : P \to [0,\infty)^*$ is a function, its *kernel* is $\ker \mu = \{x \in P : \mu x = 0\}$.

Definition 3.15. A Scott continuous map $\mu : P \to [0,\infty)^*$ on a continuous poset P is called a *measurement* if it induces the Scott topology near its kernel.

Knowing that μ is a measurement ensures that computational observations made using μ are *reliable.* For instance, Definition 3.14 says that if we *observe* that a sequence (x_n) of approximations calculate x, then they *actually do* calculate x. For much more on this, see [19] and [20].

Example 3.16. Domains and their standard measurements.

(i) $(\mathbf{I}\mathbb{R}, \mu)$ the interval domain with the length measurement $\mu[a,b] = b - a$.

(ii) $([\mathbb{N} \rightharpoonup \mathbb{N}], \mu)$ the partial functions on the naturals with

$$\mu f = |\text{dom } f|$$

where $|\cdot| : \mathcal{P}\omega \to [0, \infty)^*$ is the measurement on the algebraic lattice $\mathcal{P}\omega$ given by

$$|x| = 1 - \sum_{n \in x} \frac{1}{2^{n+1}}.$$

(iii) $(\Sigma^\infty, 1/2^{|\cdot|})$ the Cantor set model where $|\cdot| : \Sigma^\infty \to [0, \infty)^*$ is the length of a string.

(iv) $(\mathbf{U}X, \text{diam})$ the upper space of a locally compact metric space (X, d) with

$$\text{diam}\, K = \sup\{d(x, y) : x, y \in K\}.$$

(v) $(\mathbf{B}X, \pi)$ the formal ball model of a complete metric space (X, d) with $\pi(x, r) = r$.

In each example above, we have a measurement $\mu : D \to [0, \infty)^*$ on a domain with $\ker \mu = \max D$.

Proposition 3.17 (Martin [19],[20]). *If $\mu : P \to [0, \infty)^*$ is a measurement on a continuous poset P, then*

(i) *The objects in* $\ker \mu$ *are maximal elements, that is,* $\ker \mu \subseteq \max P$.

(ii) *The* $\ker \mu$ *is a G_δ subset of P in the Scott topology.*

Though we only have $\ker \mu \subseteq \max P$ in general, it can be shown that there always exists a continuous poset P_μ such that $\max P_\mu \simeq \ker \mu$. In addition, if P is a dcpo, so is P_μ. Thus, studying the topological structure of the kernel of a measurement is related to the model problem.

Definition 3.18. A sequence of open covers $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of a space X is called a *development* provided that $\{\mathrm{St}(x,\mathcal{U}_n) : n \geq 0\}$ is a basis at x where

$$\mathrm{St}(x,\mathcal{U}_n) = \bigcup\{A : x \in A \in \mathcal{U}_n\}.$$

A space with a development is termed *developable.*

Theorem 3.19 (Martin & Reed [22])**.** *A space is developable and T_1 iff it is the kernel of a measurement on a continuous poset.*

For the case of a continuous *dcpo,* see [22]. In addition, one may capture metric spaces and complete metric spaces using a special class of measurements called *Lebesgue measurements*: A space is metrizable iff it is the kernel of a Lebesgue measurement on a continuous *poset;* it is completely metrizable iff it is the kernel of a Lebesgue measurement on a continuous *dcpo* [19].

However, the reason for studying Lebesgue measurements has little to do with topology: They are precisely the measurements which extend to the convex powerdomain. As one studies computation on a space X using a model D, they study the computation of its *compact subsets* using the convex powerdomain $\mathbf{C}D$. Such ideas explain, for example, Edalat and Heckmann's treatment of fractals [5].

4. The Hierarchy for Countably Based Models

In the study of countably based models, a natural hierarchy has emerged:

$$\max D \text{ regular} \Longrightarrow \max D \text{ measurement} \Longrightarrow \max D\ G_\delta \Longrightarrow \max D.$$

At first glance, all of these implications are obvious except the first. However, by Theorem 3.11, a regular space with a countably based model is Polish, and by Example 3.16(v), a Polish space X can be modelled with the formal ball model $(\mathbf{B}X, \pi)$

as $X \simeq \ker \pi = \max \mathbf{B}X$. Thus, a Polish space is the kernel of a measurement on a countably based domain, and so the first implication holds.

Then our real question must be this: Why is Theorem 3.11 true? First, an observation long overdue.

Proposition 4.1. *For a metric space X, the following are equivalent:*

(i) *X is completely metrizable.*

(ii) *X is a G_δ subset of a locally compact sober space.*

(iii) *X is a G_δ subset of a continuous dcpo.*

(iv) *X is a G_δ subset of its metric completion $\hat{X}$.*

(v) *X is a G_δ subset of a compact Hausdorff space.*

Proof. We need only establish the equivalence of (i), (ii) and (iii). See the masterful text by Engelking [6] for the others.

(i) $\Rightarrow$ (iii): Example 3.16(v) gives a model $(\mathbf{B}X, \pi)$ of a complete metric space X with $X \simeq \ker \pi = \max \mathbf{B}X$. As the kernel of a measurement, $X \simeq \max \mathbf{B}X$ is a G_δ subset of the continuous dcpo $\mathbf{B}X$.

(iii) $\Rightarrow$ (ii): A continuous dcpo is both locally compact and sober in its Scott topology by Theorem 2.26.

(ii) $\Rightarrow$ (i): A locally compact sober space is Choquet complete by Theorem 2.32, and since Choquet completeness is inherited by G_δ sets (Theorem 2.30(iv)), such a metric space must be Choquet complete and hence completely metrizable by Theorem 2.30(iii). □

Thus, to prove Theorem 3.11, we need only show that $\max D$ is a G_δ subset of D when $\max D$ is regular. And in fact, this is exactly what one can do. But the *reason* it can be done is an interesting bit of topological magic worth emphasizing.

5. Recurrence

Recall that a subset of a space is *saturated* if it is the intersection of open sets.

Definition 5.1. A subset of a space is *recurrent* if it intersects every nonempty saturated set.

We denote spaces by ∂ and their recurrent subsets by d. The intuition is that elements in $\partial \setminus d$ are *partial,* while those in d are total, or *ideal.*

Recurrence is an extreme form of denseness for which the Baire property is actually *hereditary.*

Theorem 5.2. *A space is Baire iff all of its recurrent subsets are Baire.*

Proof. Let ∂ be a Baire space with a recurrent subset d. If $\{U_n \cap d : n \geq 1\}$ is a countable collection of open dense subsets of d, then

$$\{U_n : n \geq 1\} \text{ is a sequence of open dense subsets of } \partial,$$

as d is a dense subset of ∂. But ∂ is a Baire space, so

$$\bigcap_{n \geq 1} U_n \text{ is a dense subset of } \partial.$$

Now we claim that $(\bigcap U_n) \cap d$ is a dense subset of d. Let $U \cap d$ be a nonempty open subset of d. Then U is open in ∂ and so by the density of $\bigcap U_n$,

$$U \cap \bigcap_{n \geq 1} U_n \neq \emptyset.$$

But this set is saturated, and so by the recurrence of d,

$$(U \cap \bigcap_{n \geq 1} U_n) \cap d = (\bigcap_{n \geq 1} U_n \cap d) \cap (U \cap d) \neq \emptyset.$$

Thus, $(\bigcap U_n) \cap d$ is a dense subset of d, which proves that d is Baire.

The converse is trivial: Every space is a recurrent subset of itself. $\square$

Because recurrent sets are dense, we also have that a space is Baire iff it has a recurrent subset which is Baire. Another surprising aspect is the existence of spaces which have *smallest* recurrent sets.

Proposition 5.3. *A space ∂ has a smallest recurrent set iff it has a recurrent set d with* $\mathrm{sat}\{x\} = \{x\}$ *for all $x \in d$. In either case, its smallest recurrent set is*

$$d = \{x \in \partial : \mathrm{sat}\{x\} = \{x\}\},$$

and the relative topology on d is T_1.

Proof. ($\Rightarrow$) Let d be the smallest recurrent subset of ∂ and $x \in d$. To show that $\mathrm{sat}\{x\} = \{x\}$, let $y \in \mathrm{sat}\{x\}$. First we prove that $(d \setminus \{x\}) \cup \{y\}$ is recurrent.

To this end, let A be a saturated subset of ∂. By the recurrence of d, choose $z \in d \cap A$. If $z \neq x$, then $((d \setminus \{x\}) \cup \{y\}) \cap A \neq \emptyset$. Otherwise, $x = z \in A$. But now we have $y \in \mathrm{sat}\{x\}$ *and* $x \in A = \mathrm{sat}\, A$, which means

$$y \in \mathrm{sat}\{y\} \subseteq \mathrm{sat}\{x\} \subseteq \mathrm{sat}\, A = A.$$

Either way, $((d \setminus \{x\}) \cup \{y\}) \cap A \neq \emptyset$, for all saturated $A \subseteq \partial$, which establishes the recurrence of $(d \setminus \{x\}) \cup \{y\}$.

Finally, d is the *smallest* recurrent set, so $x \in d \subseteq (d \setminus \{x\}) \cup \{y\}$, which gives $x = y$. Thus, $\mathrm{sat}\{x\} = \{x\}$, for all $x \in d$.

($\Leftarrow$) The set $\partial^+ = \{x \in \partial : \mathrm{sat}\{x\} = \{x\}\}$ is recurrent because $d \subseteq \partial^+$. Further, it is the *smallest* recurrent subset of ∂: If $S \subseteq \partial$ is *any* recurrent set and $x \in \partial^+$, the recurrence of S implies that $S \cap \mathrm{sat}\{x\} \neq \emptyset$ and hence that $x \in S$. $\square$

Corollary 5.4. *If ∂ is a T_0 space and $d \subseteq \partial$, then d is the smallest recurrent subset of ∂ iff it is recurrent and T_1 in its relative topology.*

Proof. Let d be a recurrent subset of ∂ which is relatively T_1. By Prop. 5.3, it is enough to prove that $\mathrm{sat}\{x\} = \{x\}$, for all $x \in d$. If $x \in d$ and $y \in \mathrm{sat}\{x\}$, the recurrence of d implies that $d \cap \mathrm{sat}\{y\} \neq \emptyset$. However, $\mathrm{sat}\{y\} \subseteq \mathrm{sat}\{x\}$, and so

$$\emptyset \neq d \cap \mathrm{sat}\{y\} \subseteq d \cap \mathrm{sat}\{x\} = \{x\},$$

where the equality on the right uses the fact that d is a T_1 space. But then $x \in \mathrm{sat}\{y\}$, revealing that $\mathrm{sat}\{x\} = \mathrm{sat}\{y\}$. As ∂ is a T_0 space, we must have $x = y$, which proves $\mathrm{sat}\{x\} = \{x\}$, as desired. □

Then even the slightest bit of separation makes a recurrent set canonical.

Corollary 5.5. *The only recurrent subset of a T_1 space is itself.*

But now we definitely need an example of a nontrivial recurrent set.

Proposition 5.6. *The smallest recurrent subset of a dcpo in its Scott topology is the set of maximal elements.*

Proof. In a poset P with its Scott topology, $\mathrm{sat}\, A = \uparrow A$, for $A \subseteq P$. By Zorn's lemma, the maximal elements $\max D$ of a dcpo D are recurrent in the Scott topology. As it clear that $\mathrm{sat}\{x\} = \uparrow\{x\} = \{x\}$, for all $x \in \max D$, they are also the smallest recurrent subset of D by Prop. 5.3. □

More generally, the maximal elements of a poset are recurrent iff they are the smallest recurrent set. This gives the following example.

Example 5.7. The formal ball model $\mathbf{B}X$ of a metric space (X, d) is a continuous poset. As a space in its Scott topology, its smallest recurrent subset is $\max \mathbf{B}X = \{(x, 0) : x \in X\}$.

For a space without a smallest recurrent set, consider the poset $(\mathbb{N}, \leq)$ in its Scott topology.

Theorem 5.8. *Let ∂ be a second countable space with a smallest recurrent subset d. If d is regular, then d is a G_δ subset of ∂.*

Proof. Take a countable basis $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ for ∂ and use it to define a countable index set

$$I := \{(a, b) \in \mathcal{B}^2 : \bar{b} \cap d \subseteq a \cap d\} \subseteq \mathcal{B}^2.$$

This set is nonempty by the regularity of d, as the closure of $b \cap d$ in d is just $\bar{b} \cap d$, by basic point set topology. To each $(a, b) \in I$, we associate an open subset U_{ab} of ∂ given by $U_{ab} = (\partial \setminus \bar{b}) \cup a$. We shall prove that

$$d = \bigcap_{(a,b) \in I} U_{ab}.$$

As $d \subseteq U_{ab}$ for all $(a, b) \in I$, the inclusion $d \subseteq \bigcap U_{ab}$ is trivial. For the other, let $x \in \bigcap U_{ab}$. By the recurrence of d, choose $y \in \mathrm{sat}\{x\} \cap d$. We claim that $\mathrm{sat}\{x\} \subseteq \mathrm{sat}\{y\}$. For if U is an open set around y, there is $(a, b) \in I$ with

$$y \in \bar{b} \cap d \subseteq a \cap d \text{ and } a \subseteq U,$$

where we use that $\mathcal{B}$ is a basis for ∂, followed by the regularity of d. Now since $y \in \mathrm{sat}\{x\}$ and $y \in \bar{b}$, we must have $x \in \bar{b}$; otherwise, $x \in \partial \setminus \bar{b}$, which contradicts $y \in \mathrm{sat}\{x\}$. But $x \in U_{ab}$ and $x \in \bar{b}$ imply that $x \in a \subseteq U$. Hence, $\mathrm{sat}\{x\} \subseteq \mathrm{sat}\{y\}$.

But d is the *smallest* recurrent subset of ∂, and since $y \in d$, Prop. 5.3 ensures that $\mathrm{sat}\{y\} = \{y\}$. Thus, $x \in \mathrm{sat}\{x\} = \mathrm{sat}\{y\} = \{y\}$, which leaves us with $x = y \in d$, finishing the proof. □

Traditionally we are taught that in order to prove that a subspace Y of a space X is a G_δ subset, we must know that X is Hausdorff, that Y is dense, and more importantly, that Y is complete in some sense, usually Čech-complete.

In Theorem 5.8, the larger space ∂ need not be Hausdorff, the subspace d need not be complete, but we can nevertheless conclude that d is a G_δ subset of ∂. For example, this implies that the *rationals* are a G_δ subset of their formal ball model $\mathbf{B}\mathbb{Q}$, a continuous poset whose Scott topology is not even sober, much less Hausdorff. Recurrence is a rare form of denseness indeed.

6. Applications

We now apply the techniques discussed previously.

Theorem 6.1 (Lawson [16])**.** *The Lawson topology on an ω-continuous dcpo is Polish.*

Proposition 6.2. *Let D be an ω-continuous dcpo. If $X \subseteq \max D$ is a G_δ subset of D, then*

(i) *X is a second countable, Choquet complete T_1 space.*

(ii) *Every closed subset of X is Baire.*

(iii) *X is the continuous image of a Polish space.*

Proof. (i) Domains are Choquet complete, and Choquet completeness is hereditary for G_δ sets. (ii) Domains are completely Baire and this property is inherited by G_δ sets as well.

(iii) Consider X both as a space in its relative Lawson topology, X_λ, and as one in its inherited Scott topology, X_σ. The identity mapping

$$1 : X_\lambda \to X_\sigma$$

is continuous because the Lawson topology contains the Scott topology. However, the fact that all Scott open sets are Lawson open also means that X_λ is a G_δ set in the Lawson topology on D. But D is Polish in the Lawson topology. Thus, X_λ is Polish, and so X_σ is the continuous image of a Polish space. □

Under the assumption of analytic determinacy, we can give a very different, but interesting, "proof" of Theorem 3.11.

Corollary 6.3 (Σ^1_1 Determinacy)**.** *The maximal elements of an ω-continuous dcpo are regular iff Polish.*

Proof. Let D be an ω-continuous dcpo with $X = \max D$. By Prop 5.6, X is the smallest recurrent subset of a second countable space D. In addition, X is regular, so Theorem 5.8 implies that X is a G_δ set. By Prop. 6.2, X is completely Baire and the continuous image of a Polish space. However, as a regular second countable space, it is separable metrizable. Then it is analytic and completely Baire. By Theorem 2.40, it is Polish. □

Our next application is a result about models in general, the Baire theorem.

Proposition 6.4. *The maximal elements of a continuous dcpo are a Baire space in their relative Scott topology.*

Proof. As the Scott topology on a continuous dcpo is Baire, the maximal elements are a recurrent subset of a Baire space. By Theorem 5.2 , they are Baire. □

Corollary 6.5. *There is no model of the rationals.*

Proof. The rationals are not a Baire space. □

And now we have a better explanation for the rationals: Not only do they have no countably based model, they have no model period. In addition, the hierarchy for countably based models improves as follows:

$$\text{regularity} \Longrightarrow \text{measurement} \Longrightarrow G_\delta \Longrightarrow \text{Baire}.$$

Finally, an application to classical topology: A new way to unify the Baire theorems of analysis.

Corollary 6.6. *All complete metric spaces and all locally compact Hausdorff spaces are Baire.*

Proof. A locally compact Hausdorff space X can be modelled with its upper space $\mathbf{U}X$, while a complete metric space X can be modelled with its formal ball model $\mathbf{B}X$. By Prop. 6.4, any space with a model is Baire. □

7. Closing Remarks

First, we now have several new techniques at our disposal for studying models of spaces: The relationship between the complete Baire property and topological completeness, that domains are Choquet complete, and the fact that the maximal elements form the smallest recurrent subset of a domain. All of these properties should be regarded as fundamental.

Second, to topologists, the author urges you to take a look at this area (there are some questions at the end of this paper). The author is not a topologist, but still the following are very clear: The questions in this area are interesting, and their solutions involve new applications of certain classical ideas, applications of essentially unknown topological ideas (that are often termed "obscure"), and the introduction of new topological ideas, like recurrence, which have no classical counterparts.

Finally, to domain theorists, we have seen that in order to understand the topological structure of $\max D$ in its relative Scott topology, it is sufficient to simply study the Scott topology. In doing so, we obtain better results with proofs that are far simpler. Even in Corollary 6.3, where we explicitly make use of the Lawson topology, we only do so to express a characteristic of the Scott topology: The Scott topology has the property that it may be extended to a Polish one. As the descriptive set theorists have taught us (Theorem 2.40), this is a subtle but legitimate topological property.

8. Questions

Please send email to kmartin@comlab.ox.ac.uk immediately if you know the answer to any of these questions. Also, see www.math.tulane.edu/˜martin for much more.

(i) If X is a second countable, Choquet complete T_1 space in which all closed subsets are Baire, is X developable?

(ii) The rationals may not be embedded in a countably based domain as the set of all maximal elements, though they can be embedded as a dense subset of the maximals. Can the rationals be embedded as a *closed* subset of $\max D$, for D ω-continuous?

(iii) Let D be an ω-continuous dcpo. Prove that if $\max D$ is developable, then there is a measurement $\mu : D \to [0,\infty)^*$ with $\ker \mu = \max D$.

(iv) Let D be an ω-continuous dcpo. Is $\max D$ a G_δ subset of D with respect to the Scott topology?

(v) If X is a space with a base of countable order, and Y is a G_δ subset of X, then does Y have a base of countable order?

(vi) Let D be a continuous dcpo. Prove that if $\max D$ is a regular developable space (Moore space), then $\max D$ is a G_δ subset of D with respect to the Scott topology.

(vii) Prove that every Čech-complete space has a model by a Scott domain.

(viii) If we enlarge the usual topology on the real line, so as to include sets of the form $U \cap (\mathbb{R} \setminus \mathbb{Q})$, where $U \subseteq \mathbb{R}$ is a union of open intervals, what is obtained is a second countable Hausdorff space $\mathbb{R}_\mathbb{Q}$ that is not regular, but is Choquet complete. The space $\mathbb{R}_\mathbb{Q}$ cannot be a G_δ subset of a countably based domain: It contains a closed copy of the rationals.

Does $\mathbb{R}_\mathbb{Q}$ have a countably based model?

(ix) If $X \simeq \max D$ is a G_δ subset of an ω-continuous dcpo D, must X be the kernel of a measurement on a countably based domain? (It is known that the answer is no if D is a first countable Scott domain.)

(x) Prove Theorem 5.8 without the assumption of regularity.

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