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ON BOOLEAN ALGEBRAS OF MANY-VALUED MAPPINGS

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Abstract

Two Boolean algebras of many-valued mappings are examined. Automorphisms relatively which a free set is an invariant are defined. A number of notions proves to be in duality. Some results are applied to topology.

A set-mapping is a many-valued mapping $f\colon X \to \mathcal{P}(X)$, where X is a set, $\mathcal{P}(X)$ is the family of all subsets of X, and $x \notin f(x)$ for each $x \in X$. A subset $X' \subset X$ is called a *free set relatively* f if $x \notin f(y)$ for each $x, y \in X'$. A set-mapping f is called T_0- *separating* (resp. T_1- *separating*) if for every different points $x, y \in X$ there is $z \in X$ such that $|f(z) \cap \{x, y\}| = 1$ (resp. $f(z) \cap \{x, y\} = \{x\}$).

Denote by $\mathcal{F}$ the set of all set-mappings on a set X. Define binary operations $\wedge$, $\vee$, and an unary operation C on $\mathcal{F}$ by the formulas

$$\begin{aligned} f_1 \wedge f_2(x) &= f_1(x) \cap f_2(x),\ f_1 \vee f_2(x) \\ &= f_1(x) \cup f_2(x),\ Cf(x) \\ &= (X \setminus \ \{x\}) \setminus \ f(x). \end{aligned}$$

Theorem 1. *The algebra* $(\mathcal{F}, \vee, \wedge, C)$ *is a complete Boolean algebra with the null element* $f_\wedge(x) \equiv \emptyset$ *and the identity element* $f_\vee(x) = X \setminus \{x\}$ *for each* $x \in X$.

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A proof of the theorem is a simple check of a realization of axioms of a Boolean algebra, and we omit this step.

Define a mapping $^-\colon \mathcal{F} \to \mathcal{F}$ by the formula

$$\overline{f}(x) = \{y\colon x \in f(y)\}.$$

It is evidently that $\overline{f}\colon X \to \mathcal{P}(X)$ and $x \notin \overline{f}(x)$.

Theorem 2. *The mapping $^-\colon \mathcal{F} \to \mathcal{F}$ is an automorphism of the Boolean algebra $\mathcal{F}$.*

Proof. We have the equality $\overline{f \wedge g} = \overline{f} \wedge \overline{g}$. Indeed,

$$\begin{aligned} \overline{f \wedge g}(x) &= \{y\colon x \in f \wedge g(y)\} \\ &= \{y\colon x \in f(y) \cap g(y)\} \\ &= \{y\colon x \in f(y)\} \cap \{y\colon x \in g(y)\} \\ &= \overline{f}(x) \cap \overline{g}(x) \\ &= \overline{f} \wedge \overline{g}(x). \end{aligned}$$

Analogously the equality $\overline{f \vee g} = \overline{f} \vee \overline{g}$ is proved. Further, $\overline{f}_\wedge(x) = \{y\colon x \in f_\wedge(y)\} = \emptyset$. Hence $\overline{f}_\wedge = f_\wedge$. Besides $\overline{f}_\vee(x) = \{y\colon x \in f_\vee(y)\} = \{y\colon x \in X \setminus \{y\}\} = X \setminus \{x\}$. Thus the mapping $^-\colon \mathcal{F} \to \mathcal{F}$ is a homomorphism. But $\overline{\overline{f}}(x) = \{y\colon x \in \overline{f}(y)\} = \{y\colon y \in f(x)\} = f(x)$. Hence it is a bijection . It follows that $^-\colon \mathcal{F} \to \mathcal{F}$ is an automorphism. □

Theorem 3. *Let X be a set. Let $\mathcal{F}$ and $^-\colon \mathcal{F} \to \mathcal{F}$ are defined as above. Let $f \in \mathcal{F}$ and let κ be a cardinal number. Then the following assertions are fulfilled.*

(1) *Free sets are invariants of the automorphism $^-\colon \mathcal{F} \to \mathcal{F}$.*

(2) *The T_0-separation and injectiveness are dual properties with respect to the automorphism $^-\colon \mathcal{F} \to \mathcal{F}$.*

(3) *The T_1-separation and an incomparability (by the set − theoretic inclusion) are dual properties with respect to the automorphism $^-\colon \mathcal{F} \to \mathcal{F}$.*

(4) *The statement* $|f(x)| = \kappa$ *is dual to the statement* $|\{y \in X : x \in f(y)\}| = \kappa$ *relative to* $^{-} : \mathcal{F} \to \mathcal{F}$.

(5) *The statement* $|f(x) \cap f(y)| < \kappa$ *is dual to the statement* $|\{z \in X : \{x, y\} \subset f(z)\}| < \kappa$ *relative to* $^{-} : \mathcal{F} \to \mathcal{F}$.

Proof. We shall prove assertion (1). Let $X' \subset X$ and let X' is free relatively f. Let $x, y \in X'$ and $x \neq y$. Then $x \notin f(y)$ and $y \notin f(x)$. But $\overline{f}(x) = \{z : x \in f(z)\}$ and $\overline{f}(y) = \{z : y \in f(z)\}$. Hence $y \notin \overline{f}(x)$ and $x \notin \overline{f}(y)$. Consequently X' is a free set relatively $\overline{f}$. We shall now prove assertion (3). Let f is T_1-separating. Then there are z_1 and z_2 such that $x \in f(z_1)$, $y \notin f(z_1)$, $x \notin f(z_2)$ and $y \in f(z_2)$. But $\overline{f}(x) = \{u : x \in f(u)\}$ and $\overline{f}(y) = \{u : y \in f(u)\}$. Hence $z_1 \in \overline{f}(x)$, $z_2 \notin \overline{f}(x)$, $z_2 \in \overline{f}(y)$ and $z_1 \notin \overline{f}(y)$. Therefore $\overline{f}(x)$ and $\overline{f}(y)$ are incomparable. Conversely, let $f(x)$ and $f(y)$ are incomparable. Then there are z_1 and z_2 such that $z_1 \in f(x) \setminus f(y)$ and $z_2 \in f(y) \setminus f(x)$. But $\overline{f}(z_1) = \{u : z_1 \in f(u)\}$ and $\overline{f}(z_2) = \{u : z_2 \in f(u)\}$. Hence $x \in \overline{f}(z_1)$ and $y \notin \overline{f}(z_1)$. Also $x \notin \overline{f}(z_2)$ and $y \in \overline{f}(z_2)$. To prove assertion (4) we note that $f(x) = \overline{\overline{f}}(x) = \{y : x \in \overline{f}(y)\}$. Hence $|f(x)| = |\{y : x \in \overline{f}(y)\}|$. Analogously we can prove assertions (2) and (5). □

A. Hajnal proved in [1] the following statement:

Theorem 4 [GCH]**.** *Let* X *be a set*, $|X| = \omega_{\alpha+1}$ *and* ω_α *be a regular cardinal number. Then there exists a set-mapping* f *such that*

($\circ$) $|f(x)| = \omega_\alpha$ *for each* $x \in X$.

($\circ\circ$) $|f(x) \cap f(y)| < \omega_\alpha$ *for every* $x, y \in X$ *under* $x \neq y$.

($\circ\circ\circ$) *The cardinality of each free set relatively* f *is less than* $\omega_{\alpha+1}$.

In view of Theorem 3 the following theorem is equivalent to Theorem 4.

Theorem 5 [GCH]**.** *Let X be a set, $|X| = \omega_{\alpha+1}$ and ω_α be a regular cardinal number. Then there exists a set-mapping f such that*

($\times$) $|\{y \in X : x \in f(y)\}| = \omega_\alpha$ *for each* $x \in X$.

($\times\times$) $|\{z \in X : \{x, y\} \subset f(z)\}| < \omega_\alpha$ *for every* x, $y \in X$ *under* $x \neq y$.

($\times\times\times$) *The cardinality of each free set relatively f is less than* $\omega_{\alpha+1}$.

Under [CH] two corresponding assertions are obtained.

Theorem 4+ [CH]**.** *Let X be a set, $|X| = \omega_1$. Then there exists a set-mapping f such that*

($\circ$) $|f(x)| = \omega$ *for each* $x \in X$.

($\circ\circ$) $|f(x) \cap f(y)| < \omega$ *for every* $x, y \in X$ *under* $x \neq y$.

($\circ\circ\circ$) *The cardinality of each free set relatively f is less than* ω_1.

Theorem 5+ [CH]**.** *Let X be a set, $|X| = \omega_1$. Then there exists a set-mapping f such that*

($\times$) $|\{y \in X : x \in f(y)\}| = \omega$ *for each* $x \in X$.

($\times\times$) $|\{z \in X : \{x, y\} \subset f(z)\}| < \omega$ *for every* $x, y \in X$ *under* $x \neq y$.

($\times\times\times$) *The cardinality of each free set relatively f is less than* ω_1.

Continuum hypothesis is essential for last theorems. Indeed, assume that properties ($\circ$) and ($\circ\circ$) of Theorem 4+ are fulfilled. Under [MA + $\neg$ CH] by Theorem of M. Wage [3] there exists an uncountable set $B \subset X$ such that $|B \cap f(x)| < \omega$ for each $x \in X$. By Δ-System Lemma there is an uncountable set $A \subset B$ such that $f(x) \cap A = F$ for each $x \in A$, where F is a constant finite set. Then the set $A \setminus F$ is an uncountable set, which is free relatively f, i.e., property ($\circ \circ \circ$) is not fulfilled.

In a way analogous to this proof the following more general statement can be proved.

Theorem 4* [MA]**.** *Let* κ *be a regular cardinal number such that* $\omega < \kappa < 2^\omega$ *and let* X *be a set of the cardinality* κ. *Let* f *be a set-mapping on* X *such that*

($\circ$) $|f(x)| = \omega$ *for each* $x \in X$.

($\circ\circ$) $|f(x) \cap f(y)| < \omega$ *for every* $x, y \in X$ *under* $x \neq y$.

Then there exists a subset of X *of the cardinality* κ *which is free relatively* f.

Formulate the dual statement.

Theorem 5* [MA]**.** *Let* κ *be a regular cardinal number such that* $\omega < \kappa < 2^\omega$ *and let* X *be a set of the cardinality* κ. *Let* f *be a set-mapping on* X *such that*

($\times$) $|\{y \in X \colon x \in f(y)\}| = \omega$ *for each* $x \in X$.

($\times\times$) $|\{z \in X \colon \{x, y\} \subset f(z)\}| < \omega$ *for every* $x, y \in X$ *under* $x \neq y$.

Then there exists a subset of X *of the cardinality* κ *which is free relatively* f.

We consider also the set of all mappings $f \colon X \to \mathcal{P}(X)$ without the requirement $x \notin f(x)$. Denote this set by $\mathcal{R}$. Define binary operations $f \vee g$ and $f \wedge g$ by the formulas $f \vee g(x) = f(x) \cup g(x)$ and $f \wedge g(x) = f(x) \cap g(x)$ respectively, and an unary operation Cf by the formula $Cf(x) = X \setminus f(x)$ on the set $\mathcal{R}$.

Consider a mapping ${}^\sim \colon \mathcal{R} \to \mathcal{R}$, which is similar to the mapping ${}^- \colon \mathcal{F} \to \mathcal{F}$. More exactly, we set $\widetilde{f}(x) = \{y \colon x \in f(y)\}$. There are the following analogs of Theorems 1-3.

Theorem 6. *The algebra* $(\mathcal{R}, \vee, \wedge, C)$ *is a complete Boolean algebra with thenull element* $f_\wedge(x) \equiv \emptyset$ *and the identity element* $f_\vee(x) \equiv X$.

Theorem 7. *The mapping* ${}^\sim \colon \mathcal{R} \to \mathcal{R}$ *is an automorphism of the Boolean algebra* $\mathcal{R}$.

A mapping $h \colon \mathcal{R} \to \mathcal{F}$ *defined by the formula* $h(f)(x) = f(x) \setminus \{x\}$ *is a homomorphism of Boolean algebras.*

The Boolean algebras $\mathcal{F}$ *and* $\mathcal{R}$ *are isomorphic and their Stone's space is* βX, *where* X *is taken with the discrete topology (we concider only infinite* X).

Proof. First conclusion of the theorem is proved in just the same way as Theorem 2. Second one is trivial. Let us prove the last conclusion of the theorem. Consider the function

$$f^{(x,y)}(z) = \begin{cases} \emptyset, & \text{if } \ z \neq x; \\ \{y\}, & \text{if } \ z = x. \end{cases}$$

The function is evidently an atom of $\mathcal{R}$, and it is an atom of $\mathcal{F}$ under $x \neq y$. It is obvious that there are no other atoms of $\mathcal{R}$ and $\mathcal{F}$. Hence the Boolean algebras $\mathcal{F}$ and $\mathcal{R}$ are atomic, and the cardinalities of the sets of all atoms of $\mathcal{F}$ and $\mathcal{R}$ coincide and are equal to the cardinality of X. Then by theorem of Lindenbaum - Tarski [4] the Boolean algebras $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{P}(X)$ are isomorphic. Thus, βX is Stone's space of $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{P}(X)$ to within a homeomorphism. □

Theorem 8. *Let X be a set. Let $\mathcal{R}$ and $^\sim\colon \mathcal{R} \to \mathcal{R}$ are defined as above. Let $f \in \mathcal{R}$ and let κ be a cardinal number. Then the following assertions are fulfilled.*

(1) *Free sets are invariants of the automorphism $^\sim\colon \mathcal{R} \to \mathcal{R}$.*

(2) *The T_0-separation and injectiveness are dual properties with respect to the automorphism $^\sim\colon \mathcal{R} \to \mathcal{R}$.*

(3) *The T_1-separation and an incomparability (by the set-theoretic inclusion) are dual properties with respect to the automorphism$^\sim\colon \mathcal{R} \to \mathcal{R}$.*

(4) *The statement $|f(x)| = \kappa$ is dual to the statement $|\{y \in X\colon x \in f(y)\}| = \kappa$ relative to $^\sim\colon \mathcal{R} \to \mathcal{R}$.*

(5) *The statement $|f(x) \cap f(y)| < \kappa$ is dual to the statement $|\{z \in X\colon \{x, y\} \subset f(z)\}| < \kappa$ relative to $^\sim\colon \mathcal{R} \to \mathcal{R}$.*

(6) *Let $|X| = \kappa$. Then the statement $[\ |X \setminus \cup\{f(x)\colon x \in X'\}| < \kappa$ for each $X' \subset X$ under $|X'| = \kappa\ ]$ is an invariant of the automorphism $^\sim\colon \mathcal{R} \to \mathcal{R}$.*

Proof. Only claim (6) need be proved, the others can be proved as above, in Theorem 3. Prove claim 6. Let $|X| = \kappa$, $f \in \mathcal{R}$, and let $|X \setminus \cup\{f(x) : x \in X'\}| < \kappa$ *for each* $X' \subset X$ *under* $|X'| = \kappa$. Let us assume that there exists a set $X' \subset X$ of the cardinality κ such that $|X \setminus \cup\{\widetilde{f}(x) : x \in X'\}| = \kappa$. It follows that $|X \setminus \cup\{f(y) : y \in X \setminus \cup\{\widetilde{f}(x) : x \in X'\}\}| < \kappa$. It is easy to see that $y \in X \setminus \cup\{\widetilde{f}(x) : x \in X'\}$ iff $f(y) \cap X' = \emptyset$. Hence $X' \subset X \setminus \cup\{f(y) : y \in X \setminus \cup\{\widetilde{f}(x) : x \in X'\}\}$. Therefore $|X'| < \kappa$, a contradiction. □

A. Hajnal proved in [1] the following statement:

Theorem 9 [GCH]**.** *Let X be a set, $|X| = \omega_{\alpha+1}$ and ω_α be a regular cardinal number. Then there exists a family $\mathcal{S} \subset \mathcal{P}(X)$ such that the following assertions are fulfilled.*

(+) $|\mathcal{S}| = \omega_{\alpha+1}$.

(++) $|S| = \omega_\alpha$ *for each* $S \in \mathcal{S}$.

(+ + +) $|S_1 \cap S_2| < \omega_\alpha$ *for every* $S_1, S_2 \in \mathcal{S}$ *under* $S_1 \neq S_2$.

(++++) *If* $\mathcal{S}' \subset \mathcal{S}$ *and* $|\mathcal{S}'| = \omega_{\alpha+1}$, *then* $|X \setminus \cup\mathcal{S}'| < \omega_{\alpha+1}$.

Because $|X| = |\mathcal{S}|$, the theorem can be formulated in the following way.

Theorem 10 [GCH]**.** *Let X be a set, $|X| = \omega_{\alpha+1}$ and ω_α be a regular cardinal number. Then there exists $f \in \mathcal{R}$ such that the following assertions are fulfilled.*

(1) $|f(x)| = \omega_\alpha$ *for each* $x \in X$.

(2) $|f(x) \cap f(y)| < \omega_\alpha$ *for every* $x, y \in X$ *under* $x \neq y$.

(3) *If* $X' \subset X$ *and* $|X'| = \omega_{\alpha+1}$, *then*

$$|X \setminus \cup\{f(x) : x \in X'\}| < \omega_{\alpha+1}.$$

Theorem 4 is an immediate consequence of Theorem 10. In a different way A. Hajnal proved that Theorem 4 followed from Theorem 9 [1]. By Theorem 8 the following statement is equivalent to Theorem 10.

Theorem 11 [GCH]**.** *Let X be a set, $|X| = \omega_{\alpha+1}$ and ω_α be a regular cardinal number. Then there exists $f \in \mathcal{R}$ such that the following assertions are fulfilled.*

(i) $|\{y \in X\colon x \in f(y)\}| = \omega_\alpha$ *for each* $x \in X$.

(ii) $|\{z \in X\colon \{x, y\} \subset f(z)\}| < \omega_\alpha$ *for every* $x, y \in X$ *under* $x \neq y$.

(iii) If $X' \subset X$ *and* $|X'| = \omega_{\alpha+1}$, *then*

$$|X \setminus \cup\{f(x)\colon x \in X'\}| < \omega_{\alpha+1}.$$

The following theorem follows from Theorem 11.

Theorem 12 [GCH]**.** *Let* X *be a set,* $|X| = \omega_{\alpha+1}$ *and* ω_α *be a regular cardinal number. Then there are* $f \in \mathcal{R}$ *and a set* $Y \subset X$ *of the cardinality* $\omega_{\alpha+1}$ *such that the following assertions are fulfilled.*

(1) $|\{f(y)\colon x \in f(y)\}| = \omega_\alpha$ *for each* $x \in Y$.

(2) $|\{f(y)\colon y \in Y\}| = \omega_{\alpha+1}$.

(3) $|\{f(z)\colon \{x, y\} \subset f(z)\}| < \omega_\alpha$ *for every* $x, y \in X$ *under* $x \neq y$.

(4) *If* $X' \subset X$ *and* $|X'| = \omega_{\alpha+1}$, *then*

$$|X \setminus \cup\{f(x)\colon x \in X'\}| < \omega_{\alpha+1}.$$

Proof. We only have to prove claims (1) and (2). Denote by D the set $\{x \in X\colon (\exists y \in X)\ f(y) = \{x\}\ \}$. It follows from Theorem 11 (iii) that $|D| \leq \omega_\alpha$. Hence by Theorem 11 (i) $|\{y \in X\colon f(y) \cap D \neq \emptyset\}| \leq \omega_\alpha$. Then $|\{y \in X\colon f(y) \cap D = \emptyset\}| = \omega_{\alpha+1}$. Therefore by Theorem 11 (iii) the set $X \setminus \cup\{f(y)\colon f(y) \cap D = \emptyset\}$ has the cardinality less than $\omega_{\alpha+1}$. It follows that the set $\cup\{f(y)\colon f(y) \cap D = \emptyset\}$ has the cardinality $\omega_{\alpha+1}$. Put $Y = \cup\{f(y)\colon f(y) \cap D = \emptyset\}$. In view of Theorem 11 (i) and (iii) the cardinality of the set $f^{-1}(f(y))$ is at most ω_α (Theorem 11 (iii) is used for the case $f(y) = \emptyset$). Hence $|\{f(y)\colon y \in Y\}| = \omega_{\alpha+1}$. Thus claim (2) is fulfilled. Now let $x \in Y$. By definition $Y \cap D = \emptyset$. Then $f(y)$ has at least two points whenever $x \in f(y)$. In view of Theorem 11 (ii) the set $f^{-1}(f(y))$ has the cardinality less than ω_α for each point y whose image has at least two points. Since ω_α is a regular cardinal number, from Theorem 11 (i) follows that $|\{f(y)\colon x \in f(y)\}| = \omega_\alpha$. Claim (1) is proved. □

The last theorem can be formulated in the following way.

Theorem 13 [GCH]**.** *Let X be a set, $|X| = \omega_{\alpha+1}$ and ω_α be a regular cardinal number. Then there exists a family $\mathcal{C} \subset \mathcal{P}(X)$ such that the following assertions are fulfilled.*

(1) $|\mathcal{C}| = \omega_{\alpha+1}$.

(2) $|\{C \in \mathcal{C} \colon x \in C\}| = \omega_\alpha$ *for each* $x \in \cup\mathcal{C}$.

(3) $|\{C \in \mathcal{C} \colon \{x, y\} \subset C\}| < \omega_\alpha$ *for every* $x, y \in X$ *under* $x \neq y$.

(4) *if* $\mathcal{C}' \subset \mathcal{C}$ *and* $|\mathcal{C}'| = \omega_{\alpha+1}$, *then* $|X \setminus \cup\mathcal{C}'| < \omega_{\alpha+1}$.

We now prove that Theorem 12 implies Theorem 9. Choose a set $X' \subset X$ such that f/X' would be a bijection with $rng(f/X') = rng(f)$. It is obvious that $|X'| = \omega_{\alpha+1}$. Choosing a bijection $a \colon X' \to X$, we obtain the numbering of X by elements of X', i.e., $X = \{a_x \colon x \in X'\}$, where $a_x \neq a_y$ under $x \neq y$. Define a mapping $g \colon X \to \mathcal{P}(X)$ by the formula $g(a_x) = f(x)$. By definition we have

$$\widetilde{g}(a_x) = \{a_y \colon a_x \in g(a_y)\} = \{a_y \colon a_x \in f(y)\}.$$

Show that the family $\{\widetilde{g}(a_x) \colon a_x \in Y\}$ satisfies Theorem 9.

$$\begin{aligned} |\widetilde{g}(a_x)| &= |\{a_y \colon a_x \in f(y)\}| \\ &= |\{y \in X' \colon a_x \in f(y)\}| \\ &= |\{f(y) \colon a_x \in f(y)\}|. \end{aligned}$$

By Theorem 12 (1) it follows that $|\widetilde{g}(a_x)| = \omega_\alpha$. Hence property (++) of Theorem 9 is fulfilled. Further,

$$\widetilde{g}(a_x) \cap \widetilde{g}(a_y) = \{a_z \colon \{a_x, a_y\} \subset f(z)\}.$$

But

$$\begin{aligned} |\{a_z \colon \{a_x, a_y\} \subset f(z)\}| &= |\{z \in X' \colon \{a_x, a_y\} \subset f(z)\}| \\ &= |\{f(z) \colon \{a_x, a_y\} \subset f(z)\}|. \end{aligned}$$

By Theorem 12 (3) it follows that $|\widetilde{g}(a_x) \cap \widetilde{g}(a_y)| < \omega_\alpha$ under $a_x \neq a_y$. Hence property (+++) of Theorem 9 is fulfilled. Because Y has the cardinality $\omega_{\alpha+1}$, from (++) and (+++) follows that property (+) of Theorem 9 is fulfilled. Indeed, from

(++) and (+++) follows that $\widetilde{g}(a_x) \neq \widetilde{g}(a_y)$ under $a_x \neq a_y$. Let now $Y' \subset Y$ and let $|Y'| = \omega_{\alpha+1}$. Consider the set $Z = \{x \in X' \colon a_x \in Y'\}$. Then $|Z| = \omega_{\alpha+1}$ and $Y' = \{a_x \colon x \in Z\}$. $a_z \in X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\}$ iff $a_z \notin \widetilde{g}(a_x)$ for each $x \in Z$, i.e., $a_z \notin \{a_y \colon a_x \in g(a_y)\}$ for each $x \in Z$ or, that is the same, $a_x \notin f(z)$ for each $x \in Z$. Hence $\{a_x \colon x \in Z\} \cap f(z) = \emptyset$ for each z such that $a_z \in X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\}$. Then $\{a_x \colon x \in Z\} \cap \cup\{f(y) \colon a_y \in (X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\})\} = \emptyset$. Let us assume that $|X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\}| = \omega_{\alpha+1}$. Then by Theorem 12 (4) we have $|X \setminus \cup\{f(y) \colon a_y \in (X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\})\}| < \omega_{\alpha+1}$. But $\{a_x \colon x \in Z\} \subset X \setminus \cup\{f(y) \colon a_y \in (X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\})\}$. Therefore $|Y'| < \omega_{\alpha+1}$, contradiction. Hence $|X \setminus \cup\{\widetilde{g}(a_x) \colon x \in Z\}| < \omega_{\alpha+1}$ and property (++++) of Theorem 9 is fulfilled.

So it has been shown that Theorems 9 $\div$ 13 are equivalent. In fact, we have proved that statements of Theorems 9 $\div$ 13 are equivalent in ZFC provided ω_α is a regular cardinal number. Select two special cases equivalent theorems.

Theorem 9+[CH]**.** *Let X be a set, $|X| = \omega_1$. Then there exists a family $\mathcal{S} \subset \mathcal{P}(X)$ such that the following assertions are fulfilled.*

(+) $|\mathcal{S}| = \omega_1$.

(++) $|S| = \omega$ *for each* $S \in \mathcal{S}$.

(+ + +) $|S_1 \cap S_2| < \omega$ *for every* $S_1, S_2 \in \mathcal{S}$ *under* $S_1 \neq S_2$.

(+ + ++) *If* $\mathcal{S}' \subset \mathcal{S}$ *and* $|\mathcal{S}'| = \omega_1$, *then* $|X \setminus \cup\mathcal{S}'| < \omega_1$.

Theorem 13+ [CH]**.** *Let X be a set, $|X| = \omega_1$. Then there exists a family $\mathcal{C} \subset \mathcal{P}(X)$ such that the following assertions are fulfilled.*

(1) $|\mathcal{C}| = \omega_1$.

(2) $|\{C \in \mathcal{C} \colon x \in C\}| = \omega$ *for each* $x \in \cup\mathcal{C}$.

(3) $|\{C \in \mathcal{C} \colon \{x, y\} \subset C\}| < \omega$ *for every* $x, y \in X$ *under* $x \neq y$.

(4) *if* $\mathcal{C}' \subset \mathcal{C}$ *and* $|\mathcal{C}'| = \omega_1$, *then* $|X \setminus \cup\mathcal{C}'| < \omega_1$.

Either of these theorems implies Theorem 4+ and Theorem 5+ because, as we have observed above, a statement of Theorem 4 follows from a statement of Theorem 10. In view of Theorems 4* and 5* statements of Theorems 9+ and 13+ are false under $MA + \neg CH$.

Finally we consider a topological aspect . A base $\mathcal{B}$ for a topological space X is a *weakly uniform base* if no two points of X belong to infinitely many members of $\mathcal{B}$. More generally, any family of sets with this property is called *weakly uniform* (also known as a *pair-finite family*). A further generalization of the weakly uniform base is obtained if in the definition, "infinitely" is replaced by "uncountably". These notions were introduced in [2] by R.W.Heath and W.F.Lindgren in 1976 year, and they have been investigated in General Topology since then.

Corollary 1 [CH]**.** *There exists a Hausdorff space Y with a point-countable weakly uniform subbase $\mathcal{H}$ such that the following conditions are fulfilled.*

(*a*) $|\mathcal{H}| = \omega_1$.

(*b*) *for every $Z \subset Y$ each point-finite subfamily of the family $\mathcal{H}/Z$ is countable.*

(*c*) *for every $Z \subset Y$ each cover of Z by elements of $\mathcal{H}/Z$ contains a countable subcover.*

It is evidently that this space has a point-countable weakly uniform base and is not separable.

Proof. Use Theorem 13+. Put $Y = \cup\mathcal{C}$. It is obvious that $|Y| = \omega_1$. Identify Y with the real numbers. Let $\mathcal{R}$ be a countable uniform base for the usual topology of real line. Consider the topology $\mathcal{T}$ that is defined by the subbase $\mathcal{H} = \mathcal{C} \cup \mathcal{R}$. It is obvious that $(Y, \mathcal{T})$ is a Hausdorff nonseparable space with a point-countable weakly uniform base. Condition (a) is fulfilled by construction. $(c) \Rightarrow (b)$ follows from Δ-System Lemma. It remains to prove assertion (c). Let $\mathcal{E}$ be an uncountable cover of a set $Z \subset Y$ and $\mathcal{E} \subset \mathcal{C}/Z$. Suppose that any countable subfamily of $\mathcal{E}$ is not a cover of Z. Pick a point $x_0 \in Z$. Because the family

$\mathcal{E}_{x_0}$ is not a cover of Z, there is a point $x_1 \in Z \setminus \cup\mathcal{E}_{x_0}$. The set $\mathcal{E}_{x_0} \cup \mathcal{E}_{x_1}$ is also not a cover of Z. Continuing by induction, we construct an ω_1-sequence $< x_\xi \colon \xi \in \omega_1 >$ of points of Z such that $x_\xi \notin \cup\mathcal{E}_{x_\eta}$ under $\xi \neq \eta$. Choose $E_\xi \in \mathcal{E}_{x_\xi}$. Existence of the uncountable family $\{E_\xi \colon \xi \in \omega_1\}$ contradicts to Theorem 13+ (4). □

Theorem 14 [MA+ ¬CH]**.** *Let X be a set and $\mathcal{S}$ be a point-countable weakly uniform family of subsets of X. If for every $Y \subset X$ each point-finite subfamily of the family $\mathcal{S}/Y$ is countable, then $\mathcal{S}$ is countable.*

Proof. Suppose $\mathcal{S}$ is uncountable. Pick a point $x_0 \in X$. Because $\mathcal{S}_{x_0}$ is countable, there exists a point $x_1 \in X$ such that $\mathcal{S}_{x_1} \setminus \mathcal{S}_{x_0} \neq \emptyset$. Continuing by induction, we construct an ω_1-sequence $< x_\xi \colon \xi \in \omega_1 >$ points of X such that $\mathcal{S}_{x_\xi} \setminus \cup\{\mathcal{S}_{x_\nu} \colon \nu < \xi\} \neq \emptyset$ for each $\xi \in \omega_1$. Denote by Y the set $\{x_\xi \colon \xi \in \omega_1\}$. It is obvious that $| \cup \{\mathcal{S}_{x_\xi} \colon x_\xi \in Y\}| = \omega_1$. Besides we may assume that $|\mathcal{S}_{x_\xi}| = \omega$ for each $\xi \in \omega_1$. Since $x_\xi \neq x_\eta$ under $\xi \neq \eta$ and $\mathcal{S}$ is weakly uniform, $|\mathcal{S}_{x_\xi} \cap \mathcal{S}_{x_\eta}| < \omega$. Use Wage's theorem that has been used above. By this theorem there exists $\mathcal{S}' \subset \cup\{\mathcal{S}_{x_\xi} \colon \xi \in \omega_1\}$ such that $|\mathcal{S}'| = \omega_1$ and $|\mathcal{S}' \cap \mathcal{S}_{x_\xi}| < \omega$ for each $\xi \in \omega_1$. Then $\mathcal{S}'$ is point-finite on Y. Consequently, by the supposition, the family $\mathcal{S}'/Y$ is countable. If $S \in \mathcal{S}'$, then $S \cap Y \neq \emptyset$ because there is $\xi \in \omega_1$ such that $S \in \mathcal{S}_{x_\xi}$. Then, being point-countable, the family $\mathcal{S}'$ is countable; so, we have a contradiction. □

Corollary 2 [MA+ ¬CH]**.** *There is no a space Y with a point-countable weakly uniform subbase $\mathcal{H}$ such that the properties $(a), (b)$, and (c) of corollary 1 are fulfilled.*

Corollary 3 [MA+ ¬CH]**.** *If X is a hereditarily Lindelof T_1-space that has a weakly uniform base, then X is separable.*

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